ON AUTOMORPHISM GROUPS OF $C^*$-ALGEBRAS

BY

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I. Introduction. Let $X$ be a compact Hausdorff space and $\mathcal{L}(H)$ the algebra of all bounded operators on a Hilbert space $H$. $C(X; \mathcal{L}(H))$ is the $C^*$-algebra of all continuous $\mathcal{L}(H)$-valued functions defined on $X$, with the sup-norm. The study of automorphisms of such algebras has been initiated by Kadison and Ringrose in [13, IV, Example d] where the case $H$ finite dimensional is studied in some detail. A similar study of the case $H$ infinite dimensional (and $X$ a separable compact Hausdorff space) has been made by E. C. Lance [18] and independently by moi-même.

In the present work we study center-fixing automorphisms of $\mathfrak{A} = C(X; B)$, where $X$ is an arbitrary compact Hausdorff space, and $B$ a $C^*$-algebra. We introduce the notion of "locally-inner" automorphisms of $C(X; B)$, by "localizing" the definition of inner automorphisms. The precise definition requires several preliminary properties of center-fixing automorphisms and is given in §III.

The locally-inner automorphisms form a subgroup of $\text{Aut} (\mathfrak{A})$, the automorphism group of $\mathfrak{A}$, which we denote by $\text{loc-Inn} (\mathfrak{A})$. By the definition of $\text{loc-Inn} (\mathfrak{A})$ we will have an inclusion $\text{loc-Inn} (\mathfrak{A}) \subseteq \text{CF} (\mathfrak{A})$, the center-fixing automorphism group of $\mathfrak{A}$. If $H$ is finite dimensional, then $\text{loc-Inn} (C(X; \mathcal{L}(H))) = \text{CF} (C(X; \mathcal{L}(H)))$ [13]. For a general Hilbert space $H$ we establish (Theorem 3.5) an isomorphism of topological groups

$$\text{loc-Inn} (C(X; \mathcal{L}(H))) = C(X; \text{Aut} (\mathcal{L}(H)))$$

where $C(X; \text{Aut} (\mathcal{L}(H)))$ denotes the group of continuous maps $f: X \to \text{Aut} (\mathcal{L}(H))$. This occupies the major portion of §III.

With the aid of a theorem of Kallman [14], we find (Corollary 3.18) that $\text{CF} (\mathfrak{A}) = \text{loc-Inn} (\mathfrak{A})$ when $X$ is a separable compact Hausdorff space and $\mathfrak{A} = C(X; \mathcal{L}(H))$. Every inner automorphism is locally-inner and $\text{Inn} (\mathfrak{A})$, the inner automorphism group of $\mathfrak{A}$, is a normal subgroup of $\text{loc-Inn} (\mathfrak{A})$. As a consequence of the results of §III (Theorem 4.1) and a result of Kuiper [16] we obtain for infinite-dimensional $H$ a natural isomorphism of groups

$$\text{loc-Inn} (\mathfrak{A})/\text{Inn} (\mathfrak{A}) = \hat{H}^2(X; \mathbb{Z}),$$

where $\hat{H}^2(X; \mathbb{Z})$ denotes the 2nd Čech cohomology group of $X$ with coefficients in

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the ring of integers $\mathbb{Z}$. When $X$ is separable, combining the above results with those of Lance [18] we obtain (Corollary 4.2)

$$\text{CF}(\mathfrak{A}) = \pi(\mathfrak{A}) = \text{loc-Inn}(\mathfrak{A})$$

as a consequence. We also obtain for separable $X$ the equality

$$\text{Inn}(\mathfrak{A}) = \text{Aut}_0(\mathfrak{A}) \quad (= \text{the identity component of Aut}(\mathfrak{A})).$$

These results overlap with those of Lance in [18].

In §V we discuss relations between the carefully ideal preserving automorphisms and $\text{CF}(\mathfrak{A})$ where $\mathfrak{A}$ is $C(X; B)$. Specifically, when $X$ is separable we obtain

$$\tau_0(\mathfrak{A}) = \pi(\mathfrak{A}) = \text{CF}(\mathfrak{A})$$

where $\tau_0(\mathfrak{A})$ denotes the group of carefully ideal preserving automorphisms of $\mathfrak{A}$ (compare the results of Lance in [17]).

The presentation below is an outgrowth of the author’s Thesis submitted to Yale University in partial fulfillment of the requirements of the Ph.D. degree.

II. Preliminaries. In this paper it is assumed that the reader is familiar with the basic results of the theory of $C^*$-algebras and their $*$-representations as may be found in [20], early chapters of [2], [3] or [4]. Also those propositions for which no proofs are given are elementary and may be verified by the reader.

Definition and Conventions. Throughout this paper a $C^*$-algebra will always mean a $C^*$-algebra over the complex numbers $C$, with identity which is usually denoted by $I$.

If $\mathfrak{A}$ is a $C^*$-algebra an automorphism of $\mathfrak{A}$ is an isomorphism of complex vector spaces $\alpha: \mathfrak{A} \to \mathfrak{A}$ such that

1. $\alpha(A^*) = \alpha(A)^*$ for all $A \in \mathfrak{A}$,
2. $\alpha(AB) = \alpha(A)\alpha(B)$ for all $A, B \in \mathfrak{A}$,
3. $\alpha(I) = I$.

The set of all automorphisms of $\mathfrak{A}$ is denoted by Aut$(\mathfrak{A})$; it is a group in a natural way, the group operation being a composition of mappings.

If $S$ is a complex Banach space then $L(S)$ denotes the algebra of all bounded linear operators on $S$. If $T \in L(S)$ we define

$$\|T\| = \sup_{s \in S, \|s\| \leq 1} \|T(s)\|.$$ 

This defines a norm on $L(S)$.

If $\mathfrak{A}$ is a $C^*$-algebra then $\mathfrak{A}$ is semisimple with unique topology (in the sense of [19]) and thus any automorphism of $\mathfrak{A}$ is continuous in the norm topology of $\mathfrak{A}$. Hence Aut$(\mathfrak{A}) \subseteq L(\mathfrak{A})$ and we may equip Aut$(\mathfrak{A})$ with the relative topology. In this topology Aut$(\mathfrak{A})$ becomes a topological group.

Note that any automorphism of $\mathfrak{A}$ is an isometry [6].
Definition of various subgroups of Aut (A). Let A be a C*-algebra acting on a Hilbert space H. An automorphism α ∈ Aut (A) is said to be extendible if there is an automorphism α of the weak operator closure A of A such that α|A = α: A → A. An automorphism α ∈ Aut (A) is said to be spatial if there is a unitary operator U on H such that for all A ∈ A, α(A) = UAU*. α ∈ Aut (A) is said to be weakly-inner if it is spatial and U can be chosen in the weak operator closure of A.

An automorphism α ∈ Aut (A) is inner if there exists a unitary element U ∈ A such that α(A) = UAU* for all A ∈ A. Note that an inner automorphism of A is weakly-inner in any faithful *-representation of A.

