TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, November 1970

SOME STRICT INCLUSIONS BETWEEN SPACES OF L^p-MULTIPLIERS

BY
J. F. PRICE

Abstract. Suppose that the Hausdorff topological group G is either compact or locally compact abelian and that C_c denotes the set of continuous complex-valued functions on G with compact supports. Let L_p^q denote the restrictions to C_c of the continuous linear operators from $L^p(G)$ into $L^q(G)$ which commute with all the right translation operators.

When $1 \leqq p < q \leqq 2$ or $2 \leqq q < p \leqq \infty$ it is known that

$$L_p^p \subset L_q^q. \tag{1}$$

The main result of this paper is that the inclusion in (1) is strict unless G is finite. In fact it will be shown, using a partly constructive proof, that when G is infinite

$$\bigcup_{1 \leqq q < p} L_q^q \subsetneqq L_p^p \subsetneqq \bigcap_{p < q \leqq 2} L_q^q$$

for $1 < p < 2$, with the first inclusion remaining strict when $p = 2$ and the second inclusion remaining strict when $p = 1$. (Similar results also hold for $2 \leqq p \leqq \infty$.)

When G is compact, simple relations will also be developed between idempotent operators in L_p^q and lacunary subsets of the dual of G which will enable us to find necessary conditions so that inclusion (1) is strict even if, for example, L_p^p and L_q^q are replaced by the sets of idempotent operators in L_p^p and L_q^q respectively.

1. General notation and preliminaries.

1.1 The notation outlined in this section will generally follow [1] where further details may be found. Let G denote a multiplicatively written locally compact group (all topological groups will be assumed to be Hausdorff), e its identity and λ_G a fixed left Haar measure on G. If G is compact, λ_G is assumed to be normalized so that $\lambda_G(G) = 1$. Let $L^p = L^p(G) - 1 \leqq p \leqq \infty$ always—denote the usual Lebesgue space of equivalence classes of functions on G. Its norm will be denoted by $\|\cdot\|_p$. When $1 < p < \infty$, p' will always satisfy $1/p + 1/p' = 1$; also $1' = \infty$ and $\infty' = 1$.

Let $C = C(G)$ denote the space of continuous complex-valued functions over G. C_0 and C_c will denote the subspaces of C comprised of functions which vanish at infinity and of functions which have compact supports, respectively; $C_0 = C_c = C$ when G is compact. Also assume that C_0 is equipped with the uniform (or supremum) norm, which makes it a Banach space.

The *left and right translation operators* τ_a and ρ_a $(a \in G)$ are defined on C by

$$\tau_a f(x) = f(a^{-1}x), \qquad \rho_a f(x) = f(xa^{-1}),$$

and may then be extended to measures.

Received by the editors June 13, 1969.

AMS 1969 *subject classifications.* Primary 4635, 4755; Secondary 2265, 4250.

Key words and phrases. Lebesgue function spaces, locally compact abelian groups, compact groups, translations, multiplier operators, idempotent multiplier operators.

1.2 *Multipliers of type* (p, q). Let L_p^q denote the set of linear operators T from C_c into L^q which satisfy the following two properties:

(i) T is continuous when C_c is equipped with the topology induced by L^p, and

(ii) T commutes with right translation, that is

$$T\rho_a = \rho_a T \quad \text{for all } a \in G. \tag{1.1}$$

We further suppose that L_p^q is a Banach space by equipping it with the norm $T \mapsto \|T\|_{p,q}$ defined as the smallest positive number K such that

$$\|Tf\|_q \leqq K\|f\|_p$$

for all $f \in C_c$. Note that when $p \neq \infty$, $T \in L_p^q$ has a unique continuous extension to L^p which also satisfies (1.1).

A suggestion by R. E. Edwards leads to the use of the Riesz-Thorin convexity theorem to prove that $L_p^p \subsetneqq \bigcap \{L_q^q : 1 \leqq p < q \leqq 2\}$—see Theorem 4.4 below. The author is very grateful to Dr. Edwards for this suggestion, and for other more general comments. Also, the author would like to thank Professors E. Hewitt and K. A. Ross for their kind permission to read and quote from sections of their forthcoming book, *Abstract harmonic analysis*, Vol. II; and Drs. A. Figà-Talamanca and G. I. Gaudry for correspondence relating to their forthcoming publication [8].

