SOME STRICT INCLUSIONS BETWEEN SPACES OF $L^p$-MULTIPLIERS

BY

J. F. PRICE

Abstract. Suppose that the Hausdorff topological group $G$ is either compact or locally compact abelian and that $C_c$ denotes the set of continuous complex-valued functions on $G$ with compact supports. Let $L^p_c$ denote the restrictions to $C_c$ of the continuous linear operators from $L^p(G)$ into $L^q(G)$ which commute with all the right translation operators.

When $1 \leq p < q \leq 2$ or $2 \leq q < p \leq \infty$ it is known that

\[ L^p_c \subseteq L^q_c. \]

The main result of this paper is that the inclusion in (1) is strict unless $G$ is finite. In fact it will be shown, using a partly constructive proof, that when $G$ is infinite

\[ \bigcup_{1 < q < p} L^q_c \nsubseteq L^p_c \nsubseteq \bigcap_{p < q \leq 2} L^q_c \]

for $1 < p < 2$, with the first inclusion remaining strict when $p = 2$ and the second inclusion remaining strict when $p = 1$. (Similar results also hold for $2 < p < \infty$.)

When $G$ is compact, simple relations will also be developed between idempotent operators in $L^p_c$ and lacunary subsets of the dual of $G$ which will enable us to find necessary conditions so that inclusion (1) is strict even if, for example, $L^p_c$ and $L^q_c$ are replaced by the sets of idempotent operators in $L^p_c$ and $L^q_c$ respectively.

1. General notation and preliminaries.

1.1 The notation outlined in this section will generally follow [1] where further details may be found. Let $G$ denote a multiplicatively written locally compact group (all topological groups will be assumed to be Hausdorff), $e$ its identity and $\lambda_G$ a fixed left Haar measure on $G$. If $G$ is compact, $\lambda_G$ is assumed to be normalized so that $\int_G \lambda_G = 1$. Let $L^p = L^p(G)$—always—denote the usual Lebesgue space of equivalence classes of functions on $G$. Its norm will be denoted by $\| \cdot \|_p$. When $1 < p < \infty$, $p'$ will always satisfy $\frac{1}{p} + \frac{1}{p'} = 1$; also $1' = \infty$ and $\infty' = 1$.

Let $C = C(G)$ denote the space of continuous complex-valued functions over $G$. $C_0$ and $C_c$ will denote the subspaces of $C$ comprised of functions which vanish at infinity and of functions which have compact supports, respectively; $C_0 = C_c = C$ when $G$ is compact. Also assume that $C_0$ is equipped with the uniform (or supremum) norm, which makes it a Banach space.

The left and right translation operators $\tau_a$ and $\rho_a$ ($a \in G$) are defined on $C$ by

\[ \tau_a f(x) = f(a^{-1}x), \quad \rho_a f(x) = f(xa^{-1}), \]

and may then be extended to measures.
1.2 Multipliers of type \((p, q)\). Let \(L_p^q\) denote the set of linear operators \(T\) from \(C_c\) into \(L^q\) which satisfy the following two properties:

(i) \(T\) is continuous when \(C_c\) is equipped with the topology induced by \(L^p\), and

(ii) \(T\) commutes with right translation, that is

\[
T\rho_a = \rho_a T \quad \text{for all} \ a \in G.
\]

We further suppose that \(L_p^q\) is a Banach space by equipping it with the norm \(T \mapsto \|T\|_{p,q}\) defined as the smallest positive number \(K\) such that

\[
\|Tf\|_q \leq K\|f\|_p
\]

for all \(f \in C_c\). Note that when \(p \neq \infty\), \(T \in L_p^q\) has a unique continuous extension to \(L^q\) which also satisfies (1.1).

A suggestion by R. E. Edwards leads to the use of the Riesz-Thorin convexity theorem to prove that \(L_p^q \subseteq \bigcap \{L^q : 1 \leq p < q \leq 2\}\)—see Theorem 4.4 below. The author is very grateful to Dr. Edwards for this suggestion, and for other more general comments. Also, the author would like to thank Professors E. Hewitt and K. A. Ross for their kind permission to read and quote from sections of their forthcoming book, *Abstract harmonic analysis*, Vol. II; and Drs. A. Figà-Talamanca and G. I. Gaudry for correspondence relating to their forthcoming publication [8].

