TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, December 1970

GROUP ALGEBRA MODULES. IV

BY

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Abstract. Let Γ be a locally compact group, Ω a measurable subset of Γ, and let L_Ω denote the subspace of $L^1(\Gamma)$ consisting of all functions vanishing off Ω. Assume that L_Ω is a subalgebra of $L^1(\Gamma)$. We discuss the collection $\mathfrak{R}_\Omega(K)$ of all module homomorphisms from L_Ω into an arbitrary Banach space K which is simultaneously a left $L^1(\Gamma)$ module. We prove that $\mathfrak{R}_\Omega(K)=\mathfrak{R}_\Omega(K_0)\oplus\mathfrak{R}_\Omega(K_{\text{abs}})$, where K_0 is the collection of all $k\in K$ such that $fk=0$, for all $f\in L^1(\Gamma)$, and where K_{abs} consists of all elements of K which can be factored with respect to the module composition. We prove that $\mathfrak{R}_\Omega(K_0)$ is the collection of linear continuous maps from L_Ω to K_0 which are zero on a certain measurable subset of X. We reduce the determination of $\mathfrak{R}_\Omega(K_{\text{abs}})$ to the determination of $\mathfrak{R}_\Gamma(K_{\text{abs}})$. Denoting the topological conjugate space of K by K^*, we prove that $(K_{\text{abs}})^*$ is isometrically isomorphic to $\mathfrak{R}_\Omega(K^*)$. Finally, we discuss module homomorphisms R from L_Ω into $L^1(X)$ such that for each $f\in L_\Omega$, Rf vanishes off Y.

1. **Introduction.** Once again we come back to the question of module homomorphisms which began our investigation of group algebra modules in the first place ([3] and [4]). The present paper descends from both these papers. If Γ is a locally compact group, $L^1(\Gamma)$ the Banach space of integrable functions on Γ, and if K is a left $L^1(\Gamma)$ module, we studied in [3] the collection of module homomorphisms from $L^1(\Gamma)$ into K, from a rather abstract vantage point. On the other hand, if Γ acts on a locally compact space X as a transformation group, m_X is a positive Radon measure on X quasi-invariant with respect to Γ, and if $L^1(X)$ is the Banach space of integrable functions on X, we showed in [4] that $L^1(X)$ can be made into a left $L^1(\Gamma)$ module, and then we examined the module homomorphisms from $L^1(\Gamma)$ into $L^1(X)$.

In the present paper we let Ω be a measurable subset of Γ, and let L_Ω denote the subspace of $L^1(\Gamma)$ consisting of all functions vanishing off Ω. We assume that L_Ω is a subalgebra of $L^1(\Gamma)$. Then we discuss the module homomorphisms from L_Ω into an arbitrary left $L^1(\Gamma)$ module K. The collection of such homomorphisms we call $\mathfrak{R}_\Omega(K)$. The fact that L_Ω need not have an approximate identity makes the

Received by the editors June 17, 1969.

AMS 1969 *subject classifications*. Primary 4680, 2220.

Key words and phrases. Transformation group, quasi-invariant measure, absolutely continuous measure, approximate identity, factorable, group algebra module, module homomorphism, absolutely continuous module, order-free module, subsemigroup, left-translation invariant, orbit topology.

(1) Research supported in part by the National Science Foundation.

problem much more difficult. The fact that L_Ω has shifted approximate identities makes the problem solvable, via several reductions.

In §3 we prove that $\mathfrak{R}_\Omega(K)=\mathfrak{R}_\Omega(K_0)+\mathfrak{R}_\Omega(K_{\text{abs}})$, where K_0 consists of all elements of K which when composed with elements of $L^1(\Gamma)$ yield the zero element in K, and where K_{abs} consists of all elements of K which can be factored into the composition of an $L^1(\Gamma)$ element and some element of K. Thus the homomorphism problem splits into two parts. Via Theorem 3.5, $\mathfrak{R}_\Omega(K_0)$ is the collection of linear, continuous maps from L_Ω to K_0 which are zero on L_T where T is a certain measurable subset of G; this set T also turns up in the previous paper [5], it involves the composition operator in a direct way.

Next in §4, we look at $\mathfrak{R}_\Omega(K_{\text{abs}})$. Let $d\Omega=\{\sigma\in\Gamma$: for every measurable neighborhood Φ of σ, $\Phi\cap\Omega$ has positive measure$\}$. We show that $d\Omega$ splits up into a collection $\mathscr{J}(d\Omega)$ of pairwise disjoint subsets. Then each $R\in\mathfrak{R}_\Omega(K_{\text{abs}})$ corresponds to a collection $(R_J)_{J\in\mathscr{J}(d\Omega)}$ where $R_J\in\mathfrak{R}_\Gamma(K_{\text{abs}})$ and such that R_J and R are identical on $\Omega\cap J$. Furthermore, for each J, the homomorphism R_J is uniquely defined. Conversely, any collection $(R_J)_{J\in\mathscr{J}(d\Omega)}$ which is norm-bounded in $\mathfrak{R}_\Gamma(K)$ gives rise to a (unique) $R\in\mathfrak{R}_\Omega(K)$. Thus the problem of finding $\mathfrak{R}_\Omega(K_{\text{abs}})$ has reduced to that of finding $\mathfrak{R}_\Gamma(K_{\text{abs}})$.

In §5 we assume that $\Omega=\Gamma$. We embed K into $\mathfrak{R}_\Gamma(K)$ by the map T_K which sends $k\in K$ into right module multiplication by k. Then K_{abs} is injected isometrically onto $[\mathfrak{R}_\Gamma(K)]_{\text{abs}}$. Denoting the topological conjugate space of K by K^*, we prove that $(K_{\text{abs}})^*$ is isometrically isomorphic as a module to $\mathfrak{R}_\Gamma(K^*)$. This paves the way for a collection of examples.

Let $Y\subseteq X$ be measurable and let $\mathscr{J}(d\Omega)$ consist of only one element. We close the paper with a study of the module homomorphisms R from L_Ω to $L^1(X)$ such that for each $f\in L_\Omega$, Rf vanishes off Y.

2. **Notations.** The notations we use are mainly those given in §2 of [5]. Most of the definitions and comments below have already appeared in our preceding texts, but we desire to have them stated here explicitly for reference.

Let Γ be a locally compact group with identity 1 and left Haar measure m. For $f\in L^1(\Gamma)$ and $\sigma\in\Gamma$ we have L^1-functions f_σ, f^σ, f' defined by

$$\begin{aligned} f_\sigma(\tau) &= f(\sigma\tau), && \tau\in\Gamma,\\ f^\sigma(\tau) &= f(\tau\sigma)\Delta(\sigma), && \tau\in\Gamma,\\ f'(\tau) &= f(\tau^{-1})\Delta(\tau^{-1}), && \tau\in\Gamma. \end{aligned}$$

These functions are connected with the convolution in $L^1(\Gamma)$ by the formulas $f_\sigma * g=(f*g)_\sigma, f^\sigma * g=f*g_\sigma, f*g^\sigma=(f*g)^\sigma$, and $(f*g)'=g'*f'$ $(f, g\in L^1(\Gamma))$.

Let $\Omega\subseteq\Gamma$ be measurable. We put $L_\Omega=\{f\in L^1(\Gamma) : f=0$ a.e. outside $\Omega\}$ and $d\Omega=\{\sigma\in\Gamma$: for every measurable neighborhood Φ of σ, $m(\Phi\cap\Omega)\neq 0\}$. Then $\Omega\subseteq d\Omega$ l.a.e. (= locally almost everywhere), i.e. $d\Omega\backslash\Omega$ is locally null. For every $\sigma\in d\Omega$, L_Γ contains approximate identities $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ such that $(u_i)^{\sigma^{-1}}\in L_\Omega$ for each i and $(v_j)_{\sigma^{-1}}\in L_\Omega$ for each j [5, Lemma 3.1].

For measurable Ω, $\Phi \subseteq \Gamma$, we let $L_\Omega * L_\Phi$ be the closed linear hull of $\{f * g : f \in L_\Omega, g \in L_\Phi\}$. Then

$$L_\Omega * L_\Phi = \mathrm{Cl}\left(\sum_{\sigma \in d\Omega} L_{\sigma\Phi}\right) = \mathrm{Cl}\left(\sum_{\tau \in d\Phi} L_{\Omega\tau}\right). \tag{2.1}$$

(The first equation is [5, Corollary 3.4]; the second is proved in a similar way as the first.) If Φ is not locally null, then $d\Phi \neq \varnothing$; taking $\Omega = \Gamma$ we obtain from (2.1) that

$$\mathrm{Cl}\left(\sum_{\sigma \in \Gamma} L_{\sigma\Phi}\right) = L_\Gamma. \tag{2.2}$$

In case L_Ω is a subalgebra of L_Γ we obtain from (2.1) the inclusions $\Omega\sigma \subseteq \Omega$ l.a.e. and $\sigma\Omega \subseteq \Omega$ l.a.e. for all $\sigma \in d\Omega$. Thus, if $f \in L_\Omega$ and $\sigma \in d\Omega$, then $f^{\sigma^{-1}} \in L_\Omega$ and $f_{\sigma^{-1}} \in L_\Omega$. By [5, Corollary 3.5(ii)] $d\Omega$ is a closed semigroup of Γ.

