GROUP ALGEBRA MODULES. IV

BY

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Abstract. Let \( \Gamma \) be a locally compact group, \( \Omega \) a measurable subset of \( \Gamma \), and let \( L_\Omega \) denote the subspace of \( L^1(\Gamma) \) consisting of all functions vanishing off \( \Omega \). Assume that \( L_\Omega \) is a subalgebra of \( L^1(\Gamma) \). We discuss the collection \( \mathcal{R}_\Omega(K) \) of all module homomorphisms from \( L_\Omega \) into an arbitrary Banach space \( K \) which is simultaneously a left \( L^1(\Gamma) \) module. We prove that \( \mathcal{R}_\Omega(K) = \mathcal{R}_\Omega(K_0) \oplus \mathcal{R}_\Omega(K_{abs}) \), where \( K_0 \) is the collection of all \( k \in K \) such that \( fk = 0 \), for all \( f \in L^1(\Gamma) \), and where \( K_{abs} \) consists of all elements of \( K \) which can be factored with respect to the module composition. We prove that \( \mathcal{R}_\Omega(K_0) \) is the collection of linear continuous maps from \( L_\Omega \) to \( K_0 \) which are zero on a certain measurable subset of \( \Omega \). We reduce the determination of \( \mathcal{R}_\Omega(K_{abs}) \) to the determination of \( \mathcal{R}_\Omega(K_0) \). Denoting the topological conjugate space of \( K \) by \( K^* \), we prove that \( (K_{abs})^* \) is isometrically isomorphic to \( \mathcal{R}_\Omega(K^*) \). Finally, we discuss module homomorphisms \( R \) from \( L_\Omega \) into \( L^1(\Omega) \) such that for each \( f \in L_\Omega \), \( RF \) vanishes off \( \Omega \).

1. Introduction. Once again we come back to the question of module homomorphisms which began our investigation of group algebra modules in the first place ([3] and [4]). The present paper descends from both these papers. If \( \Gamma \) is a locally compact group, \( L^1(\Gamma) \) the Banach space of integrable functions on \( \Gamma \), and if \( K \) is a left \( L^1(\Gamma) \) module, we studied in [3] the collection of module homomorphisms from \( L^1(\Gamma) \) into \( K \), from a rather abstract vantage point. On the other hand, if \( \Gamma \) acts on a locally compact space \( X \) as a transformation group, \( m_\chi \) is a positive Radon measure on \( X \) quasi-invariant with respect to \( \Gamma \), and if \( L^1(X) \) is the Banach space of integrable functions on \( X \), we showed in [4] that \( L^1(X) \) can be made into a left \( L^1(\Gamma) \) module, and then we examined the module homomorphisms from \( L^1(\Gamma) \) into \( L^1(X) \).

In the present paper we let \( \Omega \) be a measurable subset of \( \Gamma \), and let \( L_\Omega \) denote the subspace of \( L^1(\Gamma) \) consisting of all functions vanishing off \( \Omega \). We assume that \( L_\Omega \) is a subalgebra of \( L^1(\Gamma) \). Then we discuss the module homomorphisms from \( L_\Omega \) into an arbitrary left \( L^1(\Gamma) \) module \( K \). The collection of such homomorphisms we call \( \mathcal{R}_\Omega(K) \). The fact that \( L_\Omega \) need not have an approximate identity makes the
problem much more difficult. The fact that \( L_{\Omega} \) has shifted approximate identities makes the problem solvable, via several reductions.

In §3 we prove that \( \mathfrak{R}_{\Omega}(K) = \mathfrak{R}_{\Omega}(K_0) + \mathfrak{R}_{\Omega}(K_{\text{abs}}) \), where \( K_0 \) consists of all elements of \( K \) which when composed with elements of \( L^1(\Gamma) \) yield the zero element in \( K \), and where \( K_{\text{abs}} \) consists of all elements of \( K \) which can be factored into the composition of an \( L^1(\Gamma) \) element and some element of \( K \). Thus the homomorphism problem splits into two parts. Via Theorem 3.5, \( \mathfrak{R}_{\Omega}(K_0) \) is the collection of linear, continuous maps from \( L_{\Omega} \) to \( K_0 \) which are zero on \( L_T \) where \( T \) is a certain measurable subset of \( G \); this set \( T \) also turns up in the previous paper [5], it involves the composition operator in a direct way.

Next in §4, we look at \( \mathfrak{R}_{\Omega}^{\infty}(K_{\text{abs}}) \). Let \( d\Omega = \{ \sigma \in \Gamma : \text{for every measurable neighborhood } \Phi \text{ of } \sigma, \Phi \cap \Omega \text{ has positive measure} \} \). We show that \( d\Omega \) splits up into a collection \( \mathcal{J}(d\Omega) \) of pairwise disjoint subsets. Then each \( R \in \mathfrak{R}_{\Omega}(K_{\text{abs}}) \) corresponds to a collection \( (R_j)_{j \in \mathcal{J}(d\Omega)} \) where \( R_j \in \mathfrak{R}_{\Gamma}(K_{\text{abs}}) \) and such that \( R_j \) and \( R \) are identical on \( \Omega \cap J \). Furthermore, for each \( J \), the homomorphism \( R_j \) is uniquely defined. Conversely, any collection \( (R_j)_{j \in \mathcal{J}(d\Omega)} \) which is norm-bounded in \( \mathfrak{R}_{\Omega}(K) \) gives rise to a (unique) \( R \in \mathfrak{R}_{\Omega}(K) \). Thus the problem of finding \( \mathfrak{R}_{\Omega}(K_{\text{abs}}) \) has reduced to that of finding \( \mathfrak{R}_{\Omega}(K_{\text{abs}}) \).

In §5 we assume that \( \Omega = \Gamma \). We embed \( K \) into \( \mathfrak{R}_{\Gamma}(K) \) by the map \( T_K \) which sends \( k \in K \) into right module multiplication by \( k \). Then \( K_{\text{abs}} \) is injected isometrically onto \( [\mathfrak{R}_{\Gamma}(K)]_{\text{abs}} \). Denoting the topological conjugate space of \( K \) by \( K^* \), we prove that \((K_{\text{abs}})^* \) is isometrically isomorphic as a module to \( \mathfrak{R}_{\Gamma}(K^*) \). This paves the way for a collection of examples.

Let \( Y \subseteq X \) be measurable and let \( \mathcal{J}(d\Omega) \) consist of only one element. We close the paper with a study of the module homomorphisms \( R \) from \( L_{\Omega} \) to \( L^1(X) \) such that for each \( f \in L_{\Omega} \), \( Rf \) vanishes off \( Y \).

2. Notations. The notations we use are mainly those given in §2 of [5]. Most of the definitions and comments below have already appeared in our preceding texts, but we desire to have them stated here explicitly for reference.

Let \( \Gamma \) be a locally compact group with identity 1 and left Haar measure \( m \). For \( f \in L^1(\Gamma) \) and \( \sigma \in \Gamma \) we have \( L^1 \)-functions \( f_\sigma, f^\sigma, f' \) defined by

\[
\begin{align*}
  f_\sigma(\tau) &= f(\sigma \tau), \\
  f^\sigma(\tau) &= f(\tau \sigma) \Delta(\sigma), \\
  f'(\tau) &= f(\tau^{-1}) \Delta(\tau^{-1}),
\end{align*}
\]

These functions are connected with the convolution in \( L^1(\Gamma) \) by the formulas \( f_\sigma \ast g = (f \ast g)_\sigma, \quad f^\sigma \ast g = f \ast g^\sigma, \quad f \ast g^\sigma = (f \ast g)^\sigma, \) and \((f \ast g) = g' \ast f' \) (\( f, g \in L^1(\Gamma) \)).

Let \( \Omega \subseteq \Gamma \) be measurable. We put \( L_{\Omega} = \{ f \in L^1(\Gamma) : f = 0 \text{ a.e. outside } \Omega \} \) and \( d\Omega = \{ \sigma \in \Gamma : \text{for every measurable neighborhood } \Phi \text{ of } \sigma, m(\Phi \cap \Omega) \neq 0 \} \). Then \( \Omega \subseteq d\Omega \) i.e. (= locally almost everywhere), i.e. \( d\Omega \cap \Omega \) is locally null. For every \( \sigma \in d\Omega \), \( L_{\Gamma} \) contains approximate identities \( (u_\ell)_{\ell \in \mathbb{N}} \) and \( (v_\ell)_{\ell \in \mathbb{N}} \) such that \( (u_\ell)^{\ell-1} \in L_{\Omega} \) for each \( i \) and \( (v_\ell)_\ell^{-1} \in L_{\Omega} \) for each \( j \) [5, Lemma 3.1].
For measurable $\Omega$, $\Phi \subseteq \Gamma$, we let $L_\Omega \ast L_\Phi$ be the closed linear hull of \{f \ast g : f \in L_\Omega, g \in L_\Phi\}. Then

\[
L_\Omega \ast L_\Phi = \text{Cl} \left( \sum_{\sigma \in \Omega} L_{\sigma \Phi} \right) = \text{Cl} \left( \sum_{\tau \in \Phi} L_{\Omega \tau} \right).
\]

(The first equation is [5, Corollary 3.4]; the second is proved in a similar way as the first.) If $\Phi$ is not locally null, then $d\Phi \neq \emptyset$; taking $\Omega = \Gamma$ we obtain from (2.1) that

\[
\text{Cl} \left( \sum_{\sigma \in \Omega} L_{\sigma \Phi} \right) = L_\Gamma.
\]

In case $L_\Omega$ is a subalgebra of $L_\Gamma$ we obtain from (2.1) the inclusions $\Omega \sigma \subseteq \Omega$ l.a.e. and $\sigma \Omega \subseteq \Omega$ l.a.e. for all $\sigma \in d\Omega$. Thus, if $f \in L_\Omega$ and $\sigma \in d\Omega$, then $f^{\sigma^{-1}} \in L_\Omega$ and $f^{\sigma^{-1}} \in L_\Omega$. By [5, Corollary 3.5(iii)] $d\Omega$ is a closed semigroup of $\Gamma$.