2. Representations of \(L_p^q\) when \(G\) is compact. Throughout this section \(G\) will denote a multiplicatively written compact group (not necessarily abelian) with identity \(e\) and equipped with normalized Haar measure \(\lambda_\sigma\). If \(f \in L^1\), then \(f\) is uniquely represented by a Fourier series

\[
f \sim \sum_{\gamma \in \Gamma} d(\gamma) \text{Tr} [\hat{f}(\gamma)\gamma(\cdot)],
\]

where: \(\Gamma\) is a set of representatives, one selected from each equivalence class of continuous, irreducible, unitary representations of \(G\); \(d(\gamma)\) is the (finite) dimension of the representation \(\gamma\); \(\text{Tr}\) denotes the usual trace; and \(\hat{f}\) is the Fourier transform of \(f\), defined by

\[
\hat{f}(\gamma) = \int_G f(t) \gamma(t)^* \lambda_\sigma(t),
\]

\(\gamma(t)^*\) denoting the (Hilbert) adjoint of \(\gamma(t)\). For \(\gamma \in \Gamma\), \(H_\gamma\) will denote the corresponding Hilbert space (of dimension \(d(\gamma)\)).

The following “exchange formula” will be used occasionally: \((f * g)(\gamma) = \hat{g}(\gamma)\hat{f}(\gamma)\) for each \(\gamma \in \Gamma\), where \(f, g \in L^1\). Also the Peter-Weyl completeness theorem [10] will be frequently used without comment, generally in the form: If \(f \in L^1\) and \(\hat{f}(\gamma) = 0\) for each \(\gamma \in \Gamma\), then \(f = 0\) almost everywhere.

Let \(\mathcal{E}\) denote the set consisting of all functions \(\mu\) on \(\Gamma\) such that \(\mu(\gamma)\) is an endomorphism of \(H_\gamma\) for each \(\gamma \in \Gamma\). It is easily shown—see Theorem (35.8) of [10], for example—that to each \(T \in L_p^q\) there corresponds a unique \(\mu \in \mathcal{E}\) such that

\[
(Tf) = f^* \mu
\]
for each \( f \in C \). Let \( M_p^2 \) denote the subset of \( \mathcal{E} \) corresponding to \( L_p^q \) via (2.1). A useful formula is

\[
M_p^2 = (M_p^2)^*,
\]

where \( (M_p^2)^* = \{ \mu^* : \mu^*(\gamma) = \mu(\gamma)^* \text{ for } \gamma \in \Gamma \text{ and } \mu \in M_p^2 \} \).

For each \( \gamma \in \Gamma \) define a norm \( \| \cdot \|_{(\infty)} \) on the space of endomorphisms of \( H_{\gamma} \) as the usual norm for bounded operators. Let \( \mu \in \mathcal{E} \) and define

\[
\| \mu \|_\infty = \sup \{ \| \mu(\gamma) \|_{(\infty)} : \gamma \in \Gamma \};
\]

denote by \( \mathcal{E}_\infty \) the subspace of \( \mathcal{E} \) comprised of those elements \( \mu \) which satisfy \( \| \mu \|_\infty < \infty \). Using the Parseval formula, namely

\[
\| f \|_2 = \left\{ \sum_{\gamma \in \Gamma} d(\gamma) \text{ Tr } [\tilde{f}(\gamma)\tilde{f}(\gamma)^*] \right\}^{1/2},
\]

the following well-known result may be easily proved (see, for example, Theorem (35.4) of [10]).

2.1 Theorem.

\[
M_2^2 = \mathcal{E}_\infty.
\]

In a personal communication to the author, C. S. Herz stated that he has recently discovered a proof of

\[
L_p^q \subset L_q^q \quad \text{if } 1 \leq p \leq q \leq 2 \quad \text{or} \quad 2 \leq q \leq p \leq \infty.
\]

Theorem 2.1 and inclusion (2.4) are basic for the sequel.

2.2 Remarks on pseudomeasures. Let \( A \) denote the set of continuous functions on \( G \) whose Fourier transforms belong to \( \mathcal{E}_1 \), where

\[
\mathcal{E}_1 = \left\{ \mu \in \mathcal{E} : \| \mu \|_1 = \sum_{\gamma \in \Gamma} d(\gamma) \text{ Tr } [(\mu(\gamma)\mu(\gamma)^*)^{1/2}] < \infty \right\}.
\]

Define a norm on \( A \) by \( \| f \|_A = \| f \|_1 \). Following the procedure for LCA groups (see, for example, 4.1 of [1]) we call the normed dual \( P \) of \( A \) the space of \textit{pseudomeasures over G}. Since \( A \) and \( \mathcal{E}_1 \) are isometrically isomorphic and \( \mathcal{E}_\infty \) is isometrically isomorphic to the normed dual of \( \mathcal{E}_1 \) (see [10]), then there is an isometric isomorphism between \( P \) and \( \mathcal{E}_\infty \). One such isomorphism may be defined via an extension of the Fourier transform:

**Definition.** The Fourier transform of \( \sigma \in P \) is the unique element \( \hat{\sigma} \) of \( \mathcal{E}_\infty \) which satisfies

\[
\sum_{\gamma \in \Gamma} d(\gamma) \text{ Tr } [\hat{f}(\gamma)\hat{\sigma}(\gamma)] = \langle f, \sigma \rangle
\]

for each \( f \in A \).