We use the symbol ξ_Φ to denote the characteristic function of Φ.

If A is a Banach algebra, an A-module is a (left) module K over A which is also a Banach space, and such that $\|f * k\| \leq \|f\| \, \|k\|$ for all $f \in A$, $k \in K$ ($*$ denoting the module composition). In particular we shall consider L_Γ modules. An element k of an L_Γ module K is called factorable if there exist $f \in L_\Gamma$ and $k' \in K$ such that $k = f * k'$. The factorable elements of K form a closed submodule K_{abs} of K [5]. K is said to be absolutely continuous if $K_{\mathrm{abs}} = K$. For instance, L_Γ is an absolutely continuous L_Γ module. It follows that $(K_{\mathrm{abs}})_{\mathrm{abs}} = K_{\mathrm{abs}}$ for every K. It is clear that $\lim_i u_i * k = k$ for every $k \in K_{\mathrm{abs}}$ and every approximate identity $(u_i)_{i \in I}$ in L_Γ. For $\sigma \in \Gamma$ we define a norm-preserving left shift $k \to k_\sigma$ in K_{abs} by

$$(f * k')_\sigma = f_\sigma * k', \qquad f \in L_\Gamma,\ k' \in K \tag{2.3}$$

(see [5]). Then

$$f * k_\sigma = f^\sigma * k, \qquad f \in L_\Gamma,\ k \in K_{\mathrm{abs}},\ \sigma \in \Gamma. \tag{2.4}$$

For every $k \in K_{\mathrm{abs}}$, k_σ depends continuously on σ. For all $f \in L_\Gamma$, $k \in K_{\mathrm{abs}}$ and $k^* \in K^*$, we have by [5]

$$k^*(f * k) = \int_\Gamma f(\sigma) k^*(k_{\sigma^{-1}})\, d\sigma. \tag{2.5}$$

3. **Reduction to order-free modules.** Let K be an L_Γ module. We call K *order-free* if there is no $k \in K$ such that $k \neq 0$ and such that $f * k = 0$ for every $f \in L_\Gamma$. Every absolutely continuous module is order-free, because if K is such a module, then for all $k \in K$, $k \in$ closure $\{f * k : f \in L_\Gamma\}$.

In general, for an L_Γ module K we call $\{k \in K : f * k = 0$ for every $f \in L_\Gamma\}$ the *order submodule* K_0 of K. Note that $k \in K$ lies already in K_0 if there is a measurable $\Phi \subseteq \Gamma$, not locally null, such that $L_\Phi * k = \{0\}$. (Then for every $\sigma \in \Gamma$, $\{0\} = (L_\Phi * k)_{\sigma^{-1}}$ $= (L_\Phi)_{\sigma^{-1}} * k = (L_{\sigma\Phi}) * k$, so that, by (2.2), $L_\Gamma * k = \mathrm{Cl}\,(\sum_{\sigma \in \Gamma} L_{\sigma\Phi}) * k = \{0\}$.) The

Banach space K/K_0 is made into an L_Γ module by the definition

$$f*(k+K_0) = (f*k)+K_0 \qquad (f\in L_\Gamma,\ k\in K).$$

This K/K_0 is always order-free.

Most modules we shall deal with are order-free, e.g., $C_\infty(X)$, $M(X)$ and $L^p(X)$ $(1\leqq p\leqq\infty)$. However, modules with order sometimes arise in a natural way. Thus, if K is an L_Γ module, we may define a module composition on K^* by the following definition.

3.1. DEFINITION. $(f*k^*)(k)=k^*(f'*k)$, $(f\in L_\Gamma,\ k\in K,\ k^*\in K^*)$.

K^* is then an L_Γ module. It is not hard to prove that K^* is order-free if and only if K is absolutely continuous. In fact, there is a natural isometrical module homomorphism of $(K_{\mathrm{abs}})^*$ onto $K^*/(K^*)_0$.

Let Ω be a measurable subset of Γ that is not locally null, and assume that L_Ω is a subalgebra of L_Γ. Let K be an L_Γ module.

3.2. DEFINITION. A continuous linear map $R\colon L_\Omega\to K$ is an (L_Ω, K)-homomorphism if $R(f*g)=f*Rg$ $(f,g\in L_\Omega)$. The collection of (L_Ω, K)-homomorphisms we denote by $\mathfrak{R}_\Omega(K)$. When $\Omega=\Gamma$ we suppress the Ω and write $\mathfrak{R}(K)$.

3.3. THEOREM. *For any L_Γ module K, $\mathfrak{R}_\Omega(K)$ is the direct sum of $\mathfrak{R}_\Omega(K_{\mathrm{abs}})$ and $\mathfrak{R}_\Omega(K_0)$. In particular, in case K is order-free, then $R(f)\in K_{\mathrm{abs}}$ for all $R\in\mathfrak{R}_\Omega(K)$ and $f\in L_\Omega$.*

Proof. If $k\in K_{\mathrm{abs}}$, then $\lim_i u_i*k=k$ for every approximate identity $(u_i)_{i\in I}$ in L_Γ; therefore $k\notin K_0$ if $k\neq 0$. Then by the definitions of $\mathfrak{R}_\Omega(K_{\mathrm{abs}})$ and $\mathfrak{R}_\Omega(K_0)$, their intersection is $\{0\}$. Thus we need only show that $\mathfrak{R}_\Omega(K)=\mathfrak{R}_\Omega(K_{\mathrm{abs}})+\mathfrak{R}_\Omega(K_0)$.

Let $R\in\mathfrak{R}_\Omega(K)$ be given. Take $\sigma\in d\Omega$. As we mentioned in §2 there exists an approximate identity $(u_i)_{i\in I}$ in L_Γ such that for each $i\in I$, $(u_i)^{\sigma^{-1}}\in L_\Omega$. Then $f\in L_\Omega$ implies that $f_{\sigma^{-1}}\in L_\Omega$ and

$$\begin{aligned} R(f_{\sigma^{-1}}) &= \lim_i R(u_i*f_{\sigma^{-1}}) = \lim_i R((u_i)^{\sigma^{-1}}*f) \\ &= \lim_i\, [(u_i)^{\sigma^{-1}}*Rf]. \end{aligned}$$

Since $(u_i)^{\sigma^{-1}}*Rf\in K_{\mathrm{abs}}$ and K_{abs} is closed in K, it follows that $R(f_{\sigma^{-1}})\in K_{\mathrm{abs}}$. We define $R_a\colon L_\Omega\to K$ by $R_a(f)=[R(f_{\sigma^{-1}})]_\sigma$. Then R_a is a continuous linear map of L_Ω into K_{abs}. Take $g\in L_\Omega$. For $f\in L_{\Omega\sigma}$ we have $f^\sigma\in L_\Omega$, so that $f*R_ag=f^\sigma*R(g_{\sigma^{-1}})$ $=R(f^\sigma*g_{\sigma^{-1}})=R(f*g)=f*Rg$. Thus $L_{\Omega\sigma}*(Rg-R_ag)=\{0\}$. It follows that $Rg-R_ag\in K_0$, and we conclude that $R-R_a$ is a continuous linear map $L_\Omega\to K_0$. Moreover, for all $f,g\in L_\Omega$ we obtain

$$\begin{aligned} f*R_ag &= f*Rg = [(f*Rg)_{\sigma^{-1}}]_\sigma = (f_{\sigma^{-1}}*Rg)_\sigma \\ &= [R(f_{\sigma^{-1}}*g)]_\sigma = [R((f*g)_{\sigma^{-1}})]_\sigma \\ &= R_a(f*g). \end{aligned}$$

Thus R_a is a module homomorphism, and consequently, so is $R-R_a$. We obtain $R_a \in \mathfrak{R}_\Omega(K_{\text{abs}})$, $R-R_a \in \mathfrak{R}_\Omega(K_0)$.

The elements of $\mathfrak{R}_\Omega(K_{\text{abs}})$ can be characterized in terms of the shift in K_{abs}.