Let A be an abstract C*-algebra. If φ is a faithful *-representation of A on a Hilbert space, φ-Ext (A) denotes the group of those elements α of Aut (A) for which φ(αφ⁻¹) is extendible; φ-Inn (A) denotes the group of those elements α of Aut (A) for which φ(αφ⁻¹) is spatial; φ-Inn (A) denotes the group of those elements α of Aut (A) for which φ(αφ⁻¹) is weakly-inner. Let π(A) = ∩φ φ-Inn (A), where the intersection is taken over all faithful *-representations φ of A. π(A) is called the permanently weakly-inner (or π-inner) automorphisms of A. The group of all inner automorphisms of A is denoted by Inn (A). The connected component of I ∈ Aut (A) (in the uniform topology) is denoted by Aut₀ (A).

These subgroups of Aut (A) were defined and studied by Kadison and Ringrose in [13].

An automorphism α ∈ Aut (A) is said to be center-fixing if α leaves the elements of the center Z(A) elementwise fixed; i.e. α(A) = A for all A ∈ Z(A).

Proposition. Let A be a C*-algebra. Then the set of all center-fixing automorphisms of A forms a subgroup of Aut (A). □

The subgroup of Aut (A) consisting of all the center-fixing automorphisms of A is denoted by CF (A), and is referred to as the center-fixing automorphism group of A. Note that Inn (A) ⊆ CF (A).

Some results from the work of Kadison and Ringrose [13]. It is convenient to have available some of the results of [13] concerning inclusion relations between the subgroups of Aut (A) introduced above. The remainder of this section is devoted to a summary of the relevant facts from [13].

Theorem A [13, Theorem 7]. Let A be a C*-algebra and α ∈ Aut (A) with ∥α − I∥ < 2. Then α lies in a norm-continuous one parameter subgroup of Aut (A). Such subgroups generate Aut₀ (A). Each element of Aut₀ (A) is π-inner.

It follows from Theorem A that one has inclusions

Aut₀ (A) ⊆ π(A) ⊆ φ-Inn (A) ⊆ σφ(A) ⊆ φ-Ext (A)

where φ is any faithful *-representation of A. Since Aut₀ (A) contains an open ball with center I and radius 2 in Aut (A) it follows that each of π(A), φ-Inn (A), σφ(A), φ-Ext (A) is also an open subgroup of Aut (A). Hence they are all closed too.
Furthermore, $\text{Aut}_0(\mathcal{A})$ and $\pi(\mathcal{A})$ are normal subgroups of $\text{Aut}(\mathcal{A})$, while in general, $\varphi$-$\text{Inn}(\mathcal{A})$, $\sigma_\varphi(\mathcal{A})$ and $\varphi$-$\text{Ext}(\mathcal{A})$ are not [13, pp. 48–49 after Corollary 9].

The inner automorphism group $\text{Inn}(\mathcal{A})$ of $\mathcal{A}$ is contained in $\pi(\mathcal{A})$, and is a normal subgroup of $\text{Aut}(\mathcal{A})$. Generally, it is not true that $\text{Aut}_0(\mathcal{A})$ is contained in $\text{Inn}(\mathcal{A})$ [13, p. 49].

Various examples in [13] show that all possible equalities and inequalities among the above inclusion relations actually occur. The interesting results are as follows:

**Theorem B** [13, Corollary 9]. *If $\mathcal{A}$ is a $C^*$-algebra with a faithful $*$-representation $\varphi$ as a von Neumann algebra, then*

$$\text{Inn}(\mathcal{A}) = \text{Aut}_0(\mathcal{A}) = \pi(\mathcal{A}) = \varphi\text{-Inn}(\mathcal{A})$$

*and each element of $\text{Aut}_0(\mathcal{A})$ lies on a norm-continuous one parameter subgroup of $\text{Aut}(\mathcal{A})$.*

**Theorem C** [13, IV, Example b]. *Let $\mathcal{A}$ be the $C^*$-algebra of compact operators on a separable Hilbert space with the identity operator $I$ adjoined. (Thus every element $A \in \mathcal{A}$ has the form $al + K$, where $a \in C$ and $K$ is a compact operator.) Then $\text{Aut}_0(\mathcal{A}) = \text{Aut}(\mathcal{A}) = \pi(\mathcal{A})$, and $\mathcal{A}$ admits noninner permanently weakly-inner automorphisms, i.e. $\text{Inn}(\mathcal{A}) \subset \pi(\mathcal{A})$. Since $\text{Inn}(\mathcal{A}) = \{I\}$, $\mathcal{A}$ also provides an example where $\text{Aut}_0(\mathcal{A}) \subset \text{Inn}(\mathcal{A})$.*

**Theorem D** [13, IV, Example d]. *Let $X$ be a compact Hausdorff space, $\mathcal{M}_n$ the $C^*$-algebra of all $(n \times n)$-matrices with complex entries and $\mathcal{A} = C(X; \mathcal{M}_n)$ the $C^*$-algebra of all continuous functions from $X$ to $\mathcal{M}_n$. (The norm in $\mathcal{A}$ is the sup-norm, i.e. if $f \in C(X; \mathcal{M}_n)$ then $\|f\| = \sup_{x \in X} \|f(x)\|$.) Then*

$$\text{CF}(\mathcal{A}) = \pi(\mathcal{A}) = \varphi\text{-Inn}(\mathcal{A})$$

*and

$$\text{Aut}_0(\mathcal{A}) \subseteq \text{Inn}(\mathcal{A}) \subseteq \pi(\mathcal{A}).$$

All the possible equality and inequality relations among the inclusions in Theorem D can actually occur for a suitable choice of $X$. For example, if $X = I^n$, the $n$-dimensional cell, then [13, p. 57]

$$\text{Aut}_0(\mathcal{A}) = \text{Inn}(\mathcal{A}) = \pi(\mathcal{A}).$$

If $X = U(n)/S^1$, where $U(n)$ is the unitary group in $\mathcal{M}_n$, $S^1$ the circle group of diagonal matrices in $U(n)$, then [13, p. 67] if $n \geq 3$,

$$\text{Aut}_0(\mathcal{A}) \neq \text{Inn}(\mathcal{A}) \neq \pi(\mathcal{A}).$$

If $X = S^1$, the circle, then [13]

$$\text{Aut}_0(\mathcal{A}) \neq \text{Inn}(\mathcal{A}) = \pi(\mathcal{A}).$$
If $X = \mathbb{RP}(3) = U(2)/S^1$ the real projective 3-space, then \[ \text{Aut}_0(\mathfrak{A}) = \text{Inn}(\mathfrak{A}) \neq \pi(\mathfrak{A}). \]

**Center-fixing automorphisms.** Let $\mathfrak{A}$ be a $C^*$-algebra. As noted above the automorphisms of $\mathfrak{A}$ fixing the center of $\mathfrak{A}$ form a subgroup of $\text{Aut}(\mathfrak{A})$ denoted by $\text{CF}(\mathfrak{A})$. In this section some inclusion relations between $\text{CF}(\mathfrak{A})$ and other subgroups of $\text{Aut}(\mathfrak{A})$ will be discussed.