Finally \( P \) may be made into a Banach algebra under convolution by defining \( \sigma_1 \ast \sigma_2 \), where \( \sigma_1, \sigma_2 \in P \), as the element of \( P \) which satisfies

\[
(\sigma_1 \ast \sigma_2)^\gamma = \delta_2 \delta_1
\]

on \( \Gamma \). Then \( A \) becomes a two-sided ideal of \( P \).
With this notation, Theorem 2.1 may be expressed in a more suggestive form (cf. [1] and Chapter 16 of [4]), namely: "A continuous linear operator $T$ from $L^2$ into $L^2$ belongs to $L^2$ (that is, commutes with the $\rho_\sigma$) if and only if there exists $\sigma \in P$ such that $Tf = \sigma * f$ for all $f \in C$.

2.3 The following result provides another characterization of $L^2$, when $p \neq \infty$, and also provides some motivation for the study of idempotent multipliers in the next section.

**Proposition.** The set $L^p$, $p \neq \infty$, is the closure in the strong operator topology of the set of finite linear combinations of operators from $L^p$ into $L^2$ of the form

\[(2.5) f \mapsto \phi * f\]

where, for some $\gamma \in \Gamma$,

(i) $\phi(\gamma)$ is an orthogonal projection in $H_\gamma$, and

(ii) supp $\phi = \{\gamma\}$.

**Proof.** Let $S_\phi$ denote the set of linear combinations of operators from $L^p$ into $L^2$ of the form (2.5) where $\phi$ satisfies (i) and (ii). Clearly $S_\phi \subset L^2$ so that, since $L^2$ is closed with respect to the strong operator topology, it remains to show that every member of $L^2$ may be suitably approximated by members of $S_\phi$. Let $\{\delta_a\}$ be an approximate identity comprised of trigonometric polynomials such that $\|\delta_a\|_1 = 1$ (see [10]). Suppose $T \in L^p$, $p \neq \infty$. Then, by (2.1), $\phi_a = T\delta_a$ is also a trigonometric polynomial and

\[Tf = \lim_{a} T(\delta_a * f) = \lim_{a} \phi_a * f\]

in $L^2$ for each $f \in L^p$. (The above argument is similar to [7, Theorem 1, case (iii)].) The proof is complete if we can show that each $\phi_a$ is a finite linear combination of functions of the type specified in the theorem. Thus it suffices to show that any endomorphism $U$ of a finite-dimensional Hilbert space $H$ is a finite linear combination of orthogonal projections. By the finite-dimensional spectral theorem, this is certainly the case if $U$ is normal; and the general case is accounted for by writing $U = (U + U^*)/2 + i(U - U^*)/2i$, wherein $U + U^*$ and $i(U - U^*)$ are self-adjoint, and hence normal.

3. Idempotent multipliers and lacunary subsets of $\Gamma$. Let $I_\phi$ denote the subset of $M_\phi$ consisting of $\mu$ such for each $\gamma \in \Gamma$, $\mu(\gamma)$ is either $I_\gamma$, the identity operator on $H_\gamma$, or 0. From (2.4) we have

\[(3.0) I_1 \subset I_p \subset I_\phi \subset I_2\]

when $1 < p < q < 2$.

**Remarks.** (i) Since $\mu(\gamma)$ is selfadjoint whenever $\gamma \in \Gamma$ and $\mu \in I_\phi$, (2.2) implies that $I_\phi = I_\phi'$. Thus the Riesz-Thorin convexity theorem [4, 13.4.1] is immediately applicable and yields (3.0) directly, without relying on (2.4).
(ii) When \( \Gamma \) contains an infinite Sidon set \([9, \text{p. 511}]\), it may be shown that the first inclusion of (3.0) is proper. (A proof follows the lines of [1, Theorem 4.15] by using appropriate results in [9, \S 3] for the nonabelian case. In fact, on closer inspection of the proof, it may be seen that the result is valid if we only assume the existence of an infinite subset of \( \Gamma \) which is of type \( \Lambda(q) \)—see the definition below—for all \( q < \infty \).