3.4. THEOREM. *A continuous linear map $R\colon L_\Omega \to K_{\text{abs}}$ is in $\mathfrak{R}_\Omega(K_{\text{abs}})$ if and only if $R(f_{\sigma^{-1}})=(Rf)_{\sigma^{-1}}$ for all $f\in L_\Omega$ and $\sigma \in d\Omega$.*

Proof. Take $R\in \mathfrak{R}_\Omega(K_{\text{abs}})$, $\sigma\in d\Omega$. By the proof of the preceding theorem, $Rf-[R(f_{\sigma^{-1}})]_\sigma \in K_0$ for every $f\in L_\Omega$. But of course $Rf-[R(f_{\sigma^{-1}})]_\sigma \in K_{\text{abs}}$. Therefore $Rf=[R(f_{\sigma^{-1}})]_\sigma$. In other words, $(Rf)_{\sigma^{-1}}=R(f_{\sigma^{-1}})$.

Conversely, let $R\colon L_\Omega\to K_{\text{abs}}$ be a linear continuous map such that $R(f_{\sigma^{-1}}) =(Rf)_{\sigma^{-1}}$ for all $f\in L_\Omega$ and $\sigma\in d\Omega$. For any $f, g\in L_\Omega$ and any $k^*\in K^*$ we have by (2.5) that

$$\begin{aligned} k^*(f*Rg) &= \int_\Gamma f(\sigma)k^*((Rg)_{\sigma^{-1}})\,d\sigma \\ &= \int_\Gamma f(\sigma)\{[R^*(k^*)](g_{\sigma^{-1}})\}\,d\sigma \\ &= [R^*(k^*)](f*g) = k^*(R(f*g)), \end{aligned}$$

so that $f*Rg=R(f*g)$. Thus $R\in\mathfrak{R}_\Omega(K_{\text{abs}})$.

In order to describe $\mathfrak{R}_\Omega(K_0)$ we need a little more knowledge about the algebra L_Ω. There exists an open set $T\subset\Gamma$ such that L_T is the closed linear span of $\{f*g : f, g \text{ in } L_\Omega\}$. The proof of this statement and the explicit definition of T are in [5]. (Note that $\mathscr{T}=\emptyset$ because $X=\Gamma$.) In terms of this set T we have a simple characterization of (L_Ω, K_0)-homomorphisms.

3.5. THEOREM. *A continuous linear map $R\colon L_\Omega\to K_0$ is in $\mathfrak{R}_\Omega(K_0)$ if and only if $R=0$ on L_T.*

Proof. If $R=0$ on L_T, then $R(f*g)=0$ for all $f, g\in L_\Omega$. On the other hand, $L_\Omega * K_0=\{0\}$ and $f*Rg\in L_\Omega * K_0$. Thus $f*Rg=0$ and $R\in\mathfrak{R}_\Omega(K_0)$. Conversely, for $R\in\mathfrak{R}_\Omega(K_0)$ and $f, g\in L_\Omega$ we have $R(f*g)=f*Rg\in L_\Omega * K_0=\{0\}$. Since R is linear and continuous, $R(h)=0$ for all $h\in L_T$.

We mention that $\mathfrak{R}_\Omega(K_0)=\{0\}$ if $L_T=L_\Omega$. In particular, $L_T=L_\Omega$ if $1\in d\Omega$, because then L_Ω contains an approximate identity of L_Γ. To wit, if $\Omega=\Gamma$ we have $\mathfrak{R}(K_0)=\{0\}$.

4. **A decomposition theorem for module homomorphisms.** Let Ω be a semigroup. An equivalence relation $\sim$ in Ω is called an "ideal equivalence relation" if $\sigma\tau\sim\tau$ for all $\sigma, \tau\in\Omega$. Let us define the equivalence relation $\approx$ in Ω by $\sigma\approx\tau$ if and only if $\sigma\sim\tau$ for every ideal equivalence relation $\sim$ in Ω. Then $\approx$ is itself an ideal equivalence relation. Among all ideal equivalence relations $\approx$ is the finest, has the smallest equivalence classes. Explicitly, $\sigma\approx\tau$ if and only if there exists a finite sequence $\sigma=\sigma_1, \sigma_2, \ldots, \sigma_n=\tau$ in Ω such that $(\Omega\sigma_i)\cap(\Omega\sigma_{i+1})\neq\varnothing$ for each i.

The equivalence classes modulo $\approx$ form a set $\mathscr{J}(\Omega)$. If Ω is abelian or contains a right unit, then $\mathscr{J}(\Omega)$ consists of only one element; indeed, if Ω is abelian, then $\sigma \approx \tau\sigma = \sigma\tau \approx \tau$ for all σ, $\tau \in \Omega$, and if Ω has a right unit η, then $\sigma = \sigma\eta \approx \eta$ for all $\sigma \in \Omega$. If $\sigma \in J \in \mathscr{J}(\Omega)$ and if $\tau \in \Omega$, then $(\tau^2)\sigma = \tau(\tau\sigma)$, so $\Omega J \subseteq J$, which means that J is a left ideal of Ω.

4.1. Lemma. *Let Ω be a measurable subsemigroup of Γ, with Ω not locally null. Then there exists a neighborhood Φ of $1 \in \Gamma$ such that $\Omega \cap \Phi J \subseteq J$ for every $J \in \mathscr{J}(\Omega)$. In particular, the ideal equivalence classes are relatively open in Ω, and $f = \sum \{f\xi_J : J \in \mathscr{J}(\Omega)\}$ for all $f \in L_\Omega$. Furthermore, $\sigma^{-1}J \cap \Omega = J$ for all $\sigma \in \Omega$ and $J \in \mathscr{J}(\Omega)$.*

Proof. Inasmuch as $\Omega\Omega$ contains a nonempty open subset of Γ by [6, 20.17], there is a $\beta \in \Omega$ and a neighborhood Φ of 1 such that $\beta\Phi \subseteq \Omega$. Let $\sigma \in \Omega$. For all $\tau \in \Omega \cap \Phi\sigma$ we have $\beta\tau\sigma^{-1} \in \beta\Phi \subseteq \Omega$ and consequently $\sigma \approx (\beta\tau\sigma^{-1})\sigma = \beta\tau \approx \tau$. Thus, if $\sigma \in J \in \mathscr{J}(\Omega)$, then $\Omega \cap \Phi\sigma \subseteq J$, so that $\Omega \cap \Phi J \subseteq J$. To prove the last statement, we notice that if $\tau \in J$, $\beta \in \Omega$ and $\sigma\beta \in J$, then $\tau \approx \sigma\beta \approx \beta$; therefore $\sigma^{-1}J \cap \Omega \subseteq J$, while the converse inclusion is obvious.

4.2. Corollary. *Let Ω be as above. If Ω is connected, then $\mathscr{J}(\Omega)$ contains only one element.*

Let Ω be a measurable subset of Γ. It is known that if L_Ω is an algebra, then $d\Omega$ is a closed subsemigroup of Γ. Furthermore, for any $\sigma \in d\Omega$, we have $\sigma\Omega \subseteq \Omega$ l.a.e. and $\Omega\sigma \subseteq \Omega$ l.a.e. (see §2). Inasmuch as every $J \in \mathscr{J}(d\Omega)$ is a subsemigroup, L_J and $L_{\Omega \cap J}$ are nontrivial algebras, the latter because J is a nonempty relatively open subset of $d\Omega$.

We have sufficient machinery to decompose (L_Ω, K)-homomorphisms.

4.3. Theorem. *Let Ω be a measurable subset of Γ that is not locally null. Assume L_Ω is an algebra, and K an L_Γ module which is order-free. For every $R \in \mathfrak{R}_\Omega(K)$ there is a family $\{R_J : J \in \mathscr{J}(d\Omega)\}$ of elements of $\mathfrak{R}(K)$, such that*

$$(*) \qquad R(f) = \sum_{J \in \mathscr{J}(d\Omega)} R_J(f\xi_J), \qquad f \in L_\Omega.$$

Furthermore, $\|R\| = \sup \{\|R_J\| : J \in \mathscr{J}(d\Omega)\}$.

Conversely, for every norm-bounded family $\{R_J : J \in \mathscr{J}(d\Omega)\}$ in $\mathfrak{R}(K)$ the equation $(*)$ defines an $R \in \mathfrak{R}_\Omega(K)$.