We use the symbol $\xi_\phi$ to denote the characteristic function of $\Phi$.

If $A$ is a Banach algebra, an $A$-module is a (left) module $K$ over $A$ which is also a Banach space, and such that $\|f \ast k\| \leq \|f\| \|k\|$ for all $f \in A, k \in K$ ($\ast$ denoting the module composition). In particular we shall consider $L_\Gamma$ modules. An element $k$ of an $L_\Gamma$ module $K$ is called factorable if there exist $f \in L_\Gamma$ and $k' \in K$ such that $k = f \ast k'$. The factorable elements of $K$ form a closed submodule $K_{abs}$ of $K$ [5]. $K$ is said to be absolutely continuous if $K_{abs} = K$. For instance, $L_\Gamma$ is an absolutely continuous $L_\Gamma$ module. It follows that $(K_{abs})_{abs} = K_{abs}$ for every $K$. It is clear that $\lim_{i} u_i \ast k = k$ for every $k \in K_{abs}$ and every approximate identity $(u_i)_{i \in I}$ in $L_\Gamma$. For $\sigma \in \Gamma$ we define a norm-preserving left shift $\ast \sigma$ in $K_{abs}$ by

\[
(f \ast k')_\sigma = f_\sigma \ast k', \quad f \in L_\Gamma, k' \in K
\]

(see [5]). Then

\[
f \ast k_\sigma = f^\sigma \ast k, \quad f \in L_\Gamma, k \in K_{abs}, \sigma \in \Gamma.
\]

For every $k \in K_{abs}$, $k_\sigma$ depends continuously on $\sigma$. For all $f \in L_\Gamma, k \in K_{abs}$ and $k^* \in K^*$, we have by [5]

\[
k^*(f \ast k) = \int_\Gamma f(\sigma)k^*(k_{\sigma^{-1}}) \, d\sigma.
\]

3. Reduction to order-free modules. Let $K$ be an $L_\Gamma$ module. We call $K$ order-free if there is no $k \in K$ such that $k \neq 0$ and such that $f \ast k = 0$ for every $f \in L_\Gamma$. Every absolutely continuous module is order-free, because if $K$ is such a module, then for all $k \in K, k \in \text{closure} \{f \ast k : f \in L_\Gamma\}$.

In general, for an $L_\Gamma$ module $K$ we call $\{k \in K : f \ast k = 0 \text{ for every } f \in L_\Gamma\}$ the order submodule $K_0$ of $K$. Note that $k \in K$ lies already in $K_0$ if there is a measurable $\Phi \subseteq \Gamma$, not locally null, such that $L_\Phi \ast k = \{0\}$. (Then for every $\sigma \in \Gamma$, $\{0\} = (L_\Phi \ast k)_\sigma^{-1} = (L_\Phi)_{\sigma^{-1}} \ast k = (L_\Phi) \ast k$, so that, by (2.2), $L_\Gamma \ast k = \text{Cl} \left( \sum_{\sigma \in \Gamma} L_{\sigma \Phi} \right) \ast k = \{0\}$.) The
Banach space $K/K_0$ is made into an $L_\tau$ module by the definition

$$f * (k + K_0) = (f * k) + K_0 \quad (f \in L_\tau, k \in K).$$

This $K/K_0$ is always order-free.

Most modules we shall deal with are order-free, e.g., $C_\infty(X)$, $M(X)$ and $L^p(X)$ ($1 \leq p \leq \infty$). However, modules with order sometimes arise in a natural way. Thus, if $K$ is an $L_\tau$ module, we may define a module composition on $K^*$ by the following definition.

3.1. Definition. $(f * k^*)(k) = k^*(f * k)\,$ (fe$L_\tau$, k e K, k* e K*).

$K^*$ is then an $L_\tau$ module. It is not hard to prove that $K^*$ is order-free if and only if $K$ is absolutely continuous. In fact, there is a natural isometrical module homomorphism of $(K_{abs})^*$ onto $K^*/(K^*_0)$.

Let $\Omega$ be a measurable subset of $\Gamma$ that is not locally null, and assume that $L_\Omega$ is a subalgebra of $L_\tau$. Let $K$ be an $L_\tau$ module.

3.2. Definition. A continuous linear map $R : L_\Omega \to K$ is an $(L_\Omega, K)$-homomorphism if $R(f * g) = f * Rg$ ($f, g \in L_\Omega$). The collection of $(L_\Omega, K)$-homomorphisms we denote by $\mathcal{H}_\Omega(K)$. When $\Omega = \Gamma$ we suppress the $\Omega$ and write $\mathcal{H}(K)$.

3.3. Theorem. For any $L_\tau$ module $K$, $\mathcal{H}_\Omega(K)$ is the direct sum of $\mathcal{H}_\Omega(K_{abs})$ and $\mathcal{H}_\Omega(K_0)$. In particular, in case $K$ is order-free, then $R(f) \in K_{abs}$ for all $R \in \mathcal{H}_\Omega(K)$ and $f \in L_\Omega$.

Proof. If $k \in K_{abs}$, then $\lim_i u_i * k = k$ for every approximate identity $(u_i)_{i \in I}$ in $L_\tau$; therefore $k \notin K_0$ if $k \neq 0$. Then by the definitions of $\mathcal{H}_\Omega(K_{abs})$ and $\mathcal{H}_\Omega(K_0)$, their intersection is $\{0\}$. Thus we need only show that $\mathcal{H}_\Omega(K) = \mathcal{H}_\Omega(K_{abs}) + \mathcal{H}_\Omega(K_0)$.

Let $R \in \mathcal{H}_\Omega(K)$ be given. Take $\sigma \in d\Omega$. As we mentioned in §2 there exists an approximate identity $(u_i)_{i \in I}$ in $L_\tau$ such that for each $i \in I$, $(u_i)^{\sigma^{-1}} \in L_\Omega$. Then $f \in L_\Omega$ implies that $f^{\sigma^{-1}} \in L_\Omega$ and

$$R(f^{\sigma^{-1}}) = \lim_i R(u_i * f^{\sigma^{-1}}) = \lim_i R((u_i)^{\sigma^{-1}} * f)$$

$$= \lim_i [(u_i)^{\sigma^{-1}} * Rf].$$

Since $(u_i)^{\sigma^{-1}} * Rf \in K_{abs}$ and $K_{abs}$ is closed in $K$, it follows that $R(f^{\sigma^{-1}}) \in K_{abs}$. We define $R_\sigma : L_\Omega \to K$ by $R_\sigma(f) = [R(f^{\sigma^{-1}})]_\sigma$. Then $R_\sigma$ is a continuous linear map of $L_\Omega$ into $K_{abs}$. Take $g \in L_\Omega$. For $f \in L_\Omega$ we have $f^\sigma \in L_\Omega$, so that $f^\sigma * Rg = f^\sigma * R(g^{\sigma^{-1}}) = R(f^\sigma * g^{\sigma^{-1}}) = R(f * g^{\sigma^{-1}}) = f * Rg$. Thus $L_\Omega * (Rg - R_\sigma g) = \{0\}$. It follows that $Rg - R_\sigma g \in K_0$, and we conclude that $R - R_\sigma$ is a continuous linear map $L_\Omega \to K_0$. Moreover, for all $f, g \in L_\Omega$ we obtain

$$f * R_\sigma g = f * Rg = [(f * Rg)^{\sigma^{-1}}]_\sigma = (f^{\sigma^{-1}} * Rg)_\sigma$$

$$= [R(f^{\sigma^{-1}} * g)]_\sigma = [R((f * g)^{\sigma^{-1}})]_\sigma$$

$$= R_\sigma(f * g).$$
Thus $R_a$ is a module homomorphism, and consequently, so is $R - R_a$. We obtain $R_a \in \mathcal{R}_{\Omega}(K_{abs})$, $R - R_a \in \mathcal{R}_{\Omega}(K_0)$.

The elements of $\mathcal{R}_{\Omega}(K_{abs})$ can be characterized in terms of the shift in $K_{abs}$.