(iii) Theorem 1.1 of [3] may be used to show that the third inclusion of (3.0) is strict when \( G \) is abelian.

The main result in this section is Theorem 3.4, which gives sufficient conditions for the second inclusion to be strict.

Suppose \( 0 < q < \infty \). Following [11] and [9] we say that \( E (\subset \Gamma) \) is of type \( \Lambda(q) \) [or \( E \in \Lambda(q) \)] if there exists \( p \in (0, q) \) and a real number \( K = K(p, q, E) \) such that

\[
\| f \|_q \leq K \| f \|_p
\]

for all \( f \in L^p_E \), where \( L^p_E = \{ f \in L^p : \check{f}(\gamma) = 0 \text{ if } \gamma \notin \Gamma \} \). As in [11], it may be shown that \( E \in \Lambda(q) \) if and only if \( L^p_E = L^p \) for all \( p \in (0, q) \).

When \( E \subset \Gamma \), \( \mu_E \) will always denote the function in \( E \) satisfying \( \mu_E(\gamma) = \mathbf{1}_E(\gamma) \) when \( \gamma \in E \), and 0 otherwise. Then for all \( E \subset \Gamma \), \( \mu_E \in L^2_E \subset \mathcal{C}_w \).

3.1 Lemma. (a) Let \( 2 < q < \infty \). Then \( E \in \Lambda(q) \) implies \( \mu_E \in I^q_E \) (and hence \( \mu_E \in I^p_E \) for all \( q' \leq p \leq \infty \) and \( 1 \leq q < q' \)).

(b) Let \( \mu_E \in I^p_E \) for some \( p \in [1, q) \). Then \( E \in \Lambda(q) \) (and hence \( E \in \Lambda(r) \) for all \( 0 < r < q \)).

Remark. Combining (a) and (b) above yields: Suppose \( 2 < q < \infty \) and \( q' \leq p < q \), then \( E \in \Lambda(q) \) if and only if \( \mu_E \in I^q_E \).

Proof of Lemma 3.1. (a) Let \( E \in \Lambda(q) \) where \( 2 < q < \infty \). Then \( L^p_E \subset L^q \) and \( (L^q)^{\mu_E} \subset (L^p)^{\mu_E} \); thus \( \mu_E \in I^q_E \). Now use the fact that \( I^q_E = I^q_E \) from (2.2) to obtain

\[
(L^q)^{\mu_E} = (L^q)^{\mu_E} \cdot \mu_E \subset (L^p)^{\mu_E} \cdot \mu_E \subset (L^2)^{\mu_E}
\]

that is \( \mu_E \in I^q_E \), as required.

(b) Let \( \mu_E \in I^p_E \), where \( 1 \leq p < q \). Then \( f \in L^p \) implies \( \check{f} \cdot \mu_E \in (L^q)^{\mu_E} \). Further, if \( f \in L^p \), \( \check{f} = \check{f} \cdot \mu_E \) and so \( L^p_E \subset L^q \); thus \( E \in \Lambda(q) \), as required.

3.2 Interpolation Lemma. Suppose that \( \mu_E \in I^p_E \cap I^q_E \), where \( 1 \leq p < q \leq s \leq r \leq \infty \). Then \( E \in \Lambda(y) \) for all \( y \) satisfying \( 0 < y < q(\alpha + b)/(qa + sb) \), where \( a = 1/p - 1/q \) and \( b = 1/s - 1/r \).

Proof. From the Riesz-Thorin convexity theorem [4, 13.4.1] we have \( \mu_E \in I^y_E \) whenever \( 1/x = (1 - \alpha)/p + a/r \) and \( 1/y = (1 - \alpha)/q + a/s \) for some \( \alpha \) satisfying \( 0 \leq \alpha \leq 1 \). To be able to apply Lemma 3.1 (b) requires (i) \( x \geq 1 \), and (ii) \( x < y \). Now (i) is clearly satisfied for all \( \alpha \in [0, 1] \) since then \( p \leq x \leq r \).

In determining when (ii) is satisfied we first note that \( x = y \) when \( \alpha = a/(a + b) \). Since \( a > 0 \) and \( b \geq 0 \), then \( 0 < a/(a + b) \leq 1 \), and (ii) is satisfied for \( x = x(\alpha) \) and
\[ y = y(\alpha) \text{ where } 0 \leq \alpha < a/(a+b). \] This completes the proof, since \( 0 \leq \alpha < a/(a+b) \) is equivalent to \( q \leq y < q(a+b)/(qa+sb) \).