Proof. Let $R \in \mathfrak{R}_\Omega(K)$ be given to us. Let $\alpha \in d\Omega$, and let $(u_\iota)_{\iota \in I}$ be an approximate identity in L_Γ with norm 1 and such that $(u_\iota)_{\alpha^{-1}} \in L_\Omega$ for every ι. Next let $\sigma \in \Gamma$ and $f \in L_{\sigma\Omega}$. Then $f_\sigma \in L_\Omega$ and hence $(f_\sigma)^{\alpha^{-1}} \in L_{\Omega\alpha} \subseteq L_\Omega$. Thus

$$\begin{aligned} \{R[(f_\sigma)^{\alpha^{-1}}]\}_{\sigma^{-1}} &= \lim_\iota \{R[(f_\sigma)^{\alpha^{-1}} * u_\iota]\}_{\sigma^{-1}} \\ &= \lim_\iota \{R[f_\sigma * (u_\iota)_{\alpha^{-1}}]\}_{\sigma^{-1}} \\ &= \lim_\iota \{f_\sigma * R[(u_\iota)_{\alpha^{-1}}]\}_{\sigma^{-1}} = \lim_\iota f * R[(u_\iota)_{\alpha^{-1}}]. \end{aligned}$$

In particular, the latter limit exists for any $\sigma \in \Gamma$ and $f \in L_{\sigma\Omega}$. Since the approximate identity is bounded, so is $\{R[(u_i)_{\alpha^{-1}}] : i \in I\}$, which means that if D is defined as $\{f \in L_\Gamma : \lim_i [f * R(u_i)_{\alpha^{-1}}] \text{ exists}\}$, then D must be a closed linear subspace of L_Γ, and by the calculation above, $D \supseteq \bigcup_{\sigma\in\Gamma} L_{\sigma\Omega}$. Thus $D \supseteq \mathrm{Cl}\,(\sum_{\sigma\in\Gamma} L_{\sigma\Omega}) = L_\Gamma$ (see 2.2). Hence we can define $R_\alpha : L_\Gamma \to K$ by

$$R_\alpha(f) = \lim_i f^\alpha * R((u_i)_{\alpha^{-1}}), \qquad f \in L_\Gamma.$$

Since $f, g \in L_\Gamma$ means that $(f*g)^\alpha = f * g^\alpha$, evidently $R_\alpha \in \mathfrak{R}(K)$. Recapitulating, we have taken an $R \in \mathfrak{R}_\Omega(K)$, so that R is defined only on L_Ω, and from it we have defined a module homomorphism R_α on the whole of L_Γ. The particular R_α we obtain depends (or at least appears to depend) upon the $\alpha \in d\Omega$ picked at the beginning of the proof. In any case, we next show that R_α is an extension of R restricted to $\Omega \cap J$, where $\alpha \in J$.

It is easy to show that $R_\alpha = R$ on $L_{\Omega\alpha}$. After all, if $f \in L_{\Omega\alpha}$ then $f^\alpha \in L_\Omega$ and $R_\alpha(f) = \lim_i R(f^\alpha * (u_i)_{\alpha^{-1}}) = \lim_i R(f * u_i) = R(f)$. Now we determine an ideal equivalence relation on $d\Omega$. Write $\beta \sim \sigma$ if $R_\beta = R_\sigma$. Let $\beta, \sigma \in d\Omega$. We must show that $R_{\sigma\beta} = R_\beta$. On $L_{\Omega\beta}$, $R_\beta = R$ while on $L_{\Omega\sigma\beta}$, $R_{\sigma\beta} = R$. Since $\Omega\sigma \subseteq \Omega$ l.a.e., we have $L_{\Omega\sigma\beta} \subseteq L_{\Omega\beta}$ and $R_{\sigma\beta} = R = R_\beta$ on $L_{\Omega\sigma\beta}$. But $R_{\sigma\beta}$ and R_β are module homomorphisms on L_Γ, so by Theorem 3.4 they are left translation invariant by any element of Γ. This means that they agree not only on $L_{\Omega\sigma\beta}$ but on $L_{\tau\Omega\sigma\beta}$ for any $\tau \in \Gamma$. Hence $R_{\sigma\beta} = R_\beta$ on $\mathrm{Cl}\,(\sum_{\tau\in\Gamma} L_{\tau\Omega\sigma\beta}) = L_\Gamma$ (see 2.2). This proves that $R_{\sigma\beta} = R_\beta$ and $\sim$ is an ideal equivalence relation. Next, if $\alpha \in d\Omega$, then there is a $J \in \mathscr{J}(d\Omega)$ such that $\alpha \in J$. For any $\beta \in J$, $R_\alpha = R_\beta$, so that we may define R_J as R_α and take away the apparent dependence on the particular $\alpha \in J$. Then $R_J = R$ on $\mathrm{Cl}\,(\sum_{\beta\in J} L_{\Omega\beta})$. We note that J is a closed subset of $d\Omega$, since $d\Omega \backslash J$ is relatively open in $d\Omega$ by Lemma 4.1 (where Ω is replaced by $d\Omega$). Because $d\Omega$ is closed in Γ, we know that J is also closed in Γ, so $J \supseteq dJ$. Take $\beta \in d\Omega$. Then $R_J = R$ on

$$\mathrm{Cl}\left(\sum_{\sigma\in J} L_{\Omega\sigma}\right) \supseteq \mathrm{Cl}\left(\sum_{\sigma\in dJ} L_{\Omega\sigma}\right) = \sum_{\tau\in d\Omega} L_{\tau J} \supseteq L_{\beta(\Omega\cap J)} \qquad \text{(by 2.1)},$$

which is just perfect for us because if $f \in L_{\Omega\cap J}$ then $f_{\beta^{-1}} \in L_{\beta(\Omega\cap J)}$ and consequently $R_J(f) = [R_J(f_{\beta^{-1}})]_\beta = [R(f_{\beta^{-1}})]_\beta = R(f)$ by the translation invariance of R. Thus $R = R_J$ on $L_{\Omega\cap J}$. We have thus shown that R yields the module homomorphism R_J defined on all of L_Γ in such a way that R and R_J agree on $\Omega \cap J$. From Lemma 4.1 we infer that for any $f \in L_\Omega = L_{\Omega\cap d\Omega}$, $f = \sum_J \{f\xi_J\}$, with the result that $R(f) = \sum_J \{R(f\xi_J)\} = \sum_J \{R_J(f\xi_J)\}$, which proves (*). As for the norm inequalities, $\|R_\alpha\| \leqq \|R\|$ since the approximate identity is bounded by 1. Thus $\|R_J\| \leqq \|R\|$ for every $J \in \mathscr{J}(d\Omega)$. The inequality $\|R\| \leqq \sup\{\|R_J\| : J \in \mathscr{J}(d\Omega)\}$ follows from (*). Hence $\|R\| = \sup\{\|R_J\| : J \in \mathscr{J}(d\Omega)\}$.

We have yet to prove the converse. Let $\{R_J : J \in \mathscr{J}(d\Omega)\}$ be a family of elements of $\mathfrak{R}(K)$ such that $\{\|R_J\| : J \in \mathscr{J}(d\Omega)\}$ is bounded. Then (*) defines a continuous linear map $R : L_\Omega \to K$. Now let $\sigma \in d\Omega$. By the last part of Lemma 4.1,

$\sigma^{-1}J \cap d\Omega = J$ for every $J \in \mathscr{J}(d\Omega)$. This means that for all $f \in L_\Omega$,

$$f_{\sigma^{-1}}\xi_J = (f\xi_{\sigma^{-1}J})_{\sigma^{-1}} = (f\xi_{d\Omega}\xi_{\sigma^{-1}J})_{\sigma^{-1}} = (f\xi_J)_{\sigma^{-1}}.$$

Thus

$$R(f_{\sigma^{-1}}) = \sum_J R_J(f_{\sigma^{-1}}\,\xi_J) = \sum_J R_J((f\xi_J)_{\sigma^{-1}})$$

$$= \left(\sum_J R_J(f\xi_J)\right)_{\sigma^{-1}} = (R(f))_{\sigma^{-1}},$$

so by Theorem 3.4, R is an (L_Ω, K)-module homomorphism.

The theorem says that to any module homomorphism R defined on L_Ω there corresponds a collection $(R_J)_{J\in\mathscr{J}(d\Omega)}$ of module homomorphisms on L_Γ, and each R_J is the unique extension of R restricted to $\Omega \cap J$. Thus if we have complete knowledge of $\mathfrak{R}(K)$, then the problem of $\mathfrak{R}_\Omega(K)$ is completely solved as well. In other words, the problem of characterizing $\mathfrak{R}_\Omega(K)$ is reduced to the problem of characterizing $\mathfrak{R}(K)$. Besides what occurs in this paper, there is a discussion of $\mathfrak{R}(K)$ in [3], [8], and [9].

Sometimes Theorem 4.3 takes on a simpler form.

4.4. COROLLARY. *Assume that Ω has at least one of the following properties:*

(α) *Ω is commutative.*

(β) $1 \in d\Omega$.

(γ) *Ω is connected.*

Then every $R \in \mathfrak{R}_\Omega(K)$ has a unique extension to an $\bar{R} \in \mathfrak{R}(K)$, and $\|\bar{R}\| = \|R\|$.

Proof. $\mathscr{J}(d\Omega)$ contains but one element.