**Lemma 2.1.** Let $\mathfrak{A}$ be a $C^*$-algebra and $\varphi$ a faithful $*$-representation of $\mathfrak{A}$ on a Hilbert space $H$. Then

$$\varphi^{-1}\text{Inn}(\mathfrak{A}) \subseteq \text{CF}(\mathfrak{A}).$$

As a consequence of Lemma 2.1 and the inclusions discussed above we have inclusions

$$\text{Aut}_0(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \varphi^{-1}\text{Inn}(\mathfrak{A}) \subseteq \text{CF}(\mathfrak{A}) \subseteq \text{Aut}(\mathfrak{A})$$

and

$$\text{Inn}(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \varphi^{-1}\text{Inn}(\mathfrak{A}) \subseteq \text{CF}(\mathfrak{A}) \subseteq \text{Aut}(\mathfrak{A}).$$

Thus $\text{CF}(\mathfrak{A})$ contains an open ball in $\text{Aut}(\mathfrak{A})$ with center $I$ and radius 2. Thus $\text{CF}(\mathfrak{A})$ is open and a closed subgroup of $\text{Aut}(\mathfrak{A})$.

**Lemma 2.2.** If $\mathfrak{A}$ is a $C^*$-algebra then $\text{CF}(\mathfrak{A})$ is a normal subgroup of $\text{Aut}(\mathfrak{A})$.

We will find it convenient to make use of tensor products of $C^*$-algebras in the remaining sections. We recall some of the relevant facts here.

**Tensor products and cross-norms.** If $H$ and $K$ are Hilbert spaces then we may introduce the Hilbert space tensor product $H \otimes K$ [2, p. 23], [19]. If $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras with $*$-representations $\varphi: \mathfrak{A} \to \mathcal{L}(H)$, $\psi: \mathfrak{B} \to \mathcal{L}(K)$ then we denote by $\varphi \otimes \psi$ the algebraic tensor product $*$-representation $\varphi \otimes \psi: \mathfrak{A} \otimes \mathfrak{B} \to \mathcal{L}(H \otimes K)$

$$(\varphi \otimes \psi)(A \otimes B)(\xi \otimes \eta) = \varphi(A)(\xi) \otimes \psi(B)(\eta),$$

where $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, $\xi \in H$, $\eta \in K$.

**Theorem E (Wulfsohn [26, Theorem 1]).** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras, $\varphi$, $\varphi'$ faithful $*$-representations of $\mathfrak{A}$ on $H$, $H'$ respectively, and $\psi$, $\psi'$ faithful $*$-representations of $\mathfrak{B}$ on $K$ and $K'$ respectively. Then for any $A_i \in \mathfrak{A}$, $B_i \in \mathfrak{B}$

$$\left| \sum_i \varphi(A_i) \otimes \psi(B_i) \right| = \left| \sum_i \varphi'(A_i) \otimes \psi'(B_i) \right|$$

where the norms are the norms in $\mathcal{L}(H \otimes K)$ and $\mathcal{L}(H' \otimes K')$ respectively.

**Definition.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras with faithful $*$-representations $\varphi$, $\psi$ on $H$ and $K$ respectively. The completion of $\varphi(\mathfrak{A}) \otimes \psi(\mathfrak{B})$ in the norm topology on $\mathcal{L}(H \otimes K)$ is denoted by $\mathfrak{A} \otimes^* \mathfrak{B}$. 
Remark. By Theorem E, $\mathfrak{A} \otimes^* \mathfrak{B}$ is independent of the choice of faithful $*$-representations $\varphi$ and $\psi$.

In [27] Wulfsohn has shown that the cross-norm on $\mathfrak{A} \otimes^* \mathfrak{B}$ introduced above coincides with the $\alpha_0$-norm of Turumaru [26].

III. Locally inner automorphisms. Let $X$ be a compact Hausdorff space and $B$ be a Banach algebra over $C$. We denote by $C(X; B)$ the sup-norm algebra of all continuous functions $f: X \to B$. If $B$ is a $C^*$-algebra so is $C(X; B)$ and if $B$ has an identity so does $C(X; B)$.

Notation. Let $X$ be a compact Hausdorff space and $B$ be a Banach algebra. If $b \in B$ we denote by $\tilde{B}: X \to B$ the constant function at $b \in B$. This is clearly continuous and thus $\tilde{B} \in C(X; B)$. The function $\sim: B \to C(X; B)$ imbeds $B$ in $C(X; B)$ in a natural way.

Let $X$ be a compact Hausdorff space and $B$ a $C^*$-algebra over $C$. We define a function

$$e: C(X) \otimes B \to C(X; B)$$

by

$$e \left( \sum_{i=1}^{n} f_i \otimes B_i \right)(x) = \sum_{i=1}^{n} f_i(x)B_i.$$ 

This function is continuous in the $C^*$-topology and extends to provide an isomorphism [18], [19], [21], [24]

$$e: C(X) \otimes^* B \to C(X; B)$$

of $C^*$-algebras.

This isomorphism will prove a useful technical tool for us in this section.

We introduce, and fix, throughout this section, the following notation:

Notation. $X$ is a compact Hausdorff space;
$H$ is a Hilbert space over $C$;
$B$ denotes a $C^*$-algebra;
$C(X; B)$ is the sup-norm algebra of all continuous $f: X \to B$;
$\mathfrak{A} = C(X; B)$;
$\text{Aut} (\mathfrak{A})$ is the group of automorphisms of $\mathfrak{A}$ with the uniform topology;
$\text{Aut} (B)$ is the group of all automorphisms of $B$ with the uniform topology;
$C(X; \text{Aut} (B))$ is the group of all continuous functions $X \to \text{Aut} (B)$, under pointwise multiplication. We equip $C(X; \text{Aut} (B))$ with the compact-open topology. It is a topological group.

We turn now to the definition of locally-inner automorphisms of $\mathfrak{A}$. This will require some preparation.

Notation. $e: X \times C(X; B) \to B$ is the evaluation mapping given by $e(x, f) = f(x)$. If $x \in X$ we denote by

$$e_x: C(X; B) \to B$$

the mapping given by $e_x(f) = e(x, f)$.
Proposition 3.1. The evaluation mapping \( e: X \otimes C(X; B) \to B \) is continuous.

Proof. This follows from the fact that the norm topology contains the compact-open topology. See, for example, R. H. Fox, Topologies for function spaces, Bull. Amer. Math. Soc. 51 (1945), 429–432. □

Lemma 3.2. If \( \alpha \in CF (\mathcal{M}) \) and \( x \in X \), then \( \alpha(\ker e_x) \subseteq \ker e_x \).