2. **Representations of L_p^q when G is compact.** Throughout this section G will denote a multiplicatively written compact group (not necessarily abelian) with identity e and equipped with normalized Haar measure λ_G. If $f \in L^1$, then f is uniquely represented by a Fourier series

$$f \sim \sum_{\gamma \in \Gamma} d(\gamma) \operatorname{Tr} [\hat{f}(\gamma)\gamma(\cdot)],$$

where: Γ is a set of representatives, one selected from each equivalence class of continuous, irreducible, unitary representations of G; $d(\gamma)$ is the (finite) dimension of the representation γ; Tr denotes the usual trace; and $\hat{f}$ is the Fourier transform of f, defined by

$$\hat{f}(\gamma) = \int_G f(t)\gamma(t)^* \lambda_G(t),$$

$\gamma(t)^*$ denoting the (Hilbert) adjoint of $\gamma(t)$. For $\gamma \in \Gamma$, H_γ will denote the corresponding Hilbert space (of dimension $d(\gamma)$).

The following "exchange formula" will be used occasionally: $(f * g)\hat{}(\gamma) = \hat{g}(\gamma)\hat{f}(\gamma)$ for each $\gamma \in \Gamma$, where $f, g \in L^1$. Also the Peter-Weyl completeness theorem [10] will be frequently used without comment, generally in the form: If $f \in L^1$ and $\hat{f}(\gamma) = 0$ for each $\gamma \in \Gamma$, then $f = 0$ almost everywhere.

Let $\mathfrak{E}$ denote the set consisting of all functions μ on Γ such that $\mu(\gamma)$ is an endomorphism of H_γ for each $\gamma \in \Gamma$. It is easily shown—see Theorem (35.8) of [10], for example—that to each $T \in L_p^q$ there corresponds a unique $\mu \in \mathfrak{E}$ such that

$$(Tf)\hat{} = \hat{f} \cdot \mu \tag{2.1}$$

for each $f \in C$. Let M_p^q denote the subset of $\mathfrak{E}$ corresponding to L_p^q via (2.1). A useful formula is

$$M_p^q = (M_{q'}^{p'})^*, \tag{2.2}$$

where $(M_r^s)^* = \{\mu^* : \mu^*(\gamma) = \mu(\gamma)^* \text{ for } \gamma \in \Gamma \text{ and } \mu \in M_r^s\}$.

For each $\gamma \in \Gamma$ define a norm $\|\cdot\|_{(\infty)}$ on the space of endomorphisms of H_γ as the usual norm for bounded operators. Let $\mu \in \mathfrak{E}$ and define

$$\|\mu\|_\infty = \sup\{\|\mu(\gamma)\|_{(\infty)} : \gamma \in \Gamma\}; \tag{2.3}$$

denote by $\mathfrak{E}_\infty$ the subspace of $\mathfrak{E}$ comprised of those elements μ which satisfy $\|\mu\|_\infty < \infty$. Using the Parseval formula, namely

$$\|f\|_2 = \left\{\sum_{\gamma\in\Gamma} d(\gamma)\,\mathrm{Tr}\,[\hat{f}(\gamma)\hat{f}(\gamma)^*]\right\}^{1/2},$$

the following well-known result may be easily proved (see, for example, Theorem (35.4) of [10]).

2.1 THEOREM.

$$M_2^2 = \mathfrak{E}_\infty.$$

In a personal communication to the author, C. S. Herz stated that he has recently discovered a proof of

$$L_p^p \subset L_q^q \quad \text{if } 1 \leqq p \leqq q \leqq 2 \quad \text{or} \quad 2 \leqq q \leqq p \leqq \infty. \tag{2.4}$$

Theorem 2.1 and inclusion (2.4) are basic for the sequel.

2.2 *Remarks on pseudomeasures.* Let A denote the set of continuous functions on G whose Fourier transforms belong to $\mathfrak{E}_1$, where

$$\mathfrak{E}_1 = \left\{\mu \in \mathfrak{E} : \|\mu\|_1 = \sum_{\gamma\in\Gamma} d(\gamma)\,\mathrm{Tr}\,[(\mu(\gamma)\mu(\gamma)^*)^{1/2}] < \infty\right\}.$$

Define a norm on A by $\|f\|_A = \|\hat{f}\|_1$. Following the procedure for LCA groups (see, for example, 4.1 of [1]) we call the normed dual P of A the space of *pseudomeasures over G*. Since A and $\mathfrak{E}_1$ are isometrically isomorphic and $\mathfrak{E}_\infty$ is isometrically isomorphic to the normed dual of $\mathfrak{E}_1$ (see [10]), then there is an isometric isomorphism between P and $\mathfrak{E}_\infty$. One such isomorphism may be defined via an extension of the Fourier transform:

DEFINITION. The Fourier transform of $\sigma \in P$ is the unique element $\hat{\sigma}$ of $\mathfrak{E}_\infty$ which satisfies

$$\sum_{\gamma\in\Gamma} d(\gamma)\,\mathrm{Tr}\,[\bar{f}^\wedge(\gamma)^*\hat{\sigma}(\gamma)] = \langle f, \sigma\rangle$$

for each $f \in A$.