3.4. THEOREM. A continuous linear map $R : L_\Omega \to K_{abs}$ is in $\mathcal{R}_{\Omega}(K_{abs})$ if and only if $R(f_\sigma^{-1}) = (Rf)_\sigma^{-1}$ for all $f \in L_\Omega$ and $\sigma \in d\Omega$.

Proof. Take $R \in \mathcal{R}_{\Omega}(K_{abs})$, $\sigma \in d\Omega$. By the proof of the preceding theorem, $Rf - [R(f_\sigma^{-1})]_\sigma \in K_0$ for every $f \in L_\Omega$. But of course $Rf - [R(f_\sigma^{-1})]_\sigma \in K_{abs}$. Therefore $Rf = [R(f_\sigma^{-1})]_\sigma$. In other words, $(Rf)_\sigma^{-1} = R(f_\sigma^{-1})$.

Conversely, let $R : L_\Omega \to K_{abs}$ be a linear continuous map such that $R(f_\sigma^{-1}) = (Rf)_\sigma^{-1}$ for all $f \in L_\Omega$ and $\sigma \in d\Omega$. For any $f, g \in L_\Omega$ and any $k^* \in K^*$ we have by (2.5) that

$$k^*(f* Rg) = \int f(\sigma)k^*([R(f_\sigma^{-1})]_\sigma) \, d\sigma$$
$$= \int f(\sigma)k^*([R^*(k^*)](g_\sigma^{-1})) \, d\sigma$$
$$= [R^*(k^*)](f*g) = k^*(R(f*g)),$$

so that $f * Rg = R(f*g)$. Thus $R \in \mathcal{R}_{\Omega}(K_{abs})$.

In order to describe $\mathcal{R}_{\Omega}(K_0)$ we need a little more knowledge about the algebra $L_\Omega$. There exists an open set $T \subset \Gamma$ such that $LT$ is the closed linear span of $\{f^* g : f, g \in L_\Omega\}$. The proof of this statement and the explicit definition of $T$ are in [5]. (Note that $T = \emptyset$ because $X = \Gamma$.) In terms of this set $T$ we have a simple characterization of $(L_\Omega, K_0)$-homomorphisms.

3.5. THEOREM. A continuous linear map $R : L_\Omega \to K_0$ is in $\mathcal{R}_{\Omega}(K_0)$ if and only if $R = 0$ on $L_T$.

Proof. If $R = 0$ on $L_T$, then $R(f*g) = 0$ for all $f, g \in L_\Omega$. On the other hand, $L_\Omega * K_0 = \{0\}$ and $f * Rg \in L_\Omega * K_0$. Thus $f * Rg = 0$ and $R \in \mathcal{R}_{\Omega}(K_0)$. Conversely, for $R \in \mathcal{R}_{\Omega}(K_0)$ and $f, g \in L_\Omega$ we have $R(f*g) = f * Rg \in L_\Omega * K_0 = \{0\}$. Since $R$ is linear and continuous, $R(h) = 0$ for all $h \in L_T$.

We mention that $\mathcal{R}_{\Omega}(K_0) = \{0\}$ if $L_T = L_\Omega$. In particular, $L_T = L_\Omega$ if $1 \in d\Omega$, because then $L_\Omega$ contains an approximate identity of $L_T$. To wit, if $\Omega = \Gamma$ we have $\mathcal{R}_{\Omega}(K_0) = \{0\}$.

4. A decomposition theorem for module homomorphisms. Let $\Omega$ be a semigroup. An equivalence relation $\sim$ in $\Omega$ is called an "ideal equivalence relation" if $\sigma \tau \sim \tau$ for all $\sigma, \tau \in \Omega$. Let us define the equivalence relation $\approx$ in $\Omega$ by $\sigma \approx \tau$ if and only if $\sigma \sim \tau$ for every ideal equivalence relation $\sim$ in $\Omega$. Then $\approx$ is itself an ideal equivalence relation. Among all ideal equivalence relations $\approx$ is the finest, has the smallest equivalence classes. Explicitly, $\sigma \approx \tau$ if and only if there exists a finite sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n = \tau$ in $\Omega$ such that $(\Omega\sigma_i) \cap (\Omega\sigma_{i+1}) \neq \emptyset$ for each $i$. 


The equivalence classes modulo \( \equiv \) form a set \( \mathcal{J}(\Omega) \). If \( \Omega \) is abelian or contains a right unit, then \( \mathcal{J}(\Omega) \) consists of only one element; indeed, if \( \Omega \) is abelian, then \( \sigma \equiv \tau \equiv \sigma \tau \equiv \tau \) for all \( \sigma, \tau \in \Omega \), and if \( \Omega \) has a right unit \( \eta \), then \( \sigma = \sigma \eta \equiv \eta \) for all \( \sigma \in \Omega \). If \( \sigma \in J \in \mathcal{J}(\Omega) \) and if \( \tau \in \Omega \), then \( (\tau \sigma)^{-1} \sigma = \tau \sigma \), so \( \Omega J \subseteq J \), which means that \( J \) is a left ideal of \( \Omega \).

4.1. Lemma. Let \( \Omega \) be a measurable subsemigroup of \( \Gamma \), with \( \Omega \) not locally null. Then there exists a neighborhood \( \Phi \) of \( 1 \in \Gamma \) such that \( \Omega \cap \Phi J \subseteq J \) for every \( J \in \mathcal{J}(\Omega) \). In particular, the ideal equivalence classes are relatively open in \( \Omega \), and \( f = \sum \{ f\xi_j : J \in \mathcal{J}(\Omega) \} \) for all \( f \in L_\alpha \). Furthermore, \( \sigma^{-1}J \cap \Omega \subseteq J \) for all \( \sigma \in \Omega \) and \( J \in \mathcal{J}(\Omega) \).

Proof. Inasmuch as \( \Omega \Omega \) contains a nonempty open subset of \( \Gamma \) by [6, 20.17], there is a \( \beta \in \Omega \) and a neighborhood \( \Phi \) of \( 1 \in \Gamma \) such that \( \beta \Phi \subseteq \Omega \). Let \( \sigma \in \Omega \). For all \( \tau \in \Omega \cap \Phi \sigma \) we have \( \beta \tau \sigma^{-1} \in \beta \Phi \subseteq \Omega \) and consequently \( \sigma \equiv (\beta \tau \sigma^{-1}) \sigma = \beta \tau \equiv \tau \). Thus, if \( \sigma \in J \in \mathcal{J}(\Omega) \), then \( \Omega \cap \Phi \sigma \subseteq J \), so that \( \Omega \cap \Phi J \subseteq J \). To prove the last statement, we notice that if \( \tau \in J \), \( \beta \in \Omega \) and \( \sigma \beta \in \Omega \), then \( \tau \equiv \sigma \beta \equiv \beta \); therefore \( \sigma^{-1}J \cap \Omega \subseteq J \), while the converse inclusion is obvious.

4.2. Corollary. Let \( \Omega \) be as above. If \( \Omega \) is connected, then \( \mathcal{J}(\Omega) \) contains only one element.

Let \( \Omega \) be a measurable subset of \( \Gamma \). It is known that if \( L_\alpha \) is an algebra, then \( d\Omega \) is a closed subsemigroup of \( \Gamma \). Furthermore, for any \( \sigma \in d\Omega \), we have \( \sigma \Omega \subseteq \Omega \) l.a.e. and \( \Omega \sigma \subseteq \Omega \) l.a.e. (see 
§2). Inasmuch as every \( J \in \mathcal{J}(d\Omega) \) is a subsemigroup, \( L_J \) and \( L_{\alpha,J} \) are nontrivial algebras, the latter because \( J \) is a nonempty relatively open subset of \( d\Omega \).

We have sufficient machinery to decompose \((L_\alpha, K)\)-homomorphisms.

4.3. Theorem. Let \( \Omega \) be a measurable subset of \( \Gamma \) that is not locally null. Assume \( L_\alpha \) is an algebra, and \( K \) an \( L_\Gamma \) module which is order-free. For every \( R \in \mathfrak{R}_\alpha(K) \) there is a family \( \{ R_j : J \in \mathcal{J}(d\Omega) \} \) of elements of \( \mathfrak{R}(K) \), such that

\[
(*) \quad R(f) = \sum_{J \in \mathcal{J}(d\Omega)} R(f\xi_j), \quad f \in L_\alpha.
\]

Furthermore, \( \| R \| = \sup \{ \| R_j \| : J \in \mathcal{J}(d\Omega) \} \).

Conversely, for every norm-bounded family \( \{ R_j : J \in \mathcal{J}(d\Omega) \} \) in \( \mathfrak{R}(K) \) the equation \((*)\) defines an \( R \in \mathfrak{R}_\alpha(K) \).