Proof. We will use the tensor product representation \( C(X; B) = C(X) \otimes^* B \). Under this identification the evaluation mapping \( e_x \) is given by the extension of

\[
e_x: C(X) \otimes B \to B \mid e_x \left( \sum_i f_i \otimes B_i \right) = \sum_i f_i(x)B_i.
\]

Let \( \mathcal{M} \) be the maximal two sided ideal in \( C(X) \) determined by \( x \in X \), i.e. \( \mathcal{M} = \{ f \in C(X) \mid f(x) = 0 \} \). \( \mathcal{M} \) is again a \( C^* \)-algebra, albeit without identity. Thus \( \mathcal{M} \otimes^* B \) is defined and \( \mathcal{M} \otimes^* B \subseteq C(X) \otimes^* B \) is a two sided ideal. Direct computation shows that \( \mathcal{M} \otimes^* B = \ker e_x \).

Suppose \( M_i \in \mathcal{M}, B_i \in B, i = 1, 2, \ldots, n \). Then

\[
\alpha \left( \sum_{i=1}^n M_i \otimes B_i \right) = \sum_{i=1}^n \alpha(M_i \otimes B_i) = \sum_{i=1}^n \alpha((M_i \otimes I)(I \otimes B_i)) = \sum_{i=1}^n \alpha(M_i \otimes I)\alpha(I \otimes B_i) = \sum_{i=1}^n (M_i \otimes I)\alpha(I \otimes B_i),
\]

since \( \alpha \) is center-fixing and \( Z(C(X) \otimes^* B) = C(X) \otimes I \), and \( \mathcal{M} \otimes I \subseteq C(X) \otimes I \).

Applying the evaluation map to both sides of the above equality then gives

\[
e_x\alpha \left( \sum_{i=1}^n M_i \otimes B_i \right) = \sum_{i=1}^n M_i(x)e_x(\alpha(I \otimes B_i)) = 0,
\]

since \( M_i \in \mathcal{M} \). Since the elements of the form \( \sum_{i=1}^n M_i \otimes B_i \), for finite \( n > 0 \), are dense in \( \mathcal{M} \otimes^* B = \ker e_x \) and \( \alpha, e_x \) are continuous with respect to the \( * \)-topology on \( C(X) \otimes^* B \), it follows that \( e_x(\alpha(A)) = 0 \) for any \( A \in \mathcal{M} \otimes^* B = \ker e_x \), and thus \( \alpha(\ker e_x) \subseteq \ker e_x \), as was to be shown. □

Remark. Let \( S \) be a closed subspace of \( X \). If \( f: S \to C \) is any continuous function then by Tietze's extension theorem there exists an extension of \( f, F: X \to C \).

The inclusion \( S \subseteq X \) induces by restriction a homomorphism of \( C^* \)-algebras (notice it is norm decreasing)

\[
\rho: C(X; B) \to C(S; B).
\]
Using the tensor product identifications we see that $\rho$ is onto. For suppose $f_i \in C(S)$, $B_i \in B$, $i = 1, \ldots, n$. Choose extensions $F_i \in C(X)$ such that $F_i|_S = f_i$, $i = 1, \ldots, n$. Then

$$\rho \left( \sum_i F_i \otimes B_i \right) = \sum_i f_i \otimes B_i.$$ 

Recall $\rho$ is continuous with respect to the $*$-topology, $C(X; B)$ is complete, and the elements of the form $\sum_{i=1}^n f_i \otimes B_i$ are dense in $C(S) \otimes^* B$. It follows that $\rho$ is onto as claimed.

Thus given $f: S \rightarrow B$ there exists an extension $F: X \rightarrow B$. If $F'$ is another such extension then $F - F'|_S: S \rightarrow B$ is the constant function at $0 \in B$. Thus for any $s \in S$ and $\alpha \in \text{CF} (C(X; B))$, $(F - F')(s) = 0$. Therefore $\alpha(F)|_S = \alpha(F')|_S: S \rightarrow B$.

**Definition.** If $\alpha \in \text{CF} (\mathcal{A})$ and $S$ is a closed subspace of $X$ define $\alpha|_S: C(S; B) \rightarrow C(S; B)$ by

$$\alpha|_S (f)(x) = \alpha(F)(x)$$

where $F: X \rightarrow B$ is any extension of $f$ such that $F \in \mathcal{A}$. This is well defined by the above remark.

**Proposition 3.3.** Let $X$ be a compact Hausdorff space, $S \subseteq X$ a closed subspace. Let $B$ be a C*-algebra, and $\alpha \in \text{CF} (C(X; B))$. Then

$$\alpha|_S: C(S; B) \rightarrow C(S; B),$$

is a center-fixing automorphism of $C(S; B)$.

If $\rho: C(X; B) \rightarrow C(S; B)$ is the homomorphism of C*-algebras given by restriction then the diagram

$$\begin{array}{ccc}
C(X; B) & \xrightarrow{\alpha} & C(X; B) \\
\downarrow \rho & & \downarrow \rho \\
C(S; B) & \xrightarrow{\alpha|_S} & C(S; B)
\end{array}$$

is commutative. Thus

$$\alpha|_S: \text{CF} (C(X; B)) \rightarrow \text{CF} (C(S; B))$$

is a continuous homomorphism of topological groups.

**Proof.** The verifications are all routine. □

We introduce now a subgroup of $\text{Aut} (\mathcal{A})$ that will be of interest throughout the remainder of this paper.

**Notation.** If $X$ is a topological space and $S \subseteq X$ is a subspace, then the closure of $S$ in $X$ is denoted by $\overline{S}$. 
Definition. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra, and $\mathfrak{A} = C(X; B)$. If $\alpha \in \text{CF}(\mathfrak{A})$ then $\alpha$ is locally-inner iff there exists an open covering $S_1, \ldots, S_N$ of $X$ such that $\alpha|_{S_i}$ is inner, $i = 1, \ldots, N$. The set of all locally inner automorphisms is denoted by $\text{loc-Inn}(\mathfrak{A})$.

Proposition 3.4. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra and $\mathfrak{A} = C(X; B)$. Then $\text{loc-Inn}(\mathfrak{A})$ is a subgroup of $\text{Aut}(\mathfrak{A})$ and

$$\text{Inn}(\mathfrak{A}) \subseteq \text{loc-Inn}(\mathfrak{A}) \subseteq \text{CF}(\mathfrak{A}).$$

Proof. If $\alpha, \beta \in \text{loc-Inn}(\mathfrak{A})$ then there exist open covers $\{S_1, \ldots, S_N\}, \{T_1, \ldots, T_M\}$ of $X$ with $\alpha|_{S_i}, i = 1, \ldots, N, \beta|_{T_j}, j = 1, \ldots, M,$ inner. It then follows that $\beta^{-1}|_{T_j}, j = 1, \ldots, M,$ is inner. Let $V_{1,1}, \ldots, V_{N,M}$ be the open cover of $X$ given by $V_{i,j} = S_i \cap T_j$. It is immediate that $\alpha\beta^{-1}|_{V_{i,j}}$ is inner, and thus $\text{loc-Inn}(\mathfrak{A})$ is a subgroup of $\text{Aut}(\mathfrak{A})$. □

The main result of this section is:

Theorem 3.5. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra, and $\mathfrak{A} = C(X; B)$. Then there exists a natural isomorphism of topological groups

$$\sim: \text{loc-Inn}(\mathfrak{A}) \rightarrow C(X; \text{Aut}(B)).$$

The proof of Theorem 3.5 will occupy most of the remainder of this section.