Finally P may be made into a Banach algebra under convolution by defining $\sigma_1 * \sigma_2$, where $\sigma_1, \sigma_2 \in P$, as the element of P which satisfies

$$(\sigma_1 * \sigma_2)^\wedge = \hat{\sigma}_2\hat{\sigma}_1$$

on Γ. Then A becomes a two-sided ideal of P.

With this notation, Theorem 2.1 may be expressed in a more suggestive form (cf. [1] and Chapter 16 of [4]), namely: "A continuous linear operator T from L^2 into L^2 belongs to L_2^2 (that is, commutes with the ρ_a) if and only if there exists $\sigma \in P$ such that $Tf = \sigma * f$ for all $f \in C$."

2.3 The following result provides another characterization of L_p^q, when $p \neq \infty$, and also provides some motivation for the study of idempotent multipliers in the next section.

PROPOSITION. *The set L_p^q, $p \neq \infty$, is the closure in the strong operator topology of the set of finite linear combinations of operators from L^p into L^q of the form*

$$f \mapsto \phi * f \tag{2.5}$$

where, for some $\gamma \in \Gamma$,

(i) *$\hat{\phi}(\gamma)$ is an orthogonal projection in H_γ, and*

(ii) $\operatorname{supp} \hat{\phi} = \{\gamma\}$.

Proof. Let S_p^q denote the set of linear combinations of operators from L^p into L^q of the form (2.5) where ϕ satisfies (i) and (ii). Clearly $S_p^q \subset L_p^q$ so that, since L_p^q is closed with respect to the strong operator topology, it remains to show that every member of L_p^q may be suitably approximated by members of S_p^q.

Let $\{\delta_\alpha\}$ be an approximate identity comprised of trigonometric polynomials such that $\|\delta_\alpha\|_1 = 1$ (see [10]). Suppose $T \in L_p^q$, $p \neq \infty$. Then, by (2.1), $\phi_\alpha = T\delta_\alpha$ is also a trigonometric polynomial and

$$Tf = \lim_\alpha T(\delta_\alpha * f) = \lim_\alpha \phi_\alpha * f$$

in L^q for each $f \in L^p$. (The above argument is similar to [7, Theorem 1, case (iii)].) The proof is complete if we can show that each ϕ_α is a finite linear combination of functions of the type specified in the theorem. Thus it suffices to show that any endomorphism U of a finite-dimensional Hilbert space H is a finite linear combination of orthogonal projections. By the finite-dimensional spectral theorem, this is certainly the case if U is normal; and the general case is accounted for by writing $U = (U + U^*)/2 + i(U - U^*)/2i$, wherein $U + U^*$ and $i(U - U^*)$ are self-adjoint, and hence normal.

3. **Idempotent multipliers and lacunary subsets of Γ.** Let I_p^q denote the subset of M_p^q consisting of μ such for each $\gamma \in \Gamma$, $\mu(\gamma)$ is either I_γ, the identity operator on H_γ, or 0. From (2.4) we have

$$I_1^1 \subset I_p^p \subset I_q^q \subset I_2^2 \tag{3.0}$$

when $1 < p < q < 2$.

REMARKS. (i) Since $\mu(\gamma)$ is selfadjoint whenever $\gamma \in \Gamma$ and $\mu \in I_p^q$, (2.2) implies that $I_p^p = I_{p'}^{p'}$. Thus the Riesz-Thorin convexity theorem [4, 13.4.1] is immediately applicable and yields (3.0) directly, without relying on (2.4).

(ii) When Γ contains an infinite Sidon set [9, p. 511], it may be shown that the first inclusion of (3.0) is proper. (A proof follows the lines of [1, Theorem 4.15] by using appropriate results in [9, §3] for the nonabelian case. In fact, on closer inspection of the proof, it may be seen that the result is valid if we only assume the existence of an infinite subset of Γ which is of type $\Lambda(q)$—see the definition below—for all $q<\infty$.)

(iii) Theorem 1.1 of [3] may be used to show that the third inclusion of (3.0) is strict when G is abelian.

The main result in this section is Theorem 3.4, which gives sufficient conditions for the second inclusion to be strict.