Proof. Let \( R \in \mathfrak{R}_\alpha(K) \) be given to us. Let \( \sigma \in d\Omega \), and let \( (u_i)_{a,i} \) be an approximate identity in \( L_\Gamma \) with norm \( 1 \) and such that \( (u_i)_{a,-1} \in L_\alpha \) for every \( i \). Next let \( \sigma \in \Gamma \) and \( f \in L_{\alpha,J} \). Then \( f \sigma \in L_\alpha \) and hence \( (f \sigma)^{-1} \in L_{\alpha,J} \). Thus

\[
\{ R[(f \sigma)^{-1}]\sigma^{-1} \}
= \lim_i \{ R[(f \sigma)^{-1} * u_i] \}\sigma^{-1} 
= \lim_i \{ R[f \sigma * (u_i)_{a,-1}] \}\sigma^{-1} 
= \lim_i \{ f \sigma * R[(u_i)_{a,-1}] \}\sigma^{-1} = \lim f * R[(u_i)_{a,-1}].
\]
In particular, the latter limit exists for any \( \sigma \in \Gamma \) and \( f \in L_{\alpha} \). Since the approximate identity is bounded, so is \( \{R((u_i)a_i^{-1}) : i \in I\} \), which means that if \( D \) is defined as \( \{f \in L_\tau : \lim_i [f * R(u_i)a_i^{-1}] \text{ exists}\} \), then \( D \) must be a closed linear subspace of \( L_\tau \), and by the calculation above, \( D \supseteq \bigcup_{\sigma \in \Gamma} L_{\alpha \sigma} \). Thus \( D \supseteq \text{Cl} (\bigcup_{\sigma \in \Gamma} L_{\alpha \sigma}) = L_\tau \) (see 2.2).

Hence we can define \( R_a : L_\tau \to K \) by

\[
R_a(f) = \lim_i f^* \ast R((u_i)a_i^{-1}), \quad f \in L_\tau.
\]

Since \( f, g \in L_\tau \) means that \( (f \ast g)^a = f^a \ast g^a \), evidently \( R_a \in \mathcal{M}(K) \). Recapitulating, we have taken an \( R \in \mathcal{M}_g(K) \), so that \( R \) is defined only on \( L_\alpha \), and from it we have defined a module homomorphism \( R_a \) on the whole of \( L_\tau \). The particular \( R_a \) we obtain depends (or at least appears to depend) upon the \( a \in d_\omega \) picked at the beginning of the proof. In any case, we next show that \( R_a \) is an extension of \( R \) restricted to \( \Omega \cap J \), where \( a \in J \).

It is easy to show that \( R_a = R \) on \( L_{\alpha a} \). After all, if \( f \in L_{\alpha a} \) then \( f^a \in L_\alpha \) and \( R_a(f) = \lim_i R(f^* \ast (u_i)a_i^{-1}) = \lim_i R(f^i) = R(f) \). Now we determine an ideal equivalence relation on \( d_\Omega \). Write \( \beta \sim_\sigma \alpha \) if \( R_\beta = R_\sigma \). Let \( \beta, \sigma \in d_\Omega \). We must show that \( R_{\sigma \beta} = R_\beta \). On \( L_{\sigma \beta} \), \( R_\beta = R \) while on \( L_{\sigma \beta} \), \( R_{\sigma \beta} = R \). Since \( \Omega_\sigma \subseteq \Omega \) i.e., we have \( L_{\sigma \beta} \subseteq L_{\sigma \alpha} \) and \( R_{\sigma \beta} = R = R_\beta \) on \( L_{\sigma \alpha} \). But \( R_{\sigma \beta} \) and \( R_\beta \) are module homomorphisms on \( L_\tau \), so by Theorem 3.4 they are left translation invariant by any element of \( \Gamma \). This means that they agree not only on \( L_{\sigma \alpha} \) but on \( L_{\alpha \sigma} \) for any \( \tau \in \Gamma \). Hence \( R_{\sigma \beta} = R_\beta \) on \( \text{Cl} (\bigcup_{\sigma \in \Gamma} L_{\alpha \sigma}) = L_\tau \) (see 2.2). This proves that \( R_{\sigma \beta} = R_\beta \) and \( \sim \) is an ideal equivalence relation. Next, if \( \alpha \in d_\Omega \), then there is a \( J \in \mathcal{J}(d_\Omega) \) such that \( \alpha \in J \). For any \( \beta \in J \), \( R_\alpha = R_\beta \), so that we may define \( R_J \) as \( R_\alpha \) and take away the apparent dependence on the particular \( \alpha \in J \). Then \( R_J = R \) on \( \text{Cl} (\bigcup_{\sigma \in J} L_{\alpha \sigma}) \). We note that \( J \) is a closed subset of \( d_\Omega \), since \( d_\Omega \cap J \) is relatively open in \( d_\Omega \) by Lemma 4.1 (where \( \Omega \) is replaced by \( d_\Omega \)). Because \( d_\Omega \) is closed in \( \Gamma \), we know that \( J \) is also closed in \( \Gamma \), so \( J \supseteq dJ \). Take \( \beta \in d_\Omega \). Then \( R_J = R \) on

\[
\text{Cl} \left( \bigcup_{\sigma \in J} L_{\alpha \sigma} \right) \supseteq \text{Cl} \left( \bigcup_{\sigma \in J} L_{\alpha \sigma} \right) = \sum_{\sigma \in J} L_{\alpha \sigma} \supseteq L_{\beta (\alpha \cap J)} \quad \text{(by 2.1)},
\]

which is just perfect for us because if \( f \in L_{\alpha \cap J} \) then \( f_a^{\alpha} \in L_{\beta (\alpha \cap J)} \) and consequently \( R_J(f) = [R_J(f_a)]_\beta = [R(f_a)]_\beta = R(f) \) by the translation invariance of \( R \). Thus \( R = R_J \) on \( L_{\alpha \cap J} \). We have thus shown that \( R \) yields the module homomorphism \( R_J \) defined on all of \( L_\tau \) in such a way that \( R \) and \( R_J \) agree on \( \Omega \cap J \). From Lemma 4.1 we infer that for any \( f \in L_\tau \), \( f = \sum_j \{f_j^a\} \), with the result that \( R(f) = \sum_j \{R(f_j^a)\} = \sum_j \{R_j(f_j)\} \), which proves \( (*) \). As for the norm inequalities, \( \|R_a\| \leq \|R\| \) since the approximate identity is bounded by 1. Thus \( \|R_J\| \leq \|R\| \) for every \( J \in \mathcal{J}(d_\Omega) \). The inequality \( \|R\| \leq \sup \{\|R_J\| : J \in \mathcal{J}(d_\Omega)\} \) follows from \( (*) \). Hence \( \|R\| = \sup \{\|R_J\| : J \in \mathcal{J}(d_\Omega)\} \).

We have yet to prove the converse. Let \( \{R_J : J \in \mathcal{J}(d_\Omega)\} \) be a family of elements of \( \mathcal{M}(K) \) such that \( \{\|R_J\| : J \in \mathcal{J}(d_\Omega)\} \) is bounded. Then \( (*) \) defines a continuous linear map \( R : L_\Omega \to K \). Now let \( \sigma \in d_\Omega \). By the last part of Lemma 4.1,
\[ \sigma^{-1}J \cap d\Omega = J \text{ for every } J \in \mathcal{J}(d\Omega). \text{ This means that for all } f \in L_\Omega, \]
\[ f_{\sigma^{-1}}\xi_j = (f\xi_{e^{-1}})_{\sigma^{-1}} = (f\xi_{e\Omega e^{-1}})_{\sigma^{-1}} = (f\xi_i)_{\sigma^{-1}}. \]

Thus
\[ R(f_{\sigma^{-1}}) = \sum_j R(f\xi_j) = \sum_j R((f\xi_i)_{\sigma^{-1}}) \]
\[ = \left( \sum_j R((f\xi_i)_{\sigma^{-1}}) \right)_{\sigma^{-1}} = (R(f))_{\sigma^{-1}}, \]
so by Theorem 3.4, \( R \) is an \((L_\Omega, K)\)-module homomorphism.

The theorem says that to any module homomorphism \( R \) defined on \( L_\Omega \) there corresponds a collection \( (R_j)_{j \in \mathcal{J}(d\Omega)} \) of module homomorphisms on \( L_\Gamma \), and each \( R_j \) is the unique extension of \( R \) restricted to \( \Omega \cap J \). Thus if we have complete knowledge of \( \mathcal{M}(K) \), then the problem of \( \mathcal{M}_\Omega(K) \) is completely solved as well. In other words, the problem of characterizing \( \mathcal{M}_\Omega(K) \) is reduced to the problem of characterizing \( \mathcal{M}(K) \). Besides what occurs in this paper, there is a discussion of \( \mathcal{M}(K) \) in [3], [8], and [9].

Sometimes Theorem 4.3 takes on a simpler form.

**4.4. Corollary.** Assume that \( \Omega \) has at least one of the following properties:

(a) \( \Omega \) is commutative.

(\( \beta \)) \( 1 \in d\Omega. \)

(\( \gamma \)) \( \Omega \) is connected.