3.3 Necessary conditions for membership of \( \Lambda(q) \). Before proceeding to the main inclusion theorem of this section, we will apply Lemma 3.1 (a) to derive some necessary conditions for membership of \( \Lambda(q) \) when \( G \) is the circle group. Suppose that \( 2 < q < \infty \) and \( E \in \Lambda(q) \), then \( \mu_E \in I^q \). Application of 16.4.4 (2) and 16.4.5 (3) of Edwards [4], which give necessary conditions for \( \mu \) to belong to certain \( M\eta \), then yields: Let \( G \) be the circle group, \( q \in (2, \infty) \) and \( E \in \Lambda(q) \). Then \( \sum_{n \in E} (1 + |n|)^{-2/q - \epsilon} < \infty \) for each \( \epsilon > 0 \); or even better

\[
(3.1) \quad \sum_{n \in E} (\log (2 + |n|))^{-2/q' - \epsilon} (1 + |n|)^{-2/q} < \infty
\]

for each \( \epsilon > 0 \).

This result is well suited to handle sets of the form \( E_a = \{[n\alpha] : n = 0, 1, 2, \ldots \} \) where \( \alpha > 1 \) and \([\beta]\) is the integral part of \( \beta \). Suppose \( E_a \in \Lambda(q) \) where \( 2 < q < \infty \). From (3.1) it follows that \( 2\alpha/q > 1 \), that is \( q < 2\alpha \). Alternatively, \( E_a \) (\( \alpha > 1 \)) is not of type \( \Lambda(2\alpha) \). (Rudin [11, p. 219] also proved that \( E_2 \notin \Lambda(4) \).)

3.4 Theorem. Suppose that there exists a subset \( E \) of \( \Gamma \) belonging to \( \Lambda(p) \), but not to \( \Lambda(q) \), where \( 2 < p < q < \infty \); then \( \mu_E \) belongs to \( I^p \), but not to \( I^q \) for all \( r > q \).

3.5 Remark. Of course, \( \Lambda(p) \supset \Lambda(q) \) when \( 0 < p < q < \infty \). With the restriction that \( G \) is the circle group, Rudin [12, p. 206] conjectured that the inclusion is always strict, but the only cases known are when \( p > 2 \) is an even integer [12, Theorem 4.8].

Proof of 3.4. Assume that there exists \( E \in \Lambda(p) \setminus \Lambda(q) \) where \( 2 < p < q < \infty \). Lemma 3.1 (a) implies that \( \mu_E \notin I^q \); the proof of 3.4 is completed by showing that \( \mu_E \notin I^r \) whenever \( r > q \). Suppose the contrary, that is, that there exists \( r > q \) such that \( \mu_E \in I^r \). Let \( r \in (2, p) \); then 3.1 (a) shows that \( \mu_E \in I^q \) and hence 3.2 (with \( b = 0 \)) is applicable to show that \( E \in \Lambda(y) \) for all \( y < r \). But this conflicts with the assumption that \( E \notin \Lambda(q) \) and so completes the proof.

4. \( L_p^q \subseteq L_q^q \) whenever \( 1 \leq p < q \leq 2 \) and \( G \) is infinite. In this section the underlying group \( G \) will always be either a locally compact abelian group (that is, an LCA group) or a compact group. When \( 1 < p < q < 2 \) it is known that

\[
(4.0) \quad L_1^q \subseteq L_p^q \subseteq L_2^q,
\]

where each inclusion map is continuous. (This follows from the Riesz-Thorin convexity theorem and \( L_p^q = L^q \) when \( G \) is LCA and from (2.4) above when \( G \) is compact.) The final inclusion of (4.0) is known to be strict ([8] and [9]), and in this section we will use this information to prove the strictness of the first two inclusions.

4.1 Theorem. Let \( \{\lambda_n\}_{n=1}^{\infty} \) denote a strictly positive sequence such that \( \lambda_n/\sum_{m=1}^{n-1} \lambda_m \) is unbounded and increasing. Choose \( p \in [1, 2) \) and suppose that there exists a sequence
\{T_n\}_{n=1}^\infty \text{ in } L_p^\varphi \text{ and numbers } a, b \text{ such that}

\begin{align}
(4.1)_1 & \quad 0 < a < \|T_n\|_{p, p} < b, \\
(4.1)_2 & \quad T_n \to 0 \text{ in } L_2^\varphi \text{ as } n \to \infty.
\end{align}

Then there exists a sequence \(n_1 < n_2 < \cdots\) of positive integers such that the series

\[ \sum_{r=1}^\infty \lambda_r T_{n_r} \]

converges in \(L_q^\varphi\), for each \(q \in (p, 2]\), to a unique operator, \(T\) say, and yet \(T \notin L_p^\varphi\).