Suppose $0<q<\infty$. Following [11] and [9] we say that $E\ (\subset\Gamma)$ is *of type* $\Lambda(q)$ [or $E\in\Lambda(q)$] if there exists $p\in(0,q)$ and a real number $K=K(p,q,E)$ such that

$$\|f\|_q \leqq K\|f\|_p$$

for all $f\in L^p_E$, where $L^p_E=\{f\in L^p : \hat{f}(\gamma)=0 \text{ if } \gamma\in\Gamma\backslash E\}$. As in [11], it may be shown that $E\in\Lambda(q)$ if and only if $L^p_E=L^q_E$ for *all* $p\in(0,q)$.

When $E\subset\Gamma$, μ_E will always denote the function in E satisfying $\mu_E(\gamma)=I_\gamma$ when $\gamma\in E$, and 0 otherwise. Then for all $E\subset\Gamma$, $\mu_E\in I^2_2\subset\mathfrak{E}_\infty$.

3.1 LEMMA. (a) *Let $2<q<\infty$. Then $E\in\Lambda(q)$ implies $\mu_E\in I^q_{q'}$ (and hence $\mu_E\in I^r_p$ for all $q'\leqq p\leqq\infty$ and $1\leqq r\leqq q$).*

(b) *Let $\mu_E\in I^q_p$ for some $p\in[1,q)$. Then $E\in\Lambda(q)$ (and hence $E\in\Lambda(r)$ for all $0<r\leqq q$).*

REMARK. Combining (a) and (b) above yields: Suppose $2<q<\infty$ and $q'\leqq p<q$, then $E\in\Lambda(q)$ if and only if $\mu_E\in I^q_p$.

Proof of Lemma 3.1. (a) Let $E\in\Lambda(q)$ where $2<q<\infty$. Then $L^2_E\subset L^q$ and $(L^2)^\wedge\cdot\mu_E\subset(L^2_E)^\wedge$; thus $\mu_E\in I^q_2$. Now use the fact that $I^q_2=I^2_{q'}$ from (2.2) to obtain

$$(L^{q'})^\wedge\cdot\mu_E = (L^{q'})^\wedge\cdot\mu_E\cdot\mu_E \subset (L^2)^\wedge\cdot\mu_E \subset (L^q)^\wedge;$$

that is $\mu_E\in I^q_{q'}$, as required.

(b) Let $\mu_E\in I^q_p$, where $1\leqq p<q$. Then $f\in L^p$ implies $\hat{f}\cdot\mu_E\in(L^q)^\wedge$. Further, if $f\in L^p_E$, $\hat{f}=\hat{f}\cdot\mu_E$ and so $L^p_E\subset L^q$; thus $E\in\Lambda(q)$, as required.

3.2 INTERPOLATION LEMMA. *Suppose that $\mu_E\in I^q_p\cap I^s_r$, where $1\leqq p<q\leqq s\leqq r\leqq\infty$. Then $E\in\Lambda(y)$ for all y satisfying $0<y<qs(a+b)/(qa+sb)$, where $a=1/p-1/q$ and $b=1/s-1/r$.*

Proof. From the Riesz-Thorin convexity theorem [4, 13.4.1] we have $\mu_E\in I^y_x$ whenever $1/x=(1-\alpha)/p+\alpha/r$ and $1/y=(1-\alpha)/q+\alpha/s$ for some α satisfying $0\leqq\alpha\leqq1$. To be able to apply Lemma 3.1 (b) requires (i) $x\geqq1$, and (ii) $x<y$. Now (i) is clearly satisfied for all $\alpha\in[0,1]$ since then $p\leqq x\leqq r$.

In determining when (ii) is satisfied we first note that $x=y$ when $\alpha=a/(a+b)$. Since $a>0$ and $b\geqq0$, then $0<a/(a+b)\leqq1$, and (ii) is satisfied for $x=x(\alpha)$ and

$y=y(\alpha)$ where $0\leqq\alpha<a/(a+b)$. This completes the proof, since $0\leqq\alpha<a/(a+b)$ is equivalent to $q\leqq y<qs(a+b)/(qa+sb)$.

3.3 *Necessary conditions for membership of* $\Lambda(q)$. Before proceeding to the main inclusion theorem of this section, we will apply Lemma 3.1 (a) to derive some necessary conditions for membership of $\Lambda(q)$ when G is the circle group. Suppose that $2<q<\infty$ and $E\in\Lambda(q)$, then $\mu_E\in I_2^q$. Application of 16.4.4 (2) and 16.4.5 (3) of Edwards [4], which give necessary conditions for μ to belong to certain M_p^q, then yields: Let G be the circle group, $q\in(2,\infty)$ and $E\in\Lambda(q)$. Then $\sum_{n\in E}(1+|n|)^{-2/q-\varepsilon}<\infty$ for each $\varepsilon>0$; or even better

$$\sum_{n\in E}\{\log(2+|n|)\}^{-2/q'-\varepsilon}(1+|n|)^{-2/q}<\infty \tag{3.1}$$

for each $\varepsilon>0$.