Then every \( R \in \mathcal{M}_\Omega(K) \) has a unique extension to an \( \bar{R} \in \mathcal{M}(K) \), and \( \| \bar{R} \| = \| R \|. \)

**Proof.** \( \mathcal{J}(d\Omega) \) contains but one element.

It would be conceivable, no matter what \( \Gamma \) and \( \Omega \subseteq \Omega \) are, that \( \mathcal{J}(d\Omega) \) consisted of but one element. In fact it would be desirable, since then any module homomorphism from \( L_\Omega \) to \( K \) could be extended—uniquely—to a module homomorphism from \( L_\Gamma \) to \( K \). However, this is not the case. Let \( \Gamma \) be the free group with two generators, \( \alpha \) and \( \beta \), and let \( \Omega \) have the discrete topology. Let \( \Phi \) be the subsemigroup generated by \( 1, \alpha, \) and \( \beta \), and let \( \Omega \) be the subsemigroup \( \Phi \alpha \cup \Phi \beta \). We define the continuous, linear map \( \bar{R} : L_\Omega \to L_\Gamma \) by
\[ R(\xi_{(\sigma a)}) = \xi_{(\sigma a)}, \quad \sigma \in \Phi, \quad R(\xi_{(\sigma b)}) = \xi_{(\sigma b)}, \quad \sigma \in \Phi. \]

Then \( R \in \mathcal{M}_\Omega(L_\Gamma) \), but there is no extension \( \bar{R} \in \mathcal{M}(L_\Gamma) \) such that \( R \) and \( \bar{R} \) coincide on \( L_\Omega \).

From the definition of \( \mathcal{J}(d\Omega) \) we see that \( \Omega \cap d\Omega \) is partitioned by
\[ \{ \Omega \cap J : J \in \mathcal{J}(d\Omega) \}. \]
This means that \( L_\Omega = L_\Omega \cap d\Omega = \sum L_{\Omega \cap J} \) where the \( L_{\Omega \cap J} \) are pairwise disjoint (except for \{0\}) left ideals of \( L_\Omega \). We prove below that \( \{ L_{\Omega \cap J} : J \in \mathcal{J}(d\Omega) \} \) forms the finest decomposition of \( L_\Omega \) as the sum of disjoint left ideals of the form \( L_\sigma \).

**4.5. Theorem.** Let \( (\theta_i)_{i \in I} \) be a set of measurable subsets of \( \Omega \) such that each \( L_{\theta_i} \) is a left ideal of \( L_\Omega \) and such that \( L_\Omega = \text{Cl} \left( \sum_{i \in I} L_{\theta_i} \right) \) and \( L_{\theta_i} \cap L_{\theta_j} = \{0\} \) for \( i \neq j \). Then for each \( i \), \( L_{\theta_i} \) is the closure of \( \sum \{ L_{\Omega \cap J} : J \in \mathcal{J}(d\Omega), L_{\Omega \cap J} \subseteq L_{\theta_i} \} \).
Proof. Let $P_i: L_\alpha \to L_{\theta_i}$ be defined by $P_i(f) = f_{\xi_{\theta_i}}$. Then $P_i$ is continuous, linear, a projection from $L_\alpha$ onto $L_{\theta_i}$, and $f \in L_\alpha$ implies that $f = \sum_{i \in I} P_i(f)$. Fix $i \in I$. Let $f, g \in L_\alpha$. Since by assumption $L_{\theta_i}$ is a left ideal, and since $P_i(g) \in L_{\theta_i}$, we have $f \star [P_i(g)] \in L_{\theta_i}$, and thus

$$P_i(f \star g) - f \star [P_i(g)] = P_i(f \star g) - P_i(f \star [P_i(g)]) = P_i([f \star g - P_i(g)]).$$

But $g - [P_i(g)] \in L_{\theta_i} = \text{Cl}(\sum_{j \neq i} L_{\theta_j})$, so

$$f \star (g - [P_i(g)]) \in \text{Cl}\left(\sum_{j \neq i} f \star L_{\theta_j}\right) \subseteq \text{Cl}\left(\sum_{j \neq i} L_{\theta_j}\right)$$

since each $L_{\theta_j}$ is a left ideal in $L_\alpha$. Thus $P_i(f \star [g - P_i(g)]) = 0$. Hence $P_i \in \mathfrak{R}_\alpha(L_T)$. Hence Theorem 4.3 applies, and there exists a collection $\{R_j : J \in \mathcal{J}(d\Omega)\} \subseteq \mathfrak{R}(L_T)$ such that $P_i(f) = \sum_j R_j(f \xi_j)$, for all $f \in L_T$. By a theorem of G. Wendel (see [10]), every $R_j \in \mathfrak{R}(L_T)$ determines a $\mu_j \in M(\Gamma)$ such that $R_j(g) = g \star \mu_j$, for all $g \in L_T$. Then

$$f \xi_{\theta_i} = P_i(f) = \sum_j (f \xi_j) \star \mu_j, \quad \text{for all } f \in L_\alpha.$$

Using this decomposition we show next that for each $J \in \mathcal{J}(d\Omega)$, either $\mu_j = 0$ or $\mu_j = \delta_1$, the point mass at $1 \in 1$. To that end, let $J \in \mathcal{J}(d\Omega)$ such that $\theta_1 \cap J$ is not locally null. If $f \in L_{\theta_1 \cap J}$ then the formula displayed above yields us $f = f \star \mu_j$. Thus $L_{\theta_1 \cap J} = L_{\theta_1 \cap J} \star \mu_j$, so that $L_{\theta_1 \cap J} \star (\delta_1 - \mu_j) = 0$. Then $\delta_1 - \mu_j$ lies in the order submodule of $M(\Gamma)$. However, $M(\Gamma)$ is order-free. Consequently $\delta_1 - \mu_j = 0$, or $\delta_1 = \mu_j$. On the other hand, if $J$ is such that $J \cap \theta_1$ is locally null, then by a similar reasoning, $\mu_j = 0$. Finally, we note that $(f \xi_j) \star \delta_1 = f \xi_j$, while $(f \xi_j) \star 0 = 0$, so that $f \in L_\alpha$ implies that

$$f \xi_{\theta_1} = \sum \{f \xi_j : J \in \mathcal{J}(d\Omega), \mu_j = \delta_1\}.$$

Since every element of $L_{\theta_1}$ is of the form $f \xi_{\theta_1}$ for an appropriate $f \in L_\alpha$, we see that we have decomposed $L_{\theta_1}$ as hypothesized. (The closure appearing in the statement of the theorem merely preserves the widespread convention that the sum of a collection of spaces contains only finite sums of elements of the spaces involved.)

We remark that there may very well be finer decompositions of $L_\alpha$ into a sum of ideals not of the form $L_{\theta_i}$. Thus, if $\mu$ is an idempotent measure on $\Gamma$, then $L_T$ is the direct sum of $L_T \star \mu$ and $L_T \star (\delta_1 - \mu)$, while $\mathcal{J}(\Gamma)$ contains only one element.

Let us see how §§3 and 4 have simplified the problem of finding $\mathfrak{R}_\alpha(K)$-module homomorphisms for an arbitrary $L_T$ module $K$. In the first place, we found that $\mathfrak{R}_\alpha(K) = \mathfrak{R}_\alpha(K_0) \oplus \mathfrak{R}_\alpha(K_{aba})$, in Theorem 3.3, and then described as completely as we will here the space $\mathfrak{R}_\alpha(K_0)$, which from Theorem 3.5 turns out to be the collection of linear, continuous maps from $L_T$ to $K_0$ which map $L_T$ into 0. That done, we directed our attention to those $K$ which were order-free, showing that $R \in \mathfrak{R}_\alpha(K_{aba})$ meant that $R$ corresponded to a collection $\{R_j : R_j \in \mathfrak{R}(K_{aba}), J \in \mathcal{J}(d\Omega)\}$ (Theorem
4.3. Consequently we know $\mathcal{H}(\mathcal{K}_{abs})$ provided we know $\mathcal{H}(\mathcal{K})$ if $K$ is order-free, by Theorem 3.3. Later we will use these results in special examples.

5. Module homomorphisms on $L_\tau$. In this section we assume that $\Omega=\Gamma$. For any $L_\tau$ module $K$ whatsoever, we can determine a special subspace of $\mathcal{H}(K)$ in the following way. For $k \in K$, let $T_kk \in \mathcal{H}(K)$ be defined by $(T_kk)(f) = f * k \ (f \in L_\tau)$. Indeed $K$ is order-free if and only if $T_k$ is injective. Then we have

5.1. Theorem. (i) The definition $(f * R)(g) = R(g * f) \ (f, g \in L_\tau, R \in \mathcal{H}(K))$ turns $\mathcal{H}(K)$ into an order-free $L_\tau$ module, thereby rendering $T_k$ a module homomorphism of $K$ into $\mathcal{H}(K)$.

(ii) If $K$ is absolutely continuous, $T_k$ is an isometry.

(iii) The restriction of $T_k$ to $K_{abs}$ is an isometry of $K_{abs}$ onto $\mathcal{H}(K)_{abs}$.

(iv) $T_{\mathcal{H}(K)}$ is an isometry of $\mathcal{H}(K)$ onto $\mathcal{H}(\mathcal{H}(K))$.