Definition. Let $\alpha \in \text{CF}(\mathfrak{A})$ and $x \in X$. Define a function $\bar{\alpha}_x: B \rightarrow B$ by

$$\bar{\alpha}_x(B) = \alpha(B)(x).$$

Thus for $\alpha \in \text{CF}(\mathfrak{A})$ and $x \in X$, $\bar{\alpha}_x$ is the function given by the composition

$$B \xrightarrow{\sim} C(X; B) \xrightarrow{\alpha} C(X; B) \xrightarrow{\bar{e}_x} B.$$
i.e.
\[ e_x \beta \rho (B) = e_x \beta \rho e_x \rho (B) \]
for all \( B \in B \). From the definition of \( \tilde{\alpha} \) this means
\[ ((\beta \alpha)^{(\hat{x})})_x(B) = \beta_x \cdot \tilde{\alpha}_x(B) \]
for all \( B \in B \), and thus \( ((\beta \alpha)^{(\hat{x})})_x = \beta_x \cdot \tilde{\alpha}_x \) as was to be shown. □

**Proposition 3.8.** If \( \alpha \in \text{CF}(\mathcal{A}) \) and \( x \in X \) then \( \tilde{\alpha}_x : B \to B \) is an automorphism of \( B \).

**Proof.** It is a routine task to show that \( \tilde{\alpha}_x \) is multiplicative. From Lemma 3.7 it follows that
\[ (\tilde{\alpha}^{-1})_x = (\tilde{\alpha}_x)^{-1}. \]
Then the proposition follows from Lemma 3.6. □

**Definition.** If \( \alpha \in \text{CF}(\mathcal{A}) \) let \( \tilde{\alpha} : X \to \text{Aut}(B) \) be the function defined by \( \tilde{\alpha}(x) = \tilde{\alpha}_x. \)

**Proposition 3.9.** Let \( X, B, \mathcal{A} \) be as above. Then for any \( \alpha \in \text{CF}(\mathcal{A}) \), \( \tilde{\alpha} : X \to \text{Aut}(B) \) is strongly continuous.

**Proof.** Let \( \{ x_\lambda \mid \lambda \in \Lambda \} \) be a convergent net in \( X \) with limit \( x \). Then for any \( B \in B \), \( \{ (x_\lambda, \tilde{\alpha}_x) \mid \lambda \in \Lambda \} \) is a convergent net in \( X \times C(X; B) \) with limit \( (x, \tilde{\alpha}_x) \). From the definition of \( \tilde{\alpha} \) we have
\[ \lim_{\lambda \in \Lambda} \tilde{\alpha}(x_\lambda)(B) = \lim_{\lambda \in \Lambda} e(x_\lambda, \alpha(B)) = e(x, \alpha(B)) = \tilde{\alpha}(x)(B), \]
since \( e : X \times C(X; B) \to B \) is continuous. Thus for each \( B \in B \)
\[ \lim_{\lambda \in \Lambda} \tilde{\alpha}(x_\lambda)(B) = \tilde{\alpha}(x)(B). \]
Thus \( \alpha \) takes convergent nets in \( X \) into strongly convergent nets in \( B \) and hence \( \tilde{\alpha} \) is strongly continuous. □

Since the strong operator and uniform topologies of \( \text{Aut}(B) \) coincide when \( H \) is finite dimensional we obtain

**Corollary 3.10.** Let \( X \) be a compact Hausdorff space, \( H \) a finite-dimensional Hilbert space, \( B \subseteq \mathcal{L}(H) \) and \( \mathcal{A} = C(X; B) \). Then for any \( \alpha \in \text{CF}(\mathcal{A}) \), \( \tilde{\alpha} : X \to \text{Aut}(B) \) is continuous in the uniform topology. □

If the dimension of \( H \) is not finite we have not succeeded in showing \( \tilde{\alpha} : X \to \text{Aut}(B) \) is continuous for all \( \alpha \in \text{CF}(\mathcal{A}) \), in general. But, when \( X \) is separable and \( B = \mathcal{L}(H) \) then we may establish the continuity of \( \tilde{\alpha} \) with the aid of the following theorem of Kallman [14].
Theorem F (Kallman). Let $R$ be any von Neumann algebra, $\varphi_n$ elements of the automorphism group of $R$ $(n > 0)$ such that $\|\varphi_n(T) - T\| \to 0$ $(n \uparrow \infty)$ for all $T \in R$. Then $\|\varphi_n - 1\| \to 0$ $(n \uparrow \infty)$.

Theorem 3.11. Suppose that $X$ is a separable compact Hausdorff space, $B = \mathcal{L}(H)$ for a Hilbert space $H$, and $\mathcal{A} = C(X; B)$. If $\alpha \in \text{CF} (\mathcal{A})$ then $\tilde{\alpha} : X \to \text{Aut} (B)$ is continuous in the uniform operator topology.

Proof. As $X$ is separable it will suffice to show that for each convergent sequence $\{x_n \mid x_n \in X\}$ with limit $x \in X$ we have

$$\lim_{n \to \infty} \|\tilde{\alpha}(x_n) - \tilde{\alpha}(x)\| = 0.$$ 

By replacing $\tilde{\alpha}$ by $\tilde{\alpha}(x)^{-1}\tilde{\alpha}$ we are reduced to considering the case where $\tilde{\alpha}(x) = I$. By Proposition 3.9 we have

$$\lim_{n \to \infty} \|\tilde{\alpha}(x_n)(T) - T\| = 0$$

for all $T \in B$. Applying Kallman's theorem to the von Neumann algebra $B = \mathcal{L}(H)$ now yields the desired conclusion. \hspace{1em} \square

Attempts to extend Theorem 3.11 to arbitrary $C^*$-algebras would seem to depend on extending Kallman's theorem to a more general family of $C^*$-algebras than $\mathcal{L}(H)$.

Conjecture. Let $X$ be a compact Hausdorff space and $H$ a Hilbert space, $B = \mathcal{L}(H)$ and $\mathcal{A} = C(X; B)$. Then for any $\alpha \in \text{CF} (\mathcal{A})$, $\tilde{\alpha} : X \to \text{Aut} (B)$ is continuous in the uniform topology.

While we are unable to settle the continuity of $\sim$ in general we do have:

Theorem 3.12. Let $X$ be a compact Hausdorff space, $H$ a Hilbert space, $B = \mathcal{L}(H)$ and $\mathcal{A} = C(X; B)$. If $\alpha \in \text{CF} (\mathcal{A})$ then $\tilde{\alpha} : X \to \text{Aut} (B)$ is continuous in the uniform operator topology iff $\alpha \in \text{loc-Inn} (\mathcal{A})$.