This result is well suited to handle sets of the form $E_\alpha=\{[n^\alpha]:n=0,1,2,\ldots\}$ where $\alpha>1$ and $[\beta]$ is the integral part of β. Suppose $E_\alpha\in\Lambda(q)$ where $2<q<\infty$. From (3.1) it follows that $2\alpha/q>1$, that is $q<2\alpha$. Alternatively, E_α $(\alpha>1)$ is *not* of type $\Lambda(2\alpha)$. (Rudin [11, p. 219] also proved that $E_2\notin\Lambda(4)$.)

3.4 THEOREM. *Suppose that there exists a subset E of Γ belonging to $\Lambda(p)$, but not to $\Lambda(q)$, where $2<p<q<\infty$; then μ_E belongs to I_p^p, but not to I_r^r for all $r>q$.*

3.5 REMARK. Of course, $\Lambda(p)\supset\Lambda(q)$ when $0<p<q<\infty$. With the restriction that G is the circle group, Rudin [12, p. 206] conjectured that the inclusion is always strict, but the only cases known are when $p>2$ is an even integer [12, Theorem 4.8].

Proof of 3.4. Assume that there exists $E\in\Lambda(p)\backslash\Lambda(q)$ where $2<p<q<\infty$. Lemma 3.1 (a) implies that $\mu_E\in I_p^p$; the proof of 3.4 is completed by showing that $\mu_E\notin I_r^r$ whenever $r>q$. Suppose the contrary, that is, that there exists $r>q$ such that $\mu_E\in I_r^r$. Let $t\in(2,p)$; then 3.1 (a) shows that $\mu_E\in I_t^p$ and hence 3.2 (with $b=0$) is applicable to show that $E\in\Lambda(y)$ for all $y<r$. But this conflicts with the assumption that $E\notin\Lambda(q)$ and so completes the proof.

4. $L_p^p\subsetneqq L_q^q$ **whenever** $1\leqq p<q\leqq 2$ **and** G **is infinite.** In this section the underlying group G will always be either a locally compact abelian group (that is, an LCA group) or a compact group. When $1<p<q<2$ it is known that

$$L_1^1\subset L_p^p\subset L_q^q\subset L_2^2, \tag{4.0}$$

where each inclusion map is continuous. (This follows from the Riesz-Thorin convexity theorem and $L_p^p=L_{p'}^{p'}$ when G is LCA and from (2.4) above when G is compact.) The final inclusion of (4.0) is known to be strict ([8] and [9]), and in this section we will use this information to prove the strictness of the first two inclusions.

4.1 THEOREM. *Let $\{\lambda_n\}_{n=1}^\infty$ denote a strictly positive sequence such that $\lambda_n/\sum_{m=1}^{n-1}\lambda_m$ is unbounded and increasing. Choose $p\in[1,2)$ and suppose that there exists a sequence*

$\{T_n\}_{n=1}^{\infty}$ in L_p^p and numbers a, b such that

$$0 < a < \|T_n\|_{p,p} < b, \tag{4.1$_1$}$$

$$T_n \to 0 \text{ in } L_2^2 \text{ as } n \to \infty. \tag{4.1$_2$}$$

Then there exists a sequence $n_1 < n_2 < \cdots$ of positive integers such that the series

$$\sum_{\nu=1}^{\infty} \lambda_\nu T_{n_\nu}$$

converges in L_q^q, for each $q \in (p, 2]$, to a unique operator, T say, and yet $T \notin L_p^p$.

Proof. By considering a subsequence in place of $\{T_n\}$ we may assume, without loss of generality, that

$$\|T_n\|_{2,2} \leqq \exp(-\lambda_n) \tag{4.2}$$

for $n \in N = \{1, 2, 3, \ldots\}$, and that $T_n \neq T_m$ for all $m \neq n$.