It would be conceivable, no matter what Γ and $\Omega \subseteq \Gamma$ are, that $\mathscr{J}(d\Omega)$ consisted of but one element. In fact it would be desirable, since then any module homomorphism from L_Ω to K could be extended—uniquely—to a module homomorphism from L_Γ to K. However, this is not the case. Let Γ be the free group with two generators, α and β, and let Γ have the discrete topology. Let Φ be the subsemigroup generated by 1, α, and β, and let Ω be the subsemigroup $\Phi\alpha \cup \Phi\beta$. We define the continuous, linear map $R: L_\Omega \to L_\Gamma$ by

$$R(\xi_{\{\sigma\alpha\}}) = \xi_{\{\sigma\alpha\}}, \quad \sigma \in \Phi, \qquad R(\xi_{\{\sigma\beta\}}) = \xi_{\{\sigma\alpha\}}, \quad \sigma \in \Phi.$$

Then $R \in \mathfrak{R}_\Omega(L_\Gamma)$, but there is no extension $\bar{R} \in \mathfrak{R}(L_\Gamma)$ such that R and $\bar{R}$ coincide on L_Ω.

From the definition of $\mathscr{J}(d\Omega)$ we see that $\Omega \cap d\Omega$ is partitioned by $\{\Omega \cap J : J \in \mathscr{J}(d\Omega)\}$. This means that $L_\Omega = L_{\Omega\cap d\Omega} = \sum_J L_{\Omega\cap J}$ where the $L_{\Omega\cap J}$ are pairwise disjoint (except for $\{0\}$) left ideals of L_Ω. We prove below that $\{L_{\Omega\cap J} : J \in \mathscr{J}(d\Omega)\}$ forms the finest decomposition of L_Ω as the sum of disjoint left ideals of the form L_θ.

4.5. THEOREM. *Let $(\theta_i)_{i\in I}$ be a set of measurable subsets of Ω such that each L_{θ_i} is a left ideal of L_Ω and such that $L_\Omega = \mathrm{Cl}\,(\sum_{i\in I} L_{\theta_i})$ and $L_{\theta_i} \cap L_{\theta_j} = \{0\}$ for $i \neq j$. Then for each i, L_{θ_i} is the closure of $\sum \{L_{\Omega\cap J} : J \in \mathscr{J}(d\Omega), L_{\Omega\cap J} \subseteq L_{\theta_i}\}$.*

Proof. Let $P_i: L_\Omega \to L_{\theta_i}$ be defined by $P_i(f) = f\xi_{\theta_i}$. Then P_i is continuous, linear, a projection from L_Ω onto L_{θ_i}, and $f \in L_\Omega$ implies that $f = \sum_{i \in I} P_i(f)$. Fix $i \in I$. Let $f, g \in L_\Omega$. Since by assumption L_{θ_i} is a left ideal, and since $P_i(g) \in L_{\theta_i}$, we have $f * [P_i(g)] \in L_{\theta_i}$, and thus

$$P_i(f * g) - f * [P_i(g)] = P_i(f * g) - P_i\{f * [P_i(g)]\} = P_i\{f * [g - P_i(g)]\}.$$

But $g - [P_i(g)] \in L_{\Omega \backslash \theta_i} = \mathrm{Cl}\,(\sum_{j \neq i} L_{\theta_j})$, so

$$f * (g - [P_i(g)]) \in \mathrm{Cl}\left(\sum_{j \neq i} f * L_\theta\right) \subseteq \mathrm{Cl}\left(\sum_{j \neq i} L_{\theta_j}\right)$$

since each L_{θ_i} is a left ideal in L_Ω. Thus $P_i(f * \{g - [P_i(g)]\}) = 0$. Hence $P_i \in \mathfrak{R}_\Omega(L_\Gamma)$. Hence Theorem 4.3 applies, and there exists a collection $\{R_J : J \in \mathscr{J}(d\Omega)\} \subseteq \mathfrak{R}(L_\Gamma)$ such that $P_i(f) = \sum_J R_J(f\xi_J)$, for all $f \in L_\Gamma$. By a theorem of G. Wendel (see [10]), every $R_J \in \mathfrak{R}(L_\Gamma)$ determines a $\mu_J \in M(\Gamma)$ such that $R_J(g) = g * \mu_J$, for all $g \in L_\Gamma$. Then

$$f\xi_{\theta_i} = P_i(f) = \sum_J (f\xi_J) * \mu_J, \quad \text{for all } f \in L_\Omega.$$

Using this decomposition we show next that for each $J \in \mathscr{J}(d\Omega)$, either $\mu_J = 0$ or $\mu_J = \delta_1$, the point mass at $1 \in \Gamma$. To that end, let $J \in \mathscr{J}(d\Omega)$ such that $\theta_i \cap J$ is not locally null. If $f \in L_{\theta_i \cap J}$ then the formula displayed above yields us $f = f * \mu_J$. Thus $L_{\theta_i \cap J} = L_{\theta_i \cap J} * \mu_J$, so that $L_{\theta_i \cap J} * (\delta_1 - \mu_J) = 0$. Then $\delta_1 - \mu_J$ lies in the order submodule of $M(\Gamma)$. However, $M(\Gamma)$ is order-free. Consequently $\delta_1 - \mu_J = 0$, or $\delta_1 = \mu_J$. On the other hand, if J is such that $J \cap \theta_i$ is locally null, then by a similar reasoning, $\mu_J = 0$. Finally, we note that $(f\xi_J) * \delta_1 = f\xi_J$, while $(f\xi_J) * 0 = 0$, so that $f \in L_\Omega$ implies that

$$f\xi_{\theta_i} = \sum \{f\xi_J : J \in \mathscr{J}(d\Omega),\ \mu_J = \delta_1\}.$$

Since every element of L_{θ_i} is of the form $f\xi_{\theta_i}$ for an appropriate $f \in L_\Omega$, we see that we have decomposed L_{θ_i} as hypothesized. (The closure appearing in the statement of the theorem merely preserves the widespread convention that the sum of a collection of spaces contains only finite sums of elements of the spaces involved.)

We remark that there may very well be finer decompositions of L_Ω into a sum of ideals not of the form L_θ. Thus, if μ is an idempotent measure on Γ, then L_Γ is the direct sum of $L_\Gamma * \mu$ and $L_\Gamma * (\delta_1 - \mu)$, while $\mathscr{J}(\Gamma)$ contains only one element.

Let us see how §§3 and 4 have simplified the problem of finding $\mathfrak{R}_\Omega(K)$-module homomorphisms for an arbitrary L_Γ module K. In the first place, we found that $\mathfrak{R}_\Omega(K) = \mathfrak{R}_\Omega(K_0) \oplus \mathfrak{R}_\Omega(K_{\mathrm{abs}})$, in Theorem 3.3, and then described as completely as we will here the space $\mathfrak{R}_\Omega(K_0)$, which from Theorem 3.5 turns out to be the collection of linear, continuous maps from L_Ω to K_0 which map L_T into 0. That done, we directed our attention to those K which were order-free, showing that $R \in \mathfrak{R}_\Omega(K_{\mathrm{abs}})$ meant that R corresponded to a collection $\{R_J : R_J \in \mathfrak{R}(K_{\mathrm{abs}}), J \in \mathscr{J}(d\Omega)\}$ (Theorem

4.3). Consequently we know $\mathfrak{R}_\Omega(K_{\mathrm{abs}})$ provided we know $\mathfrak{R}(K_{\mathrm{abs}})$, which in this case is none other than $\mathfrak{R}(K)$ if K is order-free, by Theorem 3.3. Later we will use these results in special examples.

5. **Module homomorphisms on L_Γ.** In this section we assume that $\Omega=\Gamma$. For any L_Γ module K whatsoever, we can determine a special subspace of $\mathfrak{R}(K)$ in the following way. For $k\in K$, let $T_Kk\in\mathfrak{R}(K)$ be defined by $(T_Kk)(f)=f*k$ $(f\in L_\Gamma)$. Indeed K is order-free if and only if T_K is injective. Then we have

5.1. THEOREM. (i) *The definition* $(f*R)(g)=R(g*f)$ $(f, g\in L_\Gamma, R\in\mathfrak{R}(K))$ *turns* $\mathfrak{R}(K)$ *into an order-free* L_Γ *module, thereby rendering* T_K *a module homomorphism of* K *into* $\mathfrak{R}(K)$.