Proof. (i) Except for showing that $\mathcal{H}(K)$ is order-free the proof is a straightforward calculation. But if $L_\tau * R = \{0\}$, then $R(L_\tau) = R(L_\tau * L_\tau) = L_\tau * R(L_\tau) = \{0\}$, so $R = 0$. To prove (ii), we note first that if $K$ is absolutely continuous, then $K$ is order-free, so $T_k$ is injective. Next, if $(u_i)_{i \in \mathbb{I}}$ is an approximate identity of norm 1 in $L_\tau$ and if $k \in K$, then

$$
\|T_kk\| = \sup \{\|f * k\| : f \in L_\tau; \|f\|_1 \leq 1\}
\geq \lim_i \|u_i * k\| = \|k\|.
$$

On the other hand, since $K$ is a Banach module, $\|T_kk\| \leq \|k\|$. Now we prove (iii). The isometry we already have. Since $K_{abs}$ is absolutely continuous, $T_kk$ is factorable in $\mathcal{H}(K)$ for every $k \in K_{abs}$. Thus $T_k$ maps $K_{abs}$ into $\mathcal{H}(K)_{abs}$. To show the map restricted to $K_{abs}$ is onto $\mathcal{H}(K)_{abs}$, let $R \in \mathcal{H}(K)_{abs}$. Then there exist $f \in L_\tau$ and $R' \in \mathcal{H}(K)$ such that $R = f * R'$. Thus for all $g \in L_\tau$,

$$
\{T_k[R'(f)](g)\} = g * (R'(f)) = (f * R')(g) = R(g).
$$

Consequently, $R = T_k(R'(f))$. Since $L_\tau$ is factorable, there are $f_1, f_2 \in L_\tau$ such that $f = f_1 * f_2$. Then $R = T_k(R'(f_1 * f_2)) = T_k(f_1 * R'(f_2)) \in T_k(K_{abs})$, which is what we needed to prove. Finally we prove (iv), which is simple now. We have natural surjective isometries

$$
\mathcal{H}(K) \to \mathcal{H}(K_{abs}) \to \mathcal{H}(\mathcal{H}(K)_{abs}) \to \mathcal{H}(\mathcal{H}(K)),
$$

the middle one coming from (iii) and the outer two by the comment following Theorem 3.4. The composition of these isometries is just $T_{\mathcal{H}(K)}$.

5.2. Theorem. Let $K$ be an $L_\tau$ module. Every $k^* \in K_{abs}$ determines a $Qk^* \in \mathcal{H}(K^*)$ by

$$
[(Qk^*)f](k) = k^*(f' * k) \quad (f \in L_\tau, k \in K).
$$
(If $K = K_{ab}$, then $Q = T_{K*}$.) The map $Q$ defined by this equation is an isometric module homomorphism of $K_{ab}^*$ onto $\mathfrak{R}(K^*)$.

**Proof.** Certainly $Qk^*$ is linear and continuous as a map from $L_T$ to $K^*$. Also $\|Qk^*\| \leq \|k^*\|$. Once one remembers that $(f \ast g)' = g' \ast f'$, there is no trouble in showing that $Qk^* \in \mathfrak{R}(K^*)$. Then $Q$ is a linear map $K_{ab}^* \rightarrow \mathfrak{R}(K^*)$ and $\|Q\| \leq 1$. The proof that $Q$ is a module homomorphism is straightforward. Because the domain of $Q$ is $K_{ab}^*$, $Qk^* = 0$ only if $k^* = 0$. Thus $Q$ is injective and we are done if we show it is surjective and $\|Qk^*\| \geq \|k^*\|$ for all $k^* \in K_{ab}^*$. Let $R \in \mathfrak{R}(K^*)$, and let $(u_i)_{i \in I}$ be an approximate identity in $L_T$, with $\|u_i\| \leq 1$ for each $i$. If $f \in L_T$ and $j \in K$, then

$$(Rf)(j) = \lim_i (R(f \ast u_i))(j) = \lim_i (f \ast Ru_i)(j) = \lim_i (Ru_i)(f' \ast j).$$

Thus $\lim_i (Ru_i)(k)$ exists for every $k \in K_{ab}$ and

$$\left\| \lim_i (Ru_i)(k) \right\| \leq \sup_i \|R\| \|u_i\| \|k\| \leq \|R\| \|k\|.$$ 

Therefore we can define $k^* \in K_{ab}^*$ by $k^*(k) = \lim_i (Ru_i)(k)$, for $k \in K_{ab}^*$, with the result that $\|k^*\| \leq \|R\|$. Now by the existence proof of $\lim_i (Ru_i)(k)$ we have that for all $f \in L_T$ and $j \in K$, $(Rf)(j) = \lim_i (Ru_i)(f' \ast j) = k^*(f' \ast j) = (Qk^*)f(j)$. Thus $R = Qk^*$ and $\|Qk^*\| \geq \|k^*\|$. This finishes the proof of the theorem.

With the aid of Theorems 5.1 and 5.2 we can compute $\mathfrak{R}(K)$ for most of the modules described in [5]. First, assume that $X$ is a locally compact space, and $\Gamma$ a group of homeomorphisms of $X$ such that the map $(\sigma, x) \rightarrow \sigma x$ ($\sigma \in \Gamma$, $x \in X$) is jointly continuous. Let $C_X$ be the Banach space of all continuous functions $k$ on $X$ such that for every $\varepsilon > 0$ the set $\{x \in X : |k(x)| \leq \varepsilon\}$ is compact. Then $C_X$ is an $L_T$-module with the module composition defined by

$$f \ast k(x) = \int_{\Gamma} f(\sigma)k(\sigma^{-1}x) \, d\sigma, \quad x \in X,$$

for $f \in L_T$, $k \in C_X$. (For details, see [4], [5].) We can make $M(X)$ an $L_T$-module by noting that it is the dual of $C_X$. Definition 3.1 yields

$$(f \ast \mu)k = \mu(f' \ast k), \quad f \in L_T, \mu \in M(X), k \in C_X,$$

and it turns out that $(f \ast \mu)(k) = \int_X \int_{\Gamma} k(\sigma x)f(\sigma) \, d\sigma \, d\mu(x)$. Since $C_X$ is absolutely continuous, Theorem 5.2 tells us that $\mathfrak{R}(M(X))$ is canonically isomorphic to $M(X)$. (Unfortunately we have not been able to obtain a description of $\mathfrak{R}(C_X)$ itself!)

Now let $X$ possess a quasi-invariant measure $m_X$. We denote by $L^p_X$ ($1 \leq p \leq \infty$) the space usually called $L^p(X)$ or $L_p(X)$, and write $L_X$ instead of $L^1(X)$. The natural embedding $L_X \rightarrow M(X)$ makes $L_X$ a submodule of $M(X)$ (see [4]), and therefore induces an embedding $\mathfrak{R}(L_X) \rightarrow \mathfrak{R}(M(X)) = M(X)$. The image of $\mathfrak{R}(L_X)$ in $M(X)$ is
$N = \{ \mu \in M(X) : \text{for every } f \in L^r, f \ast \mu \text{ is absolutely continuous with respect to } m_x \}$.

This space has been investigated in [4], and in particular several conditions equivalent to $N$ equaling $M(X)$ appear there.

Since $L^r_x = (L_x)^*$ we can use (3.1) to make an $L^r$ module out of $L^r_x$. It turns out that for $f \in L^r$ and $k \in L^r_x$,

$$f \ast k(x) = \int f(\sigma) k(\sigma^{-1}x) \, d\sigma, \quad \text{l.a.e. } x \in X.$$  

We know $L^r_x$ is itself absolutely continuous; hence, again by Theorem 5.2, $\mathcal{R}(L^r_x)$ is isomorphic to $L^r_x$.

For $1 \leq p \leq \infty$, in [4], we introduced convolution products $L^r \times L_x^r \to L_x^r$, of which the module operations on $L^r_x$ and $L^r_x$, mentioned above, are special cases. For $1 \leq p < \infty$, $L^r_x$ is an absolutely continuous module, and in particular, if $1 < p < \infty$, then by Theorem 5.2, $\mathcal{R}(L^r_x)$ is isomorphic to $L^r_x$.

Another space whose module homomorphisms we can describe is $L^r \cap L^p_x$, $p \in (1, \infty]$, which we look at through the eyes of $K = C_r + L_x^p$, where $1/p + 1/q = 1$. Now $K$ is the linear span of $\{h + k : h \in C_r, k \in L_x^p\}$. Under the norm $\|j\| = \inf \{\|h\| + \|k\| : h \in C_r, k \in L_x^p, j = h + k\}$ and under the convolution defined by

$$f \ast j(\sigma) = \int f(\tau) j(\tau^{-1}\sigma) \, d\tau, \quad \text{l.a.e. } \sigma \in \Gamma,$$

$K$ becomes an absolutely continuous $L^r$ module. The dual space $K^* = L^r \cap L^{p'}_x$ has for its norm $\|g\| = \max (\|g\|_1, \|g\|_p)$ (see [7, Theorem 5]). The convolution in $K^*$, defined by the familiar formula in (3.1) can be reduced to the formula

$$f \ast g(\sigma) = \int f(\tau) g(\tau^{-1}\sigma) \, d\tau \quad (f \in L^r, g \in L^r \cap L^{p'}_x, \sigma \in \Gamma).$$

By Theorem 5.2, $\mathcal{R}(L^r \cap L^p_x)$ is canonically isomorphic to $L^r \cap L^p_x$.