Let Ω denote the set of monotonically increasing maps ω from N to $N \cup \{\infty\}$ such that ω is strictly monotonic on $\{n : \omega(n) \in N\}$. Let $\omega \in \Omega$ and $q \in (p, 2)$. Then, after defining $T_\infty = 0$ and $\lambda_\infty = 0$,

$$\sum_{n \in N} \lambda_n \|T_{\omega(n)}\|_{q,q} < \infty, \tag{4.3}$$

since

$$\begin{aligned} \sum \lambda_n \|T_{\omega(n)}\|_{q,q} &\leqq \sum \lambda_{\omega(n)} \|T_{\omega(n)}\|_{q,q} \\ &\leqq \sum \lambda_n \|T_n\|_{q,q} \leqq \sum \lambda_n (\|T_n\|_{2,2})^\alpha (\|T_n\|_{p,p})^{1-\alpha} \end{aligned}$$

by the Riesz-Thorin convexity theorem, where $\alpha \in (0, 1]$ and satisfies $1/q = \alpha/2 + (1-\alpha)/2$. But by (4.2) and the hypothesis (4.1$_1$), this last sum is bounded by the convergent sum $\sum \lambda_n \exp(-\alpha\lambda_n) \cdot b^{1-\alpha}$.

Let $\{q_m\}$ denote a decreasing sequence in $(p, 2]$ such that $q_m \to p$ as $m \to \infty$. Then the following is a sequence of metrics on Ω:

$$d_m(\omega, \omega') = \sum_{n \in N} \lambda_n \|T_{\omega(n)} - T_{\omega'(n)}\|_{q_m, q_m} \tag{4.4}$$

for each ω, $\omega' \in \Omega$ and $m \in N$. It is well known that the topology on Ω may be defined by a single metric, d say. We will now prove that (Ω, d) is complete.

Let $\{\omega^r\}_{r \in N}$ be a Cauchy sequence in (Ω, d). Then $\{\omega^r\}$ is Cauchy in each (Ω, d_m). Since each $L_{q_m}^{q_m}$ is a Banach space and is imbedded in L_2^2, the completeness of (weighted) l^1-spaces of Banach space-valued functions shows that there exists $\{\psi_n\}_{n \in N}$, where $\psi_n \in \bigcap \{L_{q_m}^{q_m} : m \in N\}$ for each n, such that

$$\sum_{n \in N} \lambda_n \|T_{\omega^r(n)} - \psi_n\|_{q_m, q_m} \to 0 \quad \text{as } r \to \infty$$

for each $m \in N$. It suffices to show that there exists $\omega \in \Omega$ such that $T_{\omega(n)} = \psi_n$ for each $n \in N$.

Now $T_n \to 0$ in L_2^2, $T_n \neq T_m \neq 0$ for $n \neq m$, and $\|T_{\omega^r(n)} - \psi_n\|_{2,2} \to 0$ as $r \to \infty$ for each $n \in N$. Thus *either* $\psi_n = 0$ *or* $\psi_n = \sigma_{n'}$ for some $n' \in N$. In the first case, $\omega^r(n) \to \infty$ as $r \to \infty$; and in the second case, $\omega^r(n) = n'$ for r sufficiently large. Define $\omega(n) = n'$ if n satisfies the second case, and $\omega(n) = \infty$ otherwise. Thanks to the fact that ω satisfies the two requirements of monotonicity for membership of Ω, we have $\omega \in \Omega$ and so (Ω, d) is complete.

Assume that the conclusion of the theorem is false; that is, for each $\omega \in \Omega$, the (2, 2)-multiplier $T^{(\omega)} = \sum \lambda_n T_{\omega(n)}$ belongs to L_p^p. Thus we have a mapping u from Ω to the positive reals defined by

$$u\colon \omega \mapsto \|T^{(\omega)}\|_{p,p}.$$

We will prove that the set $\{\omega : u(\omega) \leqq \beta\}$ is closed for each $\beta \geqq 0$, that is, that u is lower semicontinuous. Suppose that $\omega^r \to \omega$ in Ω and that $u(\omega^r) \leqq \beta$. Then $T^{(\omega^r)} \to T^{(\omega)}$ in $L_{q_m}^{q_m}$ for each $m \in N$. The Riesz-Thorin convexity theorem shows that for $1/q_m = \alpha_m/p + (1-\alpha_m)/2$,

$$\begin{aligned}\|T^{(\omega^r)}\|_{q_m,q_m} &\leqq (\|T^{(\omega^r)}\|_{p,p})^{\alpha_m}(\|T^{(\omega^r)}\|_{2,2})^{1-\alpha_m}\\ &\leqq \beta^{\alpha_m}(\|T^{(\omega^r)}\|_{2,2})^{1-\alpha_m}.\end{aligned}$$

Letting $r \to \infty$, it follows that

$$\|T^{(\omega)}\|_{q_m,q_m} \leqq \beta^{\alpha_n}(\|T^{(\omega)}\|_{2,2})^{1-\alpha_m};$$

that is, for $f \in C_c$,

$$\|T^{(\omega)}f\|_{q_m} \leqq \beta^{\alpha_m}(\|T^{(\omega)}\|_{2,2})^{1-\alpha_m}\|f\|_{q_m}.$$

Since $T^{(\omega)}f \in L^p \cap L^2$ when $f \in C_c$, we may let $m \to \infty$, and thus $\alpha_m \to 1$, to obtain

$$\|T^{(\omega)}f\|_p \leqq \beta\|f\|_p,$$

showing that $u(\omega) \leqq \beta$, as required.