(ii) *If K is absolutely continuous, T_K is an isometry.*

(iii) *The restriction of T_K to K_{abs} is an isometry of K_{abs} onto $\mathfrak{R}(K)_{\mathrm{abs}}$.*

(iv) *$T_{\mathfrak{R}(K)}$ is an isometry of $\mathfrak{R}(K)$ onto $\mathfrak{R}(\mathfrak{R}(K))$.*

Proof. (i) Except for showing that $\mathfrak{R}(K)$ is order-free the proof is a straightforward calculation. But if $L_\Gamma*R=\{0\}$, then $R(L_\Gamma)=R(L_\Gamma*L_\Gamma)=L_\Gamma*R(L_\Gamma)=\{0\}$, so $R=0$. To prove (ii), we note first that if K is absolutely continuous, then K is order-free, so T_K is injective. Next, if $(u_\iota)_{\iota\in I}$ is an approximate identity of norm 1 in L_Γ and if $k\in K$, then

$$\begin{aligned}\|T_Kk\| &= \sup\{\|f*k\| : f\in L_\Gamma;\ \|f\|_1\leqq 1\}\\ &\geqq \lim_\iota \|u_\iota*k\| = \|k\|.\end{aligned}$$

On the other hand, since K is a Banach module, $\|T_Kk\|\leqq\|k\|$. Now we prove (iii). The isometry we already have. Since K_{abs} is absolutely continuous, T_Kk is factorable in $\mathfrak{R}(K)$ for every $k\in K_{\mathrm{abs}}$. Thus T_K maps K_{abs} into $\mathfrak{R}(K)_{\mathrm{abs}}$. To show the map restricted to K_{abs} is onto $\mathfrak{R}(K)_{\mathrm{abs}}$, let $R\in\mathfrak{R}(K)_{\mathrm{abs}}$. Then there exist $f\in L_\Gamma$ and $R'\in\mathfrak{R}(K)$ such that $R=f*R'$. Thus for all $g\in L_\Gamma$,

$$\{T_K[R'(f)]\}(g) = g*(R'f) = (f*R')(g) = R(g).$$

Consequently, $R=T_K(R'(f))$. Since L_Γ is factorable, there are $f_1, f_2\in L_\Gamma$ such that $f=f_1*f_2$. Then $R=T_K(R'(f_1*f_2))=T_K(f_1*R'(f_2))\in T_K(K_{\mathrm{abs}})$, which is what we needed to prove. Finally we prove (iv), which is simple now. We have natural surjective isometries

$$\mathfrak{R}(K)\to\mathfrak{R}(K_{\mathrm{abs}})\to\mathfrak{R}(\mathfrak{R}(K)_{\mathrm{abs}})\to\mathfrak{R}(\mathfrak{R}(K)),$$

the middle one coming from (iii) and the outer two by the comment following Theorem 3.4. The composition of these isometries is just $T_{\mathfrak{R}(K)}$.

5.2. THEOREM. *Let K be an L_Γ module. Every $k^*\in K_{\mathrm{abs}}{}^*$ determines a $Qk^*\in\mathfrak{R}(K^*)$ by*

$$[(Qk^*)f](k) = k^*(f'*k) \qquad (f\in L_\Gamma, k\in K).$$

(If $K=K_{\mathrm{abs}}$, then $Q=T_{K^*}$.) The map Q defined by this equation is an isometric module homomorphism of ${K_{\mathrm{abs}}}^*$ onto $\mathfrak{R}(K^*)$.

Proof. Certainly Qk^* is linear and continuous as a map from L_Γ to K^*. Also $\|Qk^*\| \leqq \|k^*\|$. Once one remembers that $(f*g)'=g'*f'$, there is no trouble in showing that $Qk^* \in \mathfrak{R}(K^*)$. Then Q is a linear map ${K_{\mathrm{abs}}}^* \to \mathfrak{R}(K^*)$ and $\|Q\| \leqq 1$. The proof that Q is a module homomorphism is straightforward. Because the domain of Q is ${K_{\mathrm{abs}}}^*$, $Qk^*=0$ only if $k^*=0$. Thus Q is injective and we are done if we show it is surjective and $\|Qk^*\| \geqq \|k^*\|$ for all $k^* \in {K_{\mathrm{abs}}}^*$. Let $R \in \mathfrak{R}(K^*)$, and let $(u_i)_{i\in I}$ be an approximate identity in L_Γ, with $\|u_i\| \leqq 1$ for each i. If $f \in L_\Gamma$ and $j \in K$, then

$$\begin{aligned}(Rf)(j) &= \lim_i (R(f*u_i))(j) \\ &= \lim_i (f*Ru_i)(j) = \lim_i (Ru_i)(f'*j).\end{aligned}$$

Thus $\lim_i (Ru_i)(k)$ exists for every $k \in K_{\mathrm{abs}}$ and

$$\left\|\lim_i (Ru_i)(k)\right\| \leqq \sup_i \|R\| \, \|u_i\| \, \|k\| \leqq \|R\| \, \|k\|.$$

Therefore we can define $k^* \in {K_{\mathrm{abs}}}^*$ by $k^*(k)=\lim_i (Ru_i)(k)$, for $k \in K_{\mathrm{abs}}$, with the result that $\|k^*\| \leqq \|R\|$. Now by the existence proof of $\lim_i (Ru_i)(k)$ we have that for all $f \in L_\Gamma$ and $j \in K$, $(Rf)(j)=\lim_i (Ru_i)(f'*j)=k^*(f'*j)=[(Qk^*)f](j)$. Thus $R=Qk^*$ and $\|Qk^*\| \geqq \|k^*\|$. This finishes the proof of the theorem.

With the aid of Theorems 5.1 and 5.2 we can compute $\mathfrak{R}(K)$ for most of the modules described in [5]. First, assume that X is a locally compact space, and Γ a group of homeomorphisms of X such that the map $(\sigma, x) \to \sigma x$ $(\sigma \in \Gamma,\ x \in X)$ is jointly continuous. Let C_X be the Banach space of all continuous functions k on X such that for every $\varepsilon>0$ the set $\{x \in X : |k(x)| \geqq \varepsilon\}$ is compact. Then C_X is an L_Γ module with the module composition defined by

$$f*k(x) = \int_\Gamma f(\sigma)k(\sigma^{-1}x)\, d\sigma, \qquad x \in X,$$

for $f \in L_\Gamma$, $k \in C_X$. (For details, see [4], [5].) We can make $M(X)$ an L_Γ module by noting that it is the dual of C_X. Definition 3.1 yields

$$(f*\mu)k = \mu(f'*k), \qquad f \in L_\Gamma, \mu \in M(X), k \in C_X,$$

and it turns out that $(f*\mu)(k)=\int_X \int_\Gamma k(\sigma x)f(\sigma)\, d\sigma\, d\mu(x)$.

Since C_X is absolutely continuous, Theorem 5.2 tells us that $\mathfrak{R}(M(X))$ is canonically isomorphic to $M(X)$. (Unfortunately we have not been able to obtain a description of $\mathfrak{R}(C_X)$ itself!)

Now let X possess a quasi-invariant measure m_X. We denote by L_X^p $(1 \leqq p \leqq \infty)$ the space usually called $L^p(X)$ or $L_p(X)$, and write L_X instead of $L^1(X)$. The natural embedding $L_X \to M(X)$ makes L_X a submodule of $M(X)$ (see [4]), and therefore induces an embedding $\mathfrak{R}(L_X) \to \mathfrak{R}(M(X))=M(X)$. The image of $\mathfrak{R}(L_X)$ in $M(X)$ is

$N=\{\mu \in M(X) : \text{for every } f \in L_\Gamma, f * \mu \text{ is absolutely continuous with respect to } m_X\}$. This space has been investigated in [4], and in particular several conditions equivalent to N equaling $M(X)$ appear there.

Since $L_X^\infty=(L_X)^*$ we can use (3.1) to make an L_Γ module out of L_X^∞. It turns out that for $f \in L_\Gamma$ and $k \in L_X^\infty$,

$$f * k(x) = \int_\Gamma f(\sigma)k(\sigma^{-1}x)\, d\sigma, \qquad \text{l.a.e. } x \in X.$$

We know L_X is itself absolutely continuous; hence, again by Theorem 5.2, $\mathfrak{R}(L_X^\infty)$ is isomorphic to L_X^∞.

For $1 \leqq p \leqq \infty$, in [4], we introduced convolution products $L_\Gamma \times L_X^p \to L_X^p$, of which the module operations on L_X and L_X^∞, mentioned above, are special cases. For $1 \leqq p < \infty$, L_X^p is an absolutely continuous module, and in particular, if $1 < p < \infty$, then by Theorem 5.2, $\mathfrak{R}(L_X^p)$ is isomorphic to L_X^p.

Another space whose module homomorphisms we can describe is $L_\Gamma \cap L_\Gamma^p$, $p \in (1, \infty]$, which we look at through the eyes of $K=C_\Gamma+L_\Gamma^q$, where $1/p+1/q=1$. Now K is the linear span of $\{h+k : h \in C_\Gamma, k \in L_\Gamma^q\}$. Under the norm $\|j\|=\inf\{\|h\|+\|k\| : h \in C_\Gamma, k \in L_\Gamma^q, j=h+k\}$ and under the convolution defined by

$$f * j(\sigma) = \int_\Gamma f(\tau)j(\tau^{-1}\sigma)\, d\tau, \qquad \text{l.a.e. } \sigma \in \Gamma,$$

K becomes an absolutely continuous L_Γ module. The dual space $K^*=L_\Gamma \cap L_\Gamma^p$ has for its norm $\|g\|=\max(\|g\|_1, \|g\|_p)$ (see [7, Theorem 5]). The convolution in K^*, defined by the familiar formula in (3.1) can be reduced to the formula

$$f * g(\sigma) = \int_\Gamma f(\tau)g(\tau^{-1}\sigma)\, d\sigma \qquad (f \in L_\Gamma, g \in L_\Gamma \cap L_\Gamma^p, \sigma \in \Gamma).$$

By Theorem 5.2, $\mathfrak{R}(L_\Gamma \cap L_\Gamma^p)$ is canonically isomorphic to $L_\Gamma \cap L_\Gamma^p$.