Proof. By considering a subsequence in place of \(\{T_n\}\) we may assume, without loss of generality, that

\[ \|T_n\|_{2, 2} \leq \exp(-\lambda_n) \]

for \(n \in \mathbb{N} = \{1, 2, 3, \ldots\}\), and that \(T_n \neq T_m\) for all \(m \neq n\).

Let \(\Omega\) denote the set of monotonically increasing maps \(\omega\) from \(\mathbb{N}\) to \(\mathbb{N} \cup \{\infty\}\) such that \(\omega\) is strictly monotonic on \(\{n : \omega(n) \in \mathbb{N}\}\). Let \(\omega \in \Omega\) and \(q \in (p, 2)\). Then, after defining \(\omega_0 := 0\) and \(\lambda_0 = 0\),

\[ (4.3) \quad \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega(n)}\|_{q, q} < \infty, \]

since

\[ \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega(n)}\|_{q, q} \leq \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega(n)}\|_{q, q} \]

\[ \leq \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega(n)}\|_{q, q} \leq \sum_{n \in \mathbb{N}} \lambda_n (\|T_n\|_{2, 2})^\alpha (\|T_n\|_{p, p})^{1-\alpha} \]

by the Riesz-Thorin convexity theorem, where \(\alpha \in (0, 1]\) and satisfies \(1/q = \alpha/2 + (1-\alpha)/2\). But by (4.2) and the hypothesis (4.1), this last sum is bounded by the convergent sum \(\sum \lambda_n \exp(-\alpha n) b_1^{-n}\).

Let \(\{q_m\}\) denote a decreasing sequence in \((p, 2]\) such that \(q_m \to p\) as \(m \to \infty\). Then the following is a sequence of metrics on \(\Omega\):

\[ (4.4) \quad d_m(\omega, \omega') = \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega(n)} - T_{\omega'(n)}\|_{q_m, q_m} \]

for each \(\omega, \omega' \in \Omega\) and \(m \in \mathbb{N}\). It is well known that the topology on \(\Omega\) may be defined by a single metric, \(d\) say. We will now prove that \((\Omega, d)\) is complete.

Let \(\{\omega'_r\}_{r \in \mathbb{N}}\) be a Cauchy sequence in \((\Omega, d)\). Then \(\{\omega'_r\}\) is Cauchy in each \((\Omega, d_m)\). Since each \(L_{q_m}^\varphi\) is a Banach space and is imbedded in \(L_2^\varphi\), the completeness of \((\text{weighted}) \ell^1\)-spaces of Banach space-valued functions shows that there exists \(\{\psi_n\}_{n \in \mathbb{N}}\), where \(\psi_n \in \bigcap \{L_{q_m}^\varphi : m \in \mathbb{N}\}\) for each \(n\), such that

\[ \sum_{n \in \mathbb{N}} \lambda_n \|T_{\omega'(n)} - \psi_n\|_{q_m, q_m} \to 0 \quad \text{as} \quad r \to \infty \]

for each \(m \in \mathbb{N}\). It suffices to show that there exists \(\omega \in \Omega\) such that \(T_{\omega(n)} = \psi_n\) for each \(n \in \mathbb{N}\).
Now \( T_n \to 0 \) in \( L^2_p \), \( T_n \neq T_m \neq 0 \) for \( n \neq m \), and \( \| T_{\omega'}(n) - \psi_n \|_{2,2} \to 0 \) as \( r \to \infty \) for each \( n \in N \). Thus either \( \psi_n = 0 \) or \( \psi_n = \sigma_n \) for some \( n' \in N \). In the first case, \( \omega'(n) \to \infty \) as \( r \to \infty \); and in the second case, \( \omega'(n) = n' \) for \( r \) sufficiently large. Define \( \omega(n) = n' \) if \( n \) satisfies the second case, and \( \omega(n) = \infty \) otherwise. Thanks to the fact that \( \omega \) satisfies the two requirements of monotonicity for membership of \( \Omega \), we have \( \omega \in \Omega \) and so \( \Omega, d \) is complete.