Thus the sequence of sets $\{\omega : u(\omega) \leqq n\}$, $n \in N$, forms a countable closed covering of Ω. Since (Ω, d) is a complete metrizable space, it is nonmeagre in itself, and the Baire category theorem shows that a member of this covering has a nonvoid interior. Hence there exists $m \in N$, $\omega \in \Omega$ and positive real numbers ε, k such that

$$u(\omega') \leqq k \quad \text{whenever } d_m(\omega', \omega) < \varepsilon. \tag{4.5}$$

We will now show that this is impossible. For each $r \in N$, define $\omega^r \in \Omega$ by $\omega^r = \omega$ on $\{1, 2, \ldots, r\}$ and ∞ otherwise. From (4.3) there exists $r_0 \in N$ such that for $r > r_0$, $d_m(\omega^r, \omega) < \varepsilon$. But

$$\begin{aligned}u(\omega^r) &= \left\|\sum_{n=1}^{r} \lambda_n T_{\omega(n)}\right\|_{p,p}\\ &\geqq \lambda_r\|T_{\omega(r)}\|_{p,p} - \sum_{n=1}^{r-1} \lambda_n\|T_{\omega(n)}\|_{p,p}\\ &\geqq \lambda_r a - b\sum_{n=1}^{r-1} \lambda_n,\end{aligned}$$

which is eventually

$$\geqq (a/b)\cdot\lambda_r\Big/\sum_{n=1}^{r-1}\lambda_n-1.$$

But this is unbounded by hypothesis, and so contradicts (4.5). Thus there exists $\omega\in\Omega$ such that $\sum\lambda_n T_{\omega(n)}\notin L_p^p$. For this to be true it is impossible for either ω to be identically ∞ or for $\{n : \omega(n)\in N\}$ to be finite; thus the sequence $\omega(1)<\omega(2)<\cdots$ satisfies the conclusions of the theorem.

4.2 The following definition and lemma is needed before going to the main result. For $1<p<\infty$, let A_p denote the set of all functions on G of the form $\sum_{i=1}^{\infty} f_i * g_i$ with $f_i\in L^p$, $g_i\in L^{p'}$ and $\sum\|f_i\|_p\|g_i\|_{p'}<\infty$. Define a norm on A_p by setting

$$\|h\|_{A_p}=\inf\Big\{\sum\|f_i\|_p\|g_i\|_{p'} : h=\sum f_i * g_i\Big\},$$

for each $h\in A_p$.

The importance of A_p is shown by Theorems 1 and 6 of Figà-Talamanca [6], which together state that the normed dual of A_p, $1<p<\infty$, is isometrically isomorphic to L_p^p. Also note that $A_2=A$, the space of continuous functions on G with integrable Fourier transforms, that is, $f\in A$ if and only if f is continuous and $\hat{f}\in L^1$ if G is abelian, or $\hat{f}\in\mathfrak{E}_1$ if G is compact—see 2.2.

4.3 Lemma. *A is a dense subspace of A_p, $1<p<\infty$.*

Proof. Let $p\in(1,\infty)$. Since (i) $L_p^p\subset L_2^2$, and (ii) L_p^p and L_2^2 are isometrically isomorphic to the normed duals of A_p and A respectively, it follows readily that $A\subset A_p$. To show that A is dense in A_p only requires noting that each element of A_p can be approximated by functions of the form $\sum_{i=1}^{m} f_i * g_i$, where $f_i, g_i\in C_c$.

4.4 Theorem. *Let G be an infinite LCA group or an infinite compact group. Then*

$$\bigcup_{1\leqq q<p} L_q^q \subsetneqq L_p^p \subsetneqq \bigcup_{p<q\leqq 2} L_q^q$$

if $1<p<2$. If $p=1$, the second strict inclusion remains valid, and if $p=2$ the first strict inclusion remains valid.