There is a connection between K_{abs} and $\mathfrak{R}(K)$ deeper than a superficial appraisal might reveal. It becomes apparent if we consider $K \to K_{\mathrm{abs}}$ and $\mathfrak{R}$ as functors in the category of all L_Γ modules with continuous module homomorphisms as morphisms. It is obvious that $\mathfrak{R}$ is related to the well-known functor Hom in the category of all modules over a ring. Writing L instead of L_Γ, in homological language we may denote $\mathfrak{R}(K)$ by $\mathrm{Hom}_L(L, K)$. Less obvious is the analogy between the functor $K \to K_{\mathrm{abs}}$ and the tensor product, reflected in the following theorem.

5.3. THEOREM. *Let K be an L_Γ module. Then for any L_Γ module K' and any continuous bilinear map $T: L_\Gamma \times K \to K'$ that has the property $T(f * g, k) = f * T(g, k) = T(g, f * k)$, there is a unique continuous homomorphism T' from K_{abs} into K' such that the diagram*

$$\begin{tikzcd} L_\Gamma \times K \arrow[r] \arrow[dr, "T"'] & K_{\mathrm{abs}} \arrow[d, dashed, "T'"] \\ & K' \end{tikzcd}$$

is commutative (where the horizontal arrow represents the module composition $(f, k) \to f * k$).

Proof. Let $(u_i)_{i \in I}$ be an approximate identity in L_Γ. Take $k \in K_{\mathrm{abs}}$. There exist $f \in L_\Gamma$ and $j \in K$ such that $f * j = k$. Then

$$\begin{aligned} T(f, j) &= \lim_i T(f * u_i, j) \\ &= \lim_i T(u_i, f * j) = \lim_i T(u_i, k). \end{aligned}$$

Thus we can define a linear $T' \colon K_{\mathrm{abs}} \to K'$ by $T'(k) = \lim_i T(u_i, k)$, for $k \in K_{\mathrm{abs}}$. The rest is straightforward.

Thus it looks reasonable to write $K_{\mathrm{abs}} = L \otimes_L K$. In this terminology, Theorem 5.2 takes the form

$$\mathrm{Hom}_L(L, \mathrm{Hom}_C(K, C)) = \mathrm{Hom}_C(L \otimes_L K, C), \quad C \text{ the complex numbers,}$$

which is a well-known formula in the algebraic theory. A theory relating Banach module homomorphisms to tensor product theory has been begun by Máté [8] and developed systematically by Rieffel [9].

6. **Module homomorphisms from L_Ω to L_Y.** As we saw in §5 there is a natural embedding $\mathfrak{R}(L_X) \to M(X)$. In the sequel we identify each $R \in \mathfrak{R}(L_X)$ with the corresponding element of $M(X)$; thus $\mathfrak{R}(L_X) = N \subseteq M(X)$. Let $\Omega \subseteq \Gamma$ and $Y \subseteq X$ be measurable. We are going to consider those $\mathfrak{R}(L_X)$-module homomorphisms $L_\Gamma \to L_X$ which map L_Ω into $L_Y = \{f \in L_X : f = 0 \text{ a.e. outside } Y\}$. We denote the collection of such homomorphisms by $\mathfrak{R}_{\Omega,Y}$. To aid the discussion we make the following definition.

6.1. Definition. Let $A_{\Omega,Y} = \{x \in X : \sigma x \in Y \text{ for locally almost all } \sigma \in \Omega\}$. We note that $A_{\Omega,Y}$ is measurable, by Theorem 3.15 of [5].

6.2. Lemma. *If $\mu \in \mathfrak{R}(L_X)$ and if* $\mathrm{supp}\, \mu \subseteq A_{\Omega,Y}$, *then $\mu \in \mathfrak{R}_{\Omega,Y}$.*

Proof. We need to show that $L_\Omega * \mu \subseteq L_Y$. To do that, we let $k \in L_X^\infty$ be such that $k|_Y = 0$, and we let $f \in L_\Omega$. We will show that $f * \mu(k) = 0$. For any $x \in \mathrm{supp}\, \mu \subseteq A_{\Omega,Y}$, we have $\sigma x \in Y$ for locally almost all $\sigma \in \Omega$, so that $k(\sigma x) = 0$ l.a.e. on Ω, resulting in $\int_\Gamma f(\sigma)k(\sigma x)\, d\sigma = 0$. Therefore

$$\begin{aligned} (f * \mu)(k) &= \int_X \int_\Gamma f(\sigma)k(\sigma x)\, d\sigma\, d\mu(x) \\ &= \int_{\mathrm{supp}\, \mu} \int_\Gamma f(\sigma)k(\sigma x)\, d\sigma\, d\mu(x) = 0, \end{aligned}$$

which completes the proof.

In the event that Y is closed in X, we can give a complete description of $\mathfrak{R}_{\Omega,Y}$.

6.3. Theorem. *If Y is closed in X, then $\mathfrak{R}_{\Omega,Y} = \{\mu \in M(X) : \mu \in \mathfrak{R}(L_X)$ and $\mathrm{supp}\, \mu \subseteq A_{\Omega,Y}\}$.*

Proof. Because of Lemma 6.2 all we must prove is that if $\mu \in \mathfrak{R}_{\Omega,Y}$, then $\operatorname{supp} \mu \subseteq A_{\Omega,Y}$. By assumption, $L_\Omega * \mu \subseteq L_Y$. Then for any $k \in C_X$ such that $k|_Y = 0$, we have $(f * \mu)(k) = 0$ for all $f \in L_\Omega$. Let $\sigma \in d\Omega$, and let $(u_\iota)_{\iota \in I}$ be an approximate identity for L_Γ such that $(u_\iota)_{\sigma^{-1}} \in L_\Omega$. Then

$$\begin{aligned} 0 &= ((u_\iota)_{\sigma^{-1}} * \mu, k) = (\mu, ((u_\iota)_{\sigma^{-1}})' * k) \\ &= (\mu, (u_\iota')^{\sigma} * k) = (\mu, (u_\iota)' * k_\sigma) \end{aligned}$$

which converges to $\mu(k_\sigma)$ because C_X is absolutely continuous. Since $\sigma^{-1}Y$ is closed, $\operatorname{supp} \mu \subseteq \sigma^{-1}Y$, for any $\sigma \in d\Omega$. But $\bigcap_{\sigma \in d\Omega} \sigma^{-1}Y \subseteq A_{\Omega,Y}$. Hence $\operatorname{supp} \mu \subseteq A_{\Omega,Y}$.

6.4. Corollary. *If Y is closed in X, then $A_{\Omega,Y} = \bigcap_{\sigma \in d\Omega} \sigma^{-1}Y$, and hence $A_{\Omega,Y}$ is closed.*

Recall that we have identified $\mathfrak{R}(L_X)$ with a subspace $N = \{\mu \in M(X) : L_\Gamma * \mu \subseteq L_X\}$ of $M(X)$. Obviously this N should play an important role in our discussion of $A_{\Omega,Y}$. Under the restriction that $N = M(X)$, it is not hard to prove that $A_{\Omega,Y} = A_{\Omega,Y'}$ if $Y = Y'$ l.a.e. (see the implication (i) $\Rightarrow$ (iii) of Theorem 5.6 in [4]). Without the restriction this is not true, as the following example shows: $\Gamma = \{1\}$, $Y = Y'$ only l.a.e. (However, from $Y = Y'$ l.a.e. it always follows that $A_{\Omega,Y} = A_{\Omega,Y'}$ l.a.e.) Under the condition that $N = M(X)$ we have a neater conclusion for Theorem 6.3.

6.5. Corollary. *If Y is closed in X and if $N = M(X)$, then*

$$\mathfrak{R}_{\Omega,Y} = \{\mu \in M(X) : \operatorname{supp} \mu \subseteq A_{\Omega,Y}\}.$$

Let $\Omega \subseteq \Gamma$ be measurable and such that L_Ω is a subalgebra of L_Γ. For any Banach module K over L_Ω we denote by $\mathfrak{R}_\Omega(K)$ the space of all continuous module homomorphisms $L_\Omega \to K$ (since every L_Γ module is an L_Ω module this notation is consistent with our earlier use of the symbol $\mathfrak{R}_\Omega(K)$).