Remark. It follows from Theorem 3.11 and Corollary 3.10 that $\text{CF} (\mathcal{A}) = \text{loc-Inn} (\mathcal{A})$ when the dimension of $H$ is finite. We conjecture that this equality holds with no restriction on $H$.

The proof of Theorem 3.12 will require some preparation.

Notation. $\mathcal{U}$ is the set of unitary elements of $B$. It is a topological group when equipped with the induced topology from $B$.

Let $\omega : \mathcal{U} \to \text{Aut} (B)$ be the function defined by $\omega(U) = UBU^*$ for all $B \in B$.

It follows from Kaplansky's theorem [3], [4] that $\omega : \mathcal{U} \to \text{Aut} (B)$ is a surjection of abstract groups. The kernel of $\omega$ consists of those unitary elements $U$ such that $UBU^* = B$ for all $B \in B$, i.e. $\ker \omega = \mathcal{U} \cap Z(B)$. Since $Z(B) = C \cdot I$ it follows that $\ker \omega = S^1 I$, where $S^1$ is the circle group.

Combining these observations with some calculations we obtain:
Proposition 3.13. The homomorphism $\omega: \mathcal{U} \to \text{Aut}(B)$ is a continuous-open surjection and induces an isomorphism of topological groups $\tilde{\omega}: \mathcal{U}/S^1 \to \text{Aut}(B)$.

Proof. The fact that $\omega$ is continuous is obvious. To see that $\omega$ is open one may employ Lemma 5 of [13] or the following more elementary computation due to C. L. Fefferman.

Lemma. Let $U \in \mathcal{U}$. Then there exists $t \in S^1$ such that

$$\|tU-I\| \leq 2\pi \|\omega(U)-I\|.$$ 

Proof of Lemma. Let $U \in \mathcal{U}$. By the spectral theorem we have the spectral representation

$$U = \int_0^{2\pi} e^{i\theta} dE_{\theta}.$$ 

Let $e^{i\theta_1}, e^{i\theta_2} \in \sigma(U)$. Let $\varepsilon > 0$ and choose $\varepsilon$-eigenvectors $x, y$ corresponding to $e^{i\theta_1}, e^{i\theta_2}$ respectively. Where we mean that $x$ is an $\varepsilon$-eigenvector if

$$|Ux - e^{i\theta_1}x| < \varepsilon |x|,$$

where $|\cdot|$ denotes the norm in $H$. We may assume that $|x| = 1 = |y|$.

Let $A$ be the linear operator on $H$ defined by $Az = (z, x)y, z \in H$. Then $Ax = y$, and

$$U^*x = U^{-1}x = e^{-i\theta_1}x + z$$

where $|z| \to 0$ as $\varepsilon \to 0$.

$$AU^*x = e^{-i\theta_2}Ax + Az = e^{-i\theta_1}y + Az,$n

$$UAU^*x = e^{-i\theta_1}Uy + w = e^{i\theta_2}y + w,$$

where $|w| \to 0$ as $\varepsilon \to 0$.

By definition

$$\|\omega(U)-I\| = \sup_{\|B\| \leq 1} \|UBU^*-B\| = \sup_{\|B\| \leq 1} \sup_{\|v\| \leq 1} |UBU^{-1}v-Bv|.$$ 

Therefore

$$\|\omega(U)-I\| \geq |UAU^*x-Ax| = |e^{i\theta_2}Ax - Ax + w| \geq \left|e^{i\theta_2} - 1\right| |Ax| - |w| = \left|e^{i\theta_2} - 1\right| |Ax| - |w| = \left|e^{i\theta_2} - e^{-i\theta_1}\right| - |w|.$$ 

Thus

$$\|\omega(U)-I\| \geq |e^{i\theta_2} - e^{-i\theta_1}| - |w|.$$ 

(*)
Denote by $\Delta(U)$ the diameter of the spectrum of $U$, i.e.
$$\Delta(U) = \sup_{\exp(i\theta), \exp(i\theta') \in \sigma(U)} |e^{i\theta} - e^{i\theta'}|.$$ 
Taking the sup of both sides in (*) gives
$$\|\omega(U) - I\| \geq \Delta(U) - \eta$$
for any $\eta > 0$. Therefore
$$\|\omega(U) - I\| \geq \Delta(U).$$
On the other hand, let $t = e^{i\theta_0} \in \sigma(U)$. Then
$$\|U - tI\| = \left\| \int_0^{2\pi} (e^{i\theta} - e^{i\theta_0}) dE_\theta \right\| \leq 2\pi \sup_{\exp(i\theta) \in \sigma(U)} |e^{i\theta} - e^{i\theta_0}| \leq 2\pi \Delta(U).$$
Thus there exists $t \in \sigma(U) \subseteq S^1$, with
$$\|tU - I\| = \|U - tI\| \leq 2\pi \Delta(U) \leq 2\pi \|\omega(U) - I\|. \quad \square$$
With the aid of this lemma we obtain the openness of $\omega$ as follows. Since $\omega$ is a homomorphism it suffices to show that $\omega$ is open at $I$. Let
$$N_\varepsilon = \{U \in \mathcal{U} \mid \|U - I\| < \varepsilon\}$$
be a basic open neighbourhood of $I \in \mathcal{U}$. We wish to exhibit a small open neighbourhood of $I \in \text{Aut}(B)$ contained in $\omega(N_\varepsilon)$. Consider the open set
$$\mathcal{O}_\varepsilon = \{T \in \text{Aut}(B) \mid \|T - I\| < \varepsilon/3\pi\}.$$ 
Let $T \in \mathcal{O}_\varepsilon$. There exists $U \in \mathcal{U}$ such that $\omega(U) = T$, as we remarked above. By the Lemma above there exists $t \in S^1$ such that
$$\|tU - I\| \leq 2\pi \|\omega(U) - I\| = 2\pi \|T - I\| < \varepsilon.$$ 
But $\omega(tU) = \omega(U)$. Therefore $T \in \omega(N_\varepsilon)$. Hence $\mathcal{O}_\varepsilon \subseteq \omega(N_\varepsilon)$ as required.
Thus the induced isomorphism $\tilde{\omega}: \mathcal{U} / S^1 \rightarrow \text{Aut}(B)$ is an isomorphism of topological groups, where $\mathcal{U} / S^1$ is equipped with the quotient topology.  \(\square\)

**Notation.** If $X$ is a compact Hausdorff space and $G$ is a topological group we denote by $C(X, G)$ the group of all continuous functions $X \rightarrow G$. The group operation is the pointwise product. We equip $C(X, G)$ with the compact-open topology making it into a topological group.