Assume that the conclusion of the theorem is false; that is, for each \( \omega \in \Omega \), the \( (2, 2) \)-multiplier \( T^{(\omega)} = \sum \lambda_n T_{\omega(n)} \) belongs to \( L^p_q \). Thus we have a mapping \( u \) from \( \Omega \) to the positive reals defined by

\[
u: \omega \mapsto \| T^{(\omega)} \|_{p,p}.
\]

We will prove that the set \( \{ \omega : u(\omega) \leq \beta \} \) is closed for each \( \beta \in \mathbb{R}^+ \), that is, that \( u \) is lower semicontinuous. Suppose that \( \omega_r \to \omega \) in \( \Omega \) and that \( u(\omega_r) \leq \beta \). Then \( T^{(\omega_r)} \to T^{(\omega)} \) in \( L^q_m \) for each \( m \in N \). The Riesz-Thorin convexity theorem shows that for \( \| \psi_n \|_{p,q_m} \leq \beta^n(\| T^{(\omega_r)} \|_{p,2})^{1-a_n} \),

\[
\| T^{(\omega_r)} \|_{p,q_m} \leq \beta^n(\| T^{(\omega)} \|_{p,2})^{1-a_n}.
\]

Letting \( r \to \infty \), it follows that

\[
\| T^{(\omega)} \|_{p,q_m} \leq \beta^n(\| T^{(\omega)} \|_{p,2})^{1-a_n};
\]

that is, for \( f \in C_c \),

\[
\| T^{(\omega)} f \|_{q_m} \leq \beta^n(\| T^{(\omega)} \|_{p,2})^{1-a_n} \| f \|_{q_m}.
\]

Since \( T^{(\omega)} f \in L^p \cap L^2 \) when \( f \in C_c \), we may let \( m \to \infty \), and thus \( a_m \to 1 \), to obtain

\[
\| T^{(\omega)} f \|_p \leq \beta \| f \|_p,
\]

showing that \( u(\omega) \leq \beta \), as required.

Thus the sequence of sets \( \{ \omega : u(\omega) \leq n \} \), \( n \in N \), forms a countable closed covering of \( \Omega \). Since \( (\Omega, d) \) is a complete metrizable space, it is nonmeagre in itself, and the Baire category theorem shows that a member of this covering has a nonvoid interior. Hence there exists \( m \in N \), \( \omega \in \Omega \) and positive real numbers \( \epsilon, k \) such that

\[
u(\omega) \leq k \quad \text{whenever} \quad d_m(\omega', \omega) < \epsilon.
\]

We will now show that this is impossible. For each \( r \in N \), define \( \omega^r \in \Omega \) by \( \omega^r = \omega \) on \( \{1, 2, \ldots, r\} \) and \( \infty \) otherwise. From (4.3) there exists \( r_0 \in N \) such that for \( r > r_0 \), \( d_m(\omega^r, \omega) < \epsilon \). But

\[
u(\omega^r) = \left[ \sum_{n=1}^{r} \lambda_n T_{\omega(n)} \right]_{p,p} \geq \lambda_r \| T_{\omega(r)} \|_{p,p} - \sum_{n=1}^{r-1} \lambda_n \| T_{\omega(n)} \|_{p,p} \geq \lambda_r a - b \sum_{n=1}^{r-1} \lambda_n,
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
which is eventually
\[ \geq \frac{(a/b) \cdot \lambda}{\sum_{n=1}^{\infty} \lambda_n} - 1. \]

But this is unbounded by hypothesis, and so contradicts (4.5). Thus there exists \( \omega \in \Omega \) such that \( \sum \lambda_n \omega(n) \notin L_p, \) For this to be true it is impossible for either \( \omega \) to be identically \( \infty \) or for \( \{ n : \omega(n) \in N \} \) to be finite; thus the sequence \( \omega(1) < \omega(2) < \cdots \) satisfies the conclusions of the theorem.

4.2 The following definition and lemma is needed before going to the main result. For \( 1 < p < \infty, \) let \( A_p \) denote the set of all functions on \( G \) of the form \( \sum_{i=1}^n f_i \ast g_i \) with \( f_i \in L_p, \ g_i \in L_{p'} \) and \( \sum \| f_i \|_p \| g_i \|_{p'} < \infty. \) Define a norm on \( A_p \) by setting
\[ \| h \|_{A_p} = \inf \left\{ \sum \| f_i \|_p \| g_i \|_{p'} : h = \sum f_i \ast g_i \right\}, \]
for each \( h \in A_p. \)

The importance of \( A_p \) is shown by Theorems 1 and 6 of Figà-Talamanca [6], which together state that the normed dual of \( A_p, \ 1 < p < \infty, \) is isometrically isomorphic to \( L_{p'} \). Also note that \( A_2 = A, \) the space of continuous functions on \( G \) with integrable Fourier transforms, that is, \( f \in A \) if and only if \( f \) is continuous and \( f \in L_1 \) if \( G \) is abelian, or \( f \in \mathbb{C} \) if \( G \) is compact—see 2.2.