In particular we consider measurable subsets Y of X for which $L_\Omega * L_Y \subseteq L_Y$. For such Y, L_Y is an L_Ω module. Theorem 4.3 gives an injection $\mathfrak{R}_\Omega(L_Y) \to \prod_{J \in \mathscr{J}(d\Omega)} \mathfrak{R}_{\Omega \cap J, Y}$. In case $\mathscr{J}(d\Omega)$ consists of only one element, $\mathfrak{R}_\Omega(L_Y)$ may be identified with $\mathfrak{R}_{\Omega,Y}$. Then by Lemma 6.2, $\mathfrak{R}_\Omega(L_Y) \supseteq \{\mu \in M(X) : \operatorname{supp} \mu \subseteq A_{\Omega,Y}\}$ and if $N = M(X)$, the two sets are equal if Y is closed (Corollary 6.5). If $N = M(X)$, it seems reasonable to ask whether we have equality for all Y, still assuming $\mathscr{J}(d\Omega)$ to contain only one element.

Now $N = M(X)$ if $X = \Gamma$, and $\mathscr{J}(d\Omega)$ contains only one element if Γ is abelian (Corollary 4.4). T. A. Davis states a theorem affirming the inclusion $\mathfrak{R}_\Omega(L_Y) \subseteq \{\mu \in M(X) : \mu$ is concentrated on $A_{\Omega,Y}\}$ for the case Γ is abelian, $X = \Gamma$, and $Y = \Omega$ (Theorem 3.5(2) in [2]). Unfortunately, however, his proof seems to be faulty.

By the same Corollary 4.4, $\mathscr{J}(d\Omega)$ contains only one element if $1 \in d\Omega$. For this case F. Birtel [1] proves $\mathfrak{R}_\Omega(L_Y) = \{\mu \in M(X) : \operatorname{supp} \mu \subseteq A_{\Omega,Y}\}$ under the assumptions $X = \Gamma$, $Y = \Omega$, Ω is a closed semigroup containing 1 whose interior is dense in Ω. In Theorem 6.7 we prove $\mathfrak{R}_\Omega(L_Y)=\{\mu \in M(X) : \mu$ is concentrated on $A_{\Omega,Y}\}$ if $1 \in d\Omega$ and $N=M(X)$. In Corollary 6.8 we prove $\mathfrak{R}_\Omega(L_Y)=\{\mu \in M(X) :$ supp $\mu \subseteq A_{\Omega,Y}\}$ if $1 \in d\Omega$ and if for every $f \in L_\Gamma$ and $k \in L_X^\infty$, $\int_\Gamma f(\sigma)k(\sigma^{-1}x)\,d\sigma$ depends continuously on x, which is true for $X=\Gamma$.

We employ an auxiliary topology on X, called the orbit topology and designated by $\mathcal{O}$, which is generated by sets of the form Φx, Φ open in Γ, $x \in X$. This topology is studied in [5]. In what follows we shall use the facts that in case $N=M(X)$, $\mathcal{O}$ coincides with the original topology of X on each orbit Γx, and that for $f \in L_\Gamma$ and $k \in L_X^\infty$, the function $x \to \int_\Gamma f(\sigma)k(\sigma^{-1}x)\,d\sigma$ is $\mathcal{O}$-continuous.

6.6. Lemma. *In the topology $\mathcal{O}$, $A_{\Omega,Y}$ is closed.*

Proof. Take $x \in X$. Then $x \in A_{\Omega,Y}$ if and only if $\int_\Gamma f(\sigma)\xi_{X\setminus Y}(\sigma x)\,d\sigma=0$ for every $f \in L_\Omega$. Now

$$\int_\Gamma f(\sigma)\xi_{X\setminus Y}(\sigma x)\,d\sigma = \int_\Gamma f(\sigma^{-1})\Delta(\sigma^{-1})\xi_{X\setminus Y}(\sigma^{-1}x)\,d\sigma$$
$$= \int_\Gamma f'(\sigma)\xi_{X\setminus Y}(\sigma^{-1}x)\,d\sigma$$

is $\mathcal{O}$-continuous (see Lemma 4.9 of [5]); thus A is $\mathcal{O}$-closed.

6.7. Theorem. *Let $\Omega \subseteq \Gamma$, $Y \subseteq X$ be measurable and such that L_Ω is a subalgebra of L_Γ, and $L_\Omega * L_Y \subseteq L_Y$. Assume $N=M(X)$. If $1 \in d\Omega$, then $\mathfrak{R}_\Omega(L_Y)=\{\mu \in M(X) : \mu$ is concentrated on $A_{\Omega,Y}\}$.*

Proof. Let $\mu \in \mathfrak{R}_\Omega(L_Y)$. We note that Γx is a Borel set (Theorem 5.10 of [4]) for each $x \in X$. Since μ is bounded, μ is concentrated on a sigma-compact set. Inasmuch as any compact set can intersect only countably many orbits (see Lemma 4.6 of [5]), there exists a sequence $a_1, a_2, \ldots$ in X such that μ is concentrated on $\bigcup_n \Gamma a_n$. For each n define μ_n by $d\mu_n=\xi_{\Gamma a_n}\,d\mu$ and put $Y_n=Y \cap \Gamma a_n$. Then $L_\Omega * L_{Y_n} \subseteq L_{Y_n}$, $\mu=\sum \mu_n$ and $L_\Omega * \mu_n \subseteq L_{Y_n}$ for each n. It suffices to prove that each μ_n is concentrated on A_{Ω,Y_n}. In other words, we may assume the existence of an $a \in X$ such that μ is concentrated on Γa and $Y \subset \Gamma a$. Since $1 \in d\Omega$, by Theorem 5.6 of [5] $L_\Omega * L_Y=L_Y$. Let $\Omega_0=\Omega \cap d\Omega$, $T=\{x \in X :$ there exist compact sets $\Phi \subseteq \Omega$ and $D \subseteq Y$ such that $\int_\Gamma \xi_\Phi(\sigma)\xi_D(\sigma^{-1}x)\,d\sigma>0\}$. Clearly $T \subseteq \Gamma a$. According to the proof of Theorem 5.5 of [5], we have $\Omega_0=\Omega$ l.a.e., $T=Y$ l.a.e. and $\Omega_0 T=T$. Then $T \subseteq A_{\Omega_0,T}=A_{\Omega,T}=A_{\Omega,Y}$.

Since $1 \in d\Omega$, L_Ω contains an approximate identity $(u_i)_{i \in I}$ of L_Γ. If $k \in C_X$ and $k=0$ on $\bar{T}$, then because $u_i * \mu \in L_\Omega * \mu \subseteq L_Y=L_T$,

$$\mu(k) = \lim_i \mu(u_i' * k) = \lim_i (u_i * \mu)(k) = 0.$$

This means that supp $\mu \subseteq \bar{T}$ so that μ is concentrated on $\bar{T} \cap \Gamma a$. Now as we remarked in the preceding lemma, $A_{\Omega,Y}$ is $\mathcal{O}$-closed. Because the original topology and the $\mathcal{O}$-topology coincide on Γa this means that $A_{\Omega,Y} \cap \Gamma a$ is relatively closed in Γa. Since $T \subseteq A_{\Omega,Y} \cap \Gamma a$ we obtain $\bar{T} \cap \Gamma a \subseteq A_{\Omega,Y} \cap \Gamma a \subseteq A_{\Omega,Y}$. Thus μ is concentrated on $A_{\Omega,Y}$.

6.8. COROLLARY. *Let Ω, Y be as in the preceding theorem. Assume that for every $f \in L_\Gamma$ and every $k \in L_X^\infty$, $\int_\Gamma f(\sigma)k(\sigma^{-1}x)\,d\sigma$ depends continuously on $x \in X$. Now if $1 \in d\Omega$, then $\mathfrak{R}_\Omega(L_Y) = \{\mu \in M(X) : \operatorname{supp} \mu \subseteq A_{\Omega,Y}\}$.*

Proof. Then the orbit topology and the original topology are the same (Lemma 4.9 of [5]), and $A_{\Omega,Y}$ is thus closed in X. Furthermore, the assumption implies that $N = M(X)$ (Theorem 3.3 of [5]). Thus we can use the preceding theorem.

We have one comment: $\mu \in M(X)$ may be concentrated on $A_{\Omega,Y}$ without being supported on $A_{\Omega,Y}$. Let $\Gamma = \boldsymbol{R}$ be the additive group of the reals, $\Omega = (0, \infty)$, $X = \boldsymbol{R} \cup \{\infty\}$ the one-point compactification of $\boldsymbol{R}$, with usual action of Γ on X, and $m_X(\{\infty\}) = 1$, $m_X|\boldsymbol{R} =$ Lebesgue measure, and $Y = (0, \infty)$. Then $A_{\Omega,Y} = [0, \infty)$. Let $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_n$ where δ_n is point mass at n. Then μ is concentrated on $A_{\Omega,Y}$ but not supported on $A_{\Omega,Y}$.

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