4.5 Remarks. By proving the existence of sets of uniqueness for $L^p(G)$, $1\leqq p<2$, Figà-Talamanca and Gaudry have recently proved in [8, §2] that $L_p^p\subsetneqq L_2^2$ when $1\leqq p<2$ and G is any infinite LCA group, thus improving an earlier result of R. E. Edwards which required G to satisfy the extra condition of possessing an infinite discrete subgroup. The authors of [8] then employ the Riesz-Thorin convexity theorem to prove that

$$M_p^p\cap C_0(\hat{G})\subsetneqq M_q^q\cap C_0(\hat{G})$$

when $1\leqq p<q\leqq 2$, unless G is finite, where $\hat{G}$ denotes the character group of G. However, Theorem 4.4 generalizes the basic result "$L_p^p\subsetneqq L_q^q$ when $1\leqq p<q\leqq 2$ and

G is infinite compact or infinite LCA" in a different direction and is proved by a method which is constructive to some extent.

Proof of 4.4. When $1 \leqq p < 2$, the proof of the second strict inclusion will be based on the fact that $\|\cdot\|_{p,p}$ is a strictly stronger norm than $\|\cdot\|_{2,2}$ on L_p^p. From (4.0) we know that $\|\cdot\|_{p,p}$ is stronger than $\|\cdot\|_{2,2}$ on L_p^p. Let $A_1 = C_0$. Since A is dense in A_p, $1 \leqq p < 2$ (see [5, Proposition 3.7] and Lemma 4.3), L_p^p is the normed dual of A equipped with the A_p norm. Thus if $\|\cdot\|_{p,p}$ is equivalent to $\|\cdot\|_{2,2}$ on L_p^p, then the A_p norm is equivalent to the A norm on A, and so $L_p^p = L_2^2$. But this is impossible when G is infinite, by [8] and [9]. (When $p=1$, simpler methods suffice to show that $L_1^1 \subsetneqq L_2^2$, or equivalently, $C_0 \supsetneqq A$, when G is infinite.) Thus $\|\cdot\|_{p,p}$ is strictly stronger than $\|\cdot\|_{2,2}$ on L_p^p, and so we may choose a sequence in L_p^p satisfying (4.1_1) and (4.1_2). This leads to the existence of $T \in \bigcap \{L_q^q : p < q \leqq 2\}$ such that $T \notin L_p^p$.

To prove the first strict inequality when $1 < p \leqq 2$, note first that from the above we have $L_q^q \subsetneqq L_p^p$ when $1 \leqq q < p$. Assume $\bigcup \{L_q^q : 1 \leqq q < p\} = L_p^p$. Let $\{q_m\}$ be an increasing sequence in $[1, p)$ such that $q_m \to p$ as $m \to \infty$. Then the injection maps $i_m : L_{q_m}^{q_m} \to L_p^p$ are continuous and $\bigcup \{i_m(L_{q_m}^{q_m}) : m = 1, 2, \ldots\} = L_p^p$. Thus the hypotheses of [2, 6.5.1] are satisfied, the conclusion being that $L_{q_m}^{q_m} = L_p^p$ for some positive integer m, a contradiction.

4.6 COROLLARY. *When p satisfies $2 \leqq p \leqq \infty$, the obvious analogue to Theorem* 4.4 *is valid.*

BIBLIOGRAPHY

1. B. Brainerd and R. E. Edwards, *Linear operators which commute with translations.* I: *Representation theorems*, J. Austral. Math. Soc. **6** (1966), 289–327. MR **34** #6542.

2. R. E. Edwards, *Functional analysis: Theory and applications*, Holt, Rinehart and Winston, New York, 1965. MR **36** #4308.

3. ———, *Changing signs of Fourier coefficients*, Pacific J. Math. **15** (1965), 463–475. MR **34** #564.

4. ———, *Fourier series: A modern introduction.* Vol. II, Holt, Rinehart and Winston, New York, 1967. MR **36** #5588.

5. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236. MR **37** #4208.

6. A. Figà-Talamanca, *Translation invariant operators in L^p*, Duke Math. J. **32** (1965), 495–502. MR **31** #6095.

7. A. Figà-Talamanca and G. I. Gaudry, *Density and representation theorems for multipliers of type (p, q)*, J. Austral. Math. Soc. **7** (1967), 1–6. MR **35** #666.

8. ———, *Multipliers and sets of uniqueness of L^p*, Michigan Math. J. (to appear).

9. A. Figà-Talamanca and D. Rider, *A theorem of Littlewood and lacunary series for compact groups*, Pacific J. Math. **16** (1966), 505–514. MR **34** #6444.

10. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* Vol. II: *Structure and analysis for compact groups. Analysis on locally compact abelian groups*, Die Grundlehren der math. Wissenschaften, Band 152, Springer-Verlag, Berlin, 1970.

11. W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227. MR **22** #6972.

INSTITUTE OF ADVANCED STUDIES, AUSTRALIAN NATIONAL UNIVERSITY,
CANBERRA, AUSTRALIA