There is a connection between $K_{abs}$ and $\mathcal{R}(K)$ deeper than a superficial appraisal might reveal. It becomes apparent if we consider $K \to K_{abs}$ and $\mathcal{R}$ as functors in the category of all $L^r$ modules with continuous module homomorphisms as morphisms. It is obvious that $\mathcal{R}$ is related to the well-known functor $\text{Hom}$ in the category of all modules over a ring. Writing $L$ instead of $L^r$, in homological language we may denote $\mathcal{R}(K)$ by $\text{Hom}_L(L, K)$. Less obvious is the analogy between the functor $K \to K_{abs}$ and the tensor product, reflected in the following theorem.

5.3. Theorem. Let $K$ be an $L^r$ module. Then for any $L^r$ module $K'$ and any continuous bilinear map $T : L^r \times K \to K'$ that has the property $T(f \ast g, k) = f \ast T(g, k) = T(f, f \ast k)$, there is a unique continuous homomorphism $T'$ from $K_{abs}$ into $K'$ such that the diagram

$$\begin{array}{ccc}
L^r \times K & \longrightarrow & K_{abs} \\
\downarrow T & & \downarrow \sim \quad T' \\
K' & \longrightarrow \end{array}$$
is commutative (where the horizontal arrow represents the module composition $(f, k) \to f \ast k$).

**Proof.** Let $(u_i)_{i \in I}$ be an approximate identity in $L_\Gamma$. Take $k \in K_{\text{abs}}$. There exist $f \in L_\Gamma$ and $j \in K$ such that $f \ast j = k$. Then

$$T(f, j) = \lim_{i} T(f \ast u_i, j) = \lim_{i} T(u_i, f \ast j) = \lim_{i} T(u_i, k).$$

Thus we can define a linear $T': K_{\text{abs}} \to K' \text{ by } T'(k) = \lim_i T(u_i, k)$, for $k \in K_{\text{abs}}$. The rest is straightforward.

Thus it looks reasonable to write $K_{\text{abs}} = L \otimes_L K$. In this terminology, Theorem 5.2 takes the form

$$\text{Hom}_L(L, \text{Hom}_C(K, C)) = \text{Hom}_C(L \otimes_L K, C), \quad C \text{ the complex numbers},$$

which is a well-known formula in the algebraic theory. A theory relating Banach module homomorphisms to tensor product theory has been begun by Máté [8] and developed systematically by Rieffel [9].

**6. Module homomorphisms from $L_\Omega$ to $L_\Gamma$.** As we saw in §5 there is a natural embedding $\mathcal{R}(L_\Gamma) \to M(X)$. In the sequel we identify each $R \in \mathcal{R}(L_\Gamma)$ with the corresponding element of $M(X)$; thus $\mathcal{R}(L_\Gamma) = N \subseteq M(X)$. Let $\Omega \subseteq \Gamma$ and $Y \subseteq X$ be measurable. We are going to consider those $\mathcal{R}(L_\Gamma)$-module homomorphisms $L_\Gamma \to L_X$ which map $L_\Omega$ into $L_Y = \{f \in L_X : f = 0 \text{ a.e. outside } Y\}$. We denote the collection of such homomorphisms by $\mathcal{R}_{\Omega,Y}$. To aid the discussion we make the following definition.

**6.1. Definition.** Let $A_{\Omega,Y} = \{x \in X : \sigma x \in Y \text{ for locally almost all } \sigma \in \Omega\}$. We note that $A_{\Omega,Y}$ is measurable, by Theorem 3.15 of [5].

**6.2. Lemma.** If $\mu \in \mathcal{R}(L_X)$ and if $\text{supp } \mu \subseteq A_{\Omega,Y}$, then $\mu \in \mathcal{R}_{\Omega,Y}$.

**Proof.** We need to show that $L_\Omega \ast \mu \subseteq L_Y$. To do that, we let $k \in L_\Omega^\prime$ be such that $k|_Y = 0$, and we let $f \in L_\Omega$. We will show that $f \ast \mu(k) = 0$. For any $x \in \text{supp } \mu \subseteq A_{\Omega,Y}$, we have $\sigma x \in Y$ for locally almost all $\sigma \in \Omega$, so that $k(\sigma x) = 0$ l.a.e. on $\Omega$, resulting in $\int_{\Gamma} f(\sigma) k(\sigma x) \, d\sigma = 0$. Therefore

$$(f \ast \mu)(k) = \int_X \int_{\Gamma} f(\sigma) k(\sigma x) \, d\sigma \, d\mu(x)$$

$$= \int_{\text{supp } \mu} \int_{\Gamma} f(\sigma) k(\sigma x) \, d\sigma \, d\mu(x) = 0,$$

which completes the proof.

In the event that $Y$ is closed in $X$, we can give a complete description of $\mathcal{R}_{\Omega,Y}$.

**6.3. Theorem.** If $Y$ is closed in $X$, then $\mathcal{R}_{\Omega,Y} = \{\mu \in M(X) : \mu \in \mathcal{R}(L_X) \text{ and } \text{supp } \mu \subseteq A_{\Omega,Y}\}$. 

Proof. Because of Lemma 6.2 all we must prove is that if \( \mu \in \mathcal{R}_{\Omega,Y} \), then \( \text{supp} \, \mu \subseteq A_{\Omega,Y} \). By assumption, \( L_{\Omega} \ast \mu \subseteq L_{Y} \). Then for any \( k \in C_{X} \) such that \( k|_{Y} = 0 \), we have \( (f \ast \mu)(k) = 0 \) for all \( f \in L_{\Omega} \). Let \( \sigma \in d\Omega \), and let \((u_{i})_{i \in I}\) be an approximate identity for \( L_{\Omega} \) such that \((u_{i})_{i \in I} \in L_{\Omega} \). Then

\[
0 = (((u_{i})_{i \in I} \ast \mu, k) = (\mu, ((u_{i})_{i \in I}) \ast k)
\]

which converges to \( \mu(k) \) because \( C_{X} \) is absolutely continuous. Since \( \sigma^{-1} Y \) is closed, \( \text{supp} \, \mu \subseteq \sigma^{-1} Y \), for any \( \sigma \in d\Omega \). But \( \bigcap_{\sigma \in d\Omega} \sigma^{-1} Y \subseteq A_{\Omega,Y} \), Hence \( \text{supp} \, \mu \subseteq A_{\Omega,Y} \).

6.4. **Corollary.** If \( Y \) is closed in \( X \), then \( A_{\Omega,Y} = \bigcap_{\sigma \in d\Omega} \sigma^{-1} Y \), and hence \( A_{\Omega,Y} \) is closed.

Recall that we have identified \( \mathcal{H}(L_{\chi}) \) with a subspace \( N = \{ \mu \in M(X) : L_{\chi} \ast \mu \subseteq L_{X} \} \) of \( M(X) \). Obviously this \( N \) should play an important role in our discussion of \( A_{\Omega,Y} \). Under the restriction that \( N = M(X) \), it is not hard to prove that \( A_{\Omega,Y} = A_{\Omega,Y} \) if \( Y = Y' \) l.a.e. (see the implication (i) \( \Rightarrow \) (iii) of Theorem 5.6 in [4]). Without the restriction this is not true, as the following example shows: \( \Gamma = \{1\} \), \( Y = Y' \) only l.a.e. (However, from \( Y = Y' \) l.a.e. it always follows that \( A_{\Omega,Y} = A_{\Omega,Y} \) l.a.e.) Under the condition that \( N = M(X) \) we have a neater conclusion for Theorem 6.3.

6.5. **Corollary.** If \( Y \) is closed in \( X \) and if \( N = M(X) \), then

\[
\mathcal{N}_{\Omega,Y} = \{ \mu \in M(X) : \text{supp} \, \mu \subseteq A_{\Omega,Y} \}.
\]

Let \( \Omega \subseteq \Gamma \) be measurable and such that \( L_{\Omega} \) is a subalgebra of \( L_{\chi} \). For any Banach module \( K \) over \( L_{\Omega} \) we denote by \( \mathcal{H}_{\chi}(K) \) the space of all continuous module homomorphisms \( L_{\Omega} \to K \) (since every \( L_{\chi} \) module is an \( L_{\Omega} \) module this notation is consistent with our earlier use of the symbol \( \mathcal{H}_{\chi}(K) \)).