4.3 LEMMA. \( A \) is a dense subspace of \( A_p, \ 1 < p < \infty. \)

PROOF. Let \( p \in (1, \infty). \) Since (i) \( L_p^c \subseteq L_2^c, \) and (ii) \( L_p^c \) and \( L_2^c \) are isometrically isomorphic to the normed duals of \( A_p \) and \( A \) respectively, it follows readily that \( A \subset A_p. \) To show that \( A \) is dense in \( A_p \) only requires noting that each element of \( A_p \) can be approximated by functions of the form \( \sum_{i=1}^n f_i \ast g_i, \) where \( f_i, g_i \in C_c. \)

4.4 THEOREM. Let \( G \) be an infinite \( LCA \) group or an infinite compact group. Then
\[ \bigcup_{1 \leq q < p} L_q^p \subseteq L_p^p \subseteq \bigcup_{p < q \leq 2} L_q^q \]
if \( 1 < p < 2. \) If \( p = 1, \) the second strict inclusion remains valid, and if \( p = 2 \) the first strict inclusion remains valid.

4.5 REMARKS. By proving the existence of sets of uniqueness for \( L^p(G), \ 1 \leq p < 2, \) Figà-Talamanca and Gaudry have recently proved in [8, §2] that \( L_p^p \subseteq L_2^q \) when \( 1 \leq p < 2 \) and \( G \) is any infinite \( LCA \) group, thus improving an earlier result of R. E. Edwards which required \( G \) to satisfy the extra condition of possessing an infinite discrete subgroup. The authors of [8] then employ the Riesz-Thorin convexity theorem to prove that
\[ M_p^p \cap C_0(\hat{G}) \subseteq M_q^q \cap C_0(\hat{G}) \]
when \( 1 \leq p < q \leq 2, \) unless \( G \) is finite, where \( \hat{G} \) denotes the character group of \( G. \) However, Theorem 4.4 generalizes the basic result \( \"L_p^p \subseteq L_2^q \) when \( 1 \leq p < q \leq 2, \) and
$G$ is infinite compact or infinite LCA” in a different direction and is proved by a method which is constructive to some extent.

**Proof of 4.4.** When $1 \leq p < 2$, the proof of the second strict inclusion will be based on the fact that $\| \cdot \|_{p,p}$ is a strictly stronger norm than $\| \cdot \|_{2,2}$ on $L^p$. From (4.0) we know that $\| \cdot \|_{p,p}$ is stronger than $\| \cdot \|_{2,2}$ on $L^p$. Let $A_1 = C_0$. Since $A$ is dense in $A_p$, $1 \leq p < 2$ (see [5, Proposition 3.7] and Lemma 4.3), $L^p$ is the normed dual of $A$ equipped with the $A_p$ norm. Thus if $\| \cdot \|_{p,p}$ is equivalent to $\| \cdot \|_{2,2}$ on $L^p$, then the $A_p$ norm is equivalent to the $A$ norm on $A$, and so $L^p = L^2$. But this is impossible when $G$ is infinite, by [8] and [9]. (When $p=1$, simpler methods suffice to show that $L^1 \subseteq L^2$, or equivalently, $C_0 \subseteq A$, when $G$ is infinite.) Thus $\| \cdot \|_{p,p}$ is strictly stronger than $\| \cdot \|_{2,2}$ on $L^p$, and so we may choose a sequence in $L^p$ satisfying (4.1 x) and (4.1.2). This leads to the existence of $T \in \bigcap \{ L^q : p < q \leq 2 \}$ such that $T \notin L^p$.

To prove the first strict inequality when $1 < p \leq 2$, note first that from the above we have $L^q \subseteq L^p$ when $1 \leq q < p$. Assume $\bigcup \{ L^q : 1 \leq q < p \} = L^p$. Let $\{ q_m \}$ be an increasing sequence in $(1, p)$ such that $q_m \to p$ as $m \to \infty$. Then the injection maps $i_m : L^q_m \to L^2$ are continuous and $\bigcup \{ i_m(L^q_m) : m=1, 2, \ldots \} = L^p$. Thus the hypotheses of [2, 6.5.1] are satisfied, the conclusion being that $L^q_m = L^p$ for some positive integer $m$, a contradiction.

**4.6 Corollary.** When $p$ satisfies $2 \leq p \leq \infty$, the obvious analogue to Theorem 4.4 is valid.

**Bibliography**


**Institute of Advanced Studies, Australian National University, Canberra, Australia**