**Corollary 3.14.** Let $X$ be a compact Hausdorff space, $H$ a Hilbert space and $B = \mathcal{L}(H)$. Then $\omega$ induces a natural isomorphism of topological groups
$$\omega_*: C(X; \mathcal{U} / S^1) \xrightarrow{\sim} C(X, \text{Aut}(B)). \quad \square$$
Recollections from algebraic topology. We assume that the reader is familiar with basic ideas of fiber bundle theory [11, pp. 39–41], [23, pp. 432–437], [24, Part I]. The following fundamental property of principal $G$-bundles will be used later.

**Covering Homotopy Theorem** [23, Theorem 7.2.6] or [24, Theorem 11.3].

Let $G$ be a topological group and $(E, \mu)$ a principal $G$-bundle. Let $X$ be a topological space and suppose given

1. $f_t: X \to E/G$, $t \in [0, 1]$ a homotopy class of mappings, and
2. $f_0: X \to E$ a map such that $pf_0 = f_0$. Then there exists a homotopy $f_t: X \to E$, $t \in [0, 1]$, of $f_0$, such that the following diagram commutes for all $t \in [0, 1]$, i.e. $pf_t = f_t$ for all $t \in [0, 1]$.

![Diagram](image)

Let $H$ be a (complex) Hilbert space, $B = L(H)$ and $\mathcal{U}$ the unitary elements in $B$. Then

$$p: \mathcal{U} \to \mathcal{U}/S^1$$

is a principal $S^1$-bundle, where $S^1 = \mathbb{Z} \mathcal{U}$ acts on $\mathcal{U}$ by left translation [11, p. 41].

**Lemma 3.15.** Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra. Then an element $f \in C(X; B)$ is unitary iff $f(x)$ is unitary in $B$ for all $x \in X$. □

**Proposition 3.16.** Let $X$ be a compact Hausdorff space, $H$ a Hilbert space, $B = L(H)$ and $\mathcal{U} = C(X; B)$. Then $\alpha \in \text{Aut } (\mathcal{U})$ is inner iff there exists a map $\hat{\alpha}: X \to \mathcal{U}$ such that $\omega \hat{\alpha} = \hat{\alpha}: X \to \text{Aut } (B)$.

**Proof.** Suppose that $\alpha$ is inner. Then from Lemma 3.15 it follows that there exists $\hat{\alpha} \in \mathcal{U}$, $\hat{\alpha}: X \to \mathcal{U}$ such that $\alpha(f) = \hat{\alpha} f \hat{\alpha}^*$ for all $f \in C(X; B)$. Let $B \in B$ and $x \in X$. Then from the definition of $\hat{\alpha}$ we have

$$\tilde{\alpha}(x)(B) = \alpha(B)(x) = \hat{\alpha}(x)B(x)\hat{\alpha}(x)^* = \hat{\alpha}(x)B\hat{\alpha}(x)^*.$$

Thus for each $x \in X$ the automorphism $\tilde{\alpha}(x) \in \text{Aut } (B)$ agrees with the inner automorphism $B \to \tilde{\alpha}(x)B\tilde{\alpha}(x)^*$. Recalling our identification $\omega: \mathcal{U}/S^1 \cong \text{Aut } (B)$ this means that the diagram

![Diagram](image)

commutes, i.e. $\omega \hat{\alpha} = \tilde{\alpha}$. 
Conversely, suppose that there exists \( \hat{\alpha} : X \to \mathcal{A} \) such that \( \omega \hat{\alpha} = \hat{\alpha} : X \to \text{Aut}(\mathcal{B}) \).
Define an automorphism
\[
\beta : \mathcal{A} \to \mathcal{A} \mid \beta(f)(x) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^*.
\]
This is the inner automorphism of \( \mathcal{A} \) determined by \( \hat{\alpha} \in C(X, \mathcal{B}) = \mathcal{A} \).

We assert that \( \beta = \alpha \). For suppose that \( f \in C(X, \mathcal{B}) \). Let \( x \in X \). Then \( e_x(f - (f(x))^*) = 0 \). Therefore since \( \alpha, \beta \in \text{CF}(\mathcal{A}) \) it follows from Lemma 3.2 that
\[
e_x(\alpha f - \alpha((f(x))^*)) = 0
\]
and
\[
e_x(\beta f - \beta((f(x))^*)) = 0.
\]
Thus
\[
(\alpha f)(x) = \alpha((f(x))^*)(x) = \hat{\alpha}(x)(f(x)),
\]
\[
(\beta f)(x) = \beta((f(x))^*)(x) = \beta(x)(f(x)).
\]

Now \( \beta \) being inner we have by direct computation (compare above) that
\[
\beta(x)(f(x)) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^*.
\]
But by hypothesis
\[
\hat{\alpha}(x)f(x)\hat{\alpha}(x)^* = \hat{\alpha}(x)(f(x)).
\]
Therefore, combining these equalities yields
\[
(\alpha f)(x) = \hat{\alpha}(x)(f(x)) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^* = \beta(x)(f(x))(\beta f)(x).
\]
Hence \( (\alpha f)(x) = (\beta f)(x) \) for all \( x \in X \) and thus \( \alpha f = \beta f \). Since \( f \in C(X, \mathcal{B}) \) was arbitrary it follows that \( \alpha f = \beta f = \hat{\alpha}f\hat{\alpha}^* \) for all \( f \in C(X, \mathcal{B}) \) and hence \( \alpha \) is inner. □

**Proof of Theorem 3.12.** Suppose that \( \alpha \in \text{CF}(\mathcal{A}) \) is locally inner. Let \( S_1, \ldots, S_N \) be an open cover of \( X \) such that \( \alpha|_{S_i} \) is inner, \( i = 1, \ldots, N \).

From Proposition 3.3 it follows that we have a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & \text{Aut}(\mathcal{B}) \\
\text{inclusion} & & \\
S_i & \xrightarrow{(\alpha|_{S_i})^*} & \mathcal{A} \\
\end{array}
\]
and thus \( \alpha|_{S_i} = (\alpha|_{S_i})^* \). Since \( \alpha|_{S_i} \) is an inner automorphism of \( C(S_i, \mathcal{B}) \) it follows from Proposition 3.16 that there is a continuous function \( (\alpha|_{S_i})^* : \bar{S}_i \to \mathcal{A} \) such that the diagram
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\omega} & \text{Aut}(\mathcal{B}) \\
(\alpha|_{S_i})^* & \xrightarrow{\alpha} & \\
\bar{S}_i & \xrightarrow{(\alpha|_{S_i})^*} & \mathcal{A} \\
\end{array}
\]
commutes. Thus \((\alpha|_{S_i})^{-1} : S_i \to \text{Aut}(B)\) is the composition of two continuous maps and hence is continuous. Therefore \(\alpha|_{S_i} : S_i \to \text{Aut}(B)\) is continuous. Since \(S_1, \ldots, S_N\) is an open cover of \(X\) and \(\alpha|_{S_1}, \ldots, \alpha|_{S_N}\) are continuous, it follows that \(\alpha\) is continuous.