In particular we consider measurable subsets \( Y \) of \( X \) for which \( L_{\Omega} \ast L_{Y} \subseteq L_{Y} \). For such \( Y \), \( L_{Y} \) is an \( L_{\Omega} \) module. Theorem 4.3 gives an injection \( \mathcal{H}_{\chi}(L_{Y}) \to \prod_{\sigma \in \mathcal{F}(d\Omega)} \mathcal{H}_{\chi^{-1}Y} \). In case \( \mathcal{F}(d\Omega) \) consists of only one element, \( \mathcal{H}_{\chi}(L_{Y}) \) may be identified with \( \mathcal{H}_{\chi,Y} \). Then by Lemma 6.2, \( \mathcal{H}_{\chi}(L_{Y}) \cong \{ \mu \in M(X) : \text{supp} \, \mu \subseteq A_{\Omega,Y} \} \) and if \( N = M(X) \), the two sets are equal if \( Y \) is closed (Corollary 6.5). If \( N = M(X) \), it seems reasonable to ask whether we have equality for all \( Y \), still assuming \( \mathcal{F}(d\Omega) \) to contain only one element.

Now \( N = M(X) \) if \( X = \Gamma \), and \( \mathcal{F}(d\Omega) \) contains only one element if \( \Gamma \) is abelian (Corollary 4.4). T. A. Davis states a theorem affirming the inclusion \( \mathcal{H}_{\chi}(L_{Y}) \subseteq \{ \mu \in M(X) : \mu \text{ is concentrated on } A_{\Omega,Y} \} \) for the case \( \Gamma \) is abelian, \( X = \Gamma \), and \( Y = \Omega \) (Theorem 3.5(2) in [2]). Unfortunately, however, his proof seems to be faulty.

By the same Corollary 4.4, \( \mathcal{F}(d\Omega) \) contains only one element if \( l \in d\Omega \). For this case F. Birtel [1] proves \( \mathcal{H}_{\chi}(L_{Y}) = \{ \mu \in M(X) : \text{supp} \, \mu \subseteq A_{\Omega,Y} \} \) under the assumptions \( X = \Gamma \), \( Y = \Omega \), \( \Omega \) is a closed semigroup containing 1 whose interior is
In Theorem 6.7 we prove \( \mathcal{R}_\alpha(L_Y) = \{ \mu \in M(X) : \mu \text{ is concentrated on } A_{\alpha,Y} \} \) if \( 1 \in d\Omega \) and \( N = M(X) \). In Corollary 6.8 we prove \( \mathcal{R}_\alpha(L_Y) = \{ \mu \in M(X) : \text{supp } \mu \subseteq A_{\alpha,Y} \} \) if \( 1 \in d\Omega \) and if for every \( f \in L_t \) and \( k \in L^r_t \), \( \int f(\sigma)k(\sigma^{-1}x) \, d\sigma \) depends continuously on \( x \), which is true for \( X = \Gamma \).

We employ an auxiliary topology on \( X \), called the orbit topology and designated by \( \mathcal{O} \), which is generated by sets of the form \( \langle t \rangle_x \), open in \( Y \), \( x \in X \). This topology is studied in [5]. In what follows we shall use the facts that in case \( N = M(X) \), \( \mathcal{O} \) coincides with the original topology of \( X \) on each orbit \( \Gamma x \), and that for \( f \in L_t \) and \( k \in L^r_t \), the function \( x \to \int f(\sigma)k(\sigma^{-1}x) \, d\sigma \) is \( \mathcal{O} \)-continuous.

6.6. Lemma. In the topology \( \mathcal{O} \), \( A_{\alpha,Y} \) is closed.

Proof. Take \( x \in X \). Then \( x \in A_{\alpha,Y} \) if and only if \( \int f(\sigma)\xi_{X/Y}(\sigma x) \, d\sigma = 0 \) for every \( f \in L_\alpha \). Now

\[
\int f(\sigma)\xi_{X/Y}(\sigma x) \, d\sigma = \int f(\sigma^{-1})\Delta(\sigma^{-1})\xi_{X/Y}(\sigma^{-1}x) \, d\sigma
\]

is \( \mathcal{O} \)-continuous (see Lemma 4.9 of [5]); thus \( A \) is \( \mathcal{O} \)-closed.

6.7. Theorem. Let \( \Omega \subseteq \Gamma \), \( Y \subseteq X \) be measurable and such that \( L_\alpha \) is a subalgebra of \( L_t \), and \( L_\alpha \ast L_Y \subseteq L_Y \). Assume \( N = M(X) \). If \( 1 \in d\Omega \), then \( \mathcal{R}_\alpha(L_Y) = \{ \mu \in M(X) : \mu \text{ is concentrated on } A_{\alpha,Y} \} \).

Proof. Let \( \mu \in \mathcal{R}_\alpha(L_Y) \). We note that \( \Gamma x \) is a Borel set (Theorem 5.10 of [4]) for each \( x \in X \). Since \( \mu \) is bounded, \( \mu \) is concentrated on a sigma-compact set. Inasmuch as any compact set can intersect only countably many orbits (see Lemma 4.6 of [5]), there exists a sequence \( a_1, a_2, \ldots \) in \( X \) such that \( \mu \) is concentrated on \( \bigcap_n \Gamma a_n \). For each \( n \) define \( \mu_n \) by \( d\mu_n = \xi_{Y a_n} \, d\mu \) and put \( Y_n = Y \cap \Gamma a_n \). Then \( L_\alpha \ast L_{Y_n} \subseteq L_{Y_n} \), \( \mu = \sum \mu_n \) and \( L_\alpha \ast \mu_n \subseteq L_{Y_n} \) for each \( n \). It suffices to prove that each \( \mu_n \) is concentrated on \( A_{\alpha,Y_n} \). In other words, we may assume the existence of an \( a \in X \) such that \( \mu \) is concentrated on \( \Gamma a \) and \( Y \subseteq \Gamma a \). Since \( 1 \in d\Omega \), by Theorem 5.6 of [5] \( L_\alpha \ast L_{Y_n} = L_{Y_n} \). Let \( \Omega_0 = \Omega \cap d\Omega \), \( T = \{ x \in X : \text{there exist compact sets } \Phi \subseteq \Omega \text{ and } D \subseteq Y \text{ such that } \int \xi_{\Phi}(\sigma)\xi_{D}(\sigma^{-1}x) \, d\sigma > 0 \} \). Clearly \( T \subseteq \Gamma a \). According to the proof of Theorem 5.5 of [5], we have \( \Omega_0 = \Omega \) l.a.e., \( T = Y \) l.a.e. and \( \Omega_0 T = T \). Then \( T \subseteq A_{\Omega_0,\Gamma a} = A_{\alpha,T} = A_{\alpha,Y} \).

Since \( 1 \in d\Omega \), \( L_\alpha \) contains an approximate identity \( (u_i)_{i \in I} \) of \( L_t \). If \( k \in C_X \) and \( k = 0 \) on \( T \), then because \( u_i \ast \mu \in L_\alpha \ast L_Y \subseteq L_Y \),

\[
\mu(k) = \lim_i \mu(u_i \ast k) = \lim_i (u_i \ast \mu)(k) = 0.
\]

This means that \( \text{supp } \mu \subseteq \overline{T} \) so that \( \mu \) is concentrated on \( \overline{T} \cap \Gamma a \). Now as we remarked in the preceding lemma, \( A_{\alpha,Y} \) is \( \mathcal{O} \)-closed. Because the original topology
and the $\theta$-topology coincide on $\Gamma a$ this means that $A_{\Omega,Y} \cap \Gamma a$ is relatively closed in $\Gamma a$. Since $T \subseteq A_{\Omega,Y} \cap \Gamma a$ we obtain $T \cap \Gamma a \subseteq A_{\Omega,Y} \cap \Gamma a \subseteq A_{\Omega,Y}$. Thus $\mu$ is concentrated on $A_{\Omega,Y}$.

6.8. Corollary. Let $\Omega$, $Y$ be as in the preceding theorem. Assume that for every $f \in L^r_\Omega$ and every $k \in L^\infty_\Omega$, $\int_\Omega f(\sigma)k(\sigma^{-1}x)\,d\sigma$ depends continuously on $x \in X$. Now if $1 \in d\Omega$, then $R_\Omega(L_\Omega) = \{\mu \in M(X) : \text{supp} \, \mu \subseteq A_{\Omega,Y}\}$.

Proof. Then the orbit topology and the original topology are the same (Lemma 4.9 of [5]), and $A_{\Omega,Y}$ is thus closed in $X$. Furthermore, the assumption implies that $N = M(X)$ (Theorem 3.3 of [5]). Thus we can use the preceding theorem.

We have one comment: $\mu \in M(X)$ may be concentrated on $A_{\Omega,Y}$ without being supported on $A_{\Omega,Y}$. Let $\Gamma = \mathbb{R}$ be the additive group of the reals, $\Omega = (0, \infty)$, $X = \mathbb{R} \cup \{\omega\}$ the one-point compactification of $\mathbb{R}$, with usual action of $\Gamma$ on $X$, and $m_x(\{\omega\}) = 1$, $m_x|\mathbb{R} = \text{Lebesgue measure}$, and $Y = (0, \infty)$. Then $A_{\Omega,Y} = [0, \infty)$. Let $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_n$ where $\delta_n$ is point mass at $n$. Then $\mu$ is concentrated on $A_{\Omega,Y}$ but not supported on $A_{\Omega,Y}$.

References


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