Conversely suppose that \(\alpha : X \to \text{Aut}(B)\) is continuous. Under the identification \(\bar{\omega} : \mathcal{U}/S^1 = \text{Aut}(B)\) the mapping \(\omega : \mathcal{U} \to \text{Aut}(B)\) becomes a locally trivial fiber bundle. Thus there exists an open cover \(\{\mathcal{O}_\lambda | \lambda \in \Lambda\}\) of \(\text{Aut}(B)\) and continuous functions \(s_\lambda : \mathcal{O}_\lambda \to \mathcal{U}\) such that \(\omega s_\lambda = 1 : \mathcal{O}_\lambda \to \mathcal{O}_\lambda\).

Since \(\alpha\) is continuous \(\{\alpha^{-1}\mathcal{O}_\lambda | \lambda \in \Lambda\}\) is an open cover of \(X\). Since \(X\) is compact Hausdorff we may select a finite open covering \(S_1, \ldots, S_N\) of \(X\) with \(S_i \subseteq \alpha^{-1}\mathcal{O}_{\lambda_i}, i = 1, \ldots, N\), for some \(\lambda_1, \ldots, \lambda_N \in \Lambda\). (This process is called shrinking the cover.) Therefore \(\alpha(S_i) \subseteq \mathcal{O}_{\lambda_i}\).

Define \(\tilde{\alpha}_i : S_i \to \mathcal{U}\) by \(\tilde{\alpha}_i = s_{\lambda_i}\alpha|_{S_i}\). Then
\[
(\alpha|_{S_i})^{-1} = \tilde{\alpha}_i|_{S_i} = \omega(s_{\lambda_i}\alpha|_{S_i}) = \omega\tilde{\alpha}_i.
\]
Hence by Proposition 3.16 \(\alpha|_{S_i}\) is inner. Therefore \(S_1, \ldots, S_n\) is an open cover of \(X\) such that \(\alpha|_{S_1}, \ldots, \alpha|_{S_n}\) are inner and hence \(\alpha\) is locally inner. □

**Proof of Theorem 3.5.** It follows from Theorem 3.12 that we may define a function

\[
\sim : \text{loc-Inn}(\mathfrak{g}) \to C(X, \text{Aut}(B))
\]

by \(\alpha \mapsto \tilde{\alpha}\).

To show that \(\sim\) is a homomorphism of abstract groups recall that the group operation on \(C(X, \text{Aut}(B))\) is pointwise multiplication. Thus we need only show that for each \(x \in X\), \(\alpha, \beta \in \text{loc-Inn}(\mathfrak{g})\) that \(((\beta \alpha)^{-1})(x) = \beta(x)\alpha(x)\). But since \(\text{loc-Inn}(\mathfrak{g}) \subseteq \text{CF}(\mathfrak{g})\), this follows from Lemma 3.7.

The continuity and openness of \(\sim\) are exercises in the use of the compact-open topology.

To prove that \(\sim\) is an isomorphism of abstract groups we define a function

\[
\sim : C(X, \text{Aut}(B)) \to \text{loc-Inn}(\mathfrak{g})
\]
as follows.

Let \(\varphi \in C(X, \text{Aut}(B))\). For any \(f \in C(X, B)\) define \(\hat{\varphi}(f)(x) = (\varphi(x))(f(x))\), for any \(x \in X\). Note that \(\hat{\varphi}\) is the composite

\[
X \xrightarrow{\text{diag}} X \times X \xrightarrow{f \times \varphi} B \times \text{Aut}(B) \xrightarrow{e} B
\]

and hence is continuous (compare Proposition 3.1). Thus \(\hat{\varphi}\) defines a function \(\hat{\varphi} : C(X, B) \to C(X, B)\). Direct computation shows that \(\hat{\varphi} \in \text{Aut}(\mathfrak{g})\). If \(f \in Z(\mathfrak{g})\), then there exists a continuous function \(g : X \to C\) such that \(f = gI\). Thus for any \(x \in X\)

\[
(\hat{\varphi}(f))(x) = \varphi(x)(f(x)) = \varphi(x)(g(x)I) = g(x)\varphi(x)I = g(x)I = f(x).
\]
Therefore $\tilde{\phi}(f) = f$ and $\hat{\phi} \in \text{CF}(\mathcal{A})$.

For any $B \in \mathcal{B}$ and $x \in X$ we have

$$\tilde{\phi}(x)(B) = \phi(B)(x) = \varphi(x)(B(x)) = \varphi(x)(B).$$

and therefore

$$\tilde{\phi} = \varphi: X \to \text{Aut}(B).$$

Hence $\tilde{\phi}$ is a continuous function and therefore $\hat{\phi} \in \text{loc-Inn}(\mathcal{A})$ by Theorem 3.12.

This defines $\sim$.

Routine calculation shows that

$$\sim: C(X, \text{Aut}(B)) \to \text{loc-Inn}(\mathcal{A})$$

is a homomorphism of abstract groups.

We have already seen that $\tilde{\phi} = \varphi$ for any $\varphi \in C(X, \text{Aut}(B))$. If $\beta \in \text{loc-Inn}(\mathcal{A})$ then for any $f \in C(X, B)$, $x \in X$,

$$e_x(f - (f(x))\sim) = 0.$$

Thus (compare the proof of Proposition 3.16) by Lemma 3.2,

$$(\beta f)(x) = \beta(f(x))\sim(x).$$

Therefore from the definitions of $\hat{\sim}$ and $\tilde{\sim}$ we have for any $\alpha \in \text{loc-Inn}(\mathcal{A})$

$$(\alpha \tilde{f})(x) = \tilde{\alpha}(f(x))\sim(x) = \tilde{\alpha}(x)((f(x))\sim(x)) = \tilde{\alpha}(x)(f(x)) = \alpha(f(x))\sim(x) = (\alpha f)(x).$$

Therefore $\tilde{\alpha} \tilde{f} = \alpha f$ and hence $\hat{\sim} = \hat{\alpha}$. Therefore $\sim$ and $\tilde{\sim}$ are inverse isomorphisms of abstract groups and the result follows. □

**Corollary 3.17.** Let $X$ be a compact Hausdorff space, $H$ a Hilbert space, $B = \mathcal{L}(H)$ and $\mathcal{A} = C(X, B)$. Then there exists a natural isomorphism of topological groups $\sim: \text{loc-Inn}(\mathcal{A}) \cong C(X, \mathcal{U}/S^1)$; where $\mathcal{U} \subset B$ is the unitary group and $S^1 = Z(\mathcal{U})$.

**Proof.** This follows from Theorem 3.5 and Proposition 3.13. □

**Corollary 3.18.** Let $X$ be a separable compact Hausdorff space, $H$ a Hilbert space, $B = \mathcal{L}(H)$ and $\mathcal{A} = C(X; B)$. Then $\text{CF}(\mathcal{A}) = \text{loc-Inn}(\mathcal{A})$.

**Proof.** This follows from Theorems 3.11 and 3.12. □

**IV. A special case.** In this section we will apply the results of the previous section to the algebra $C(X; \mathcal{L}(H))$ when $H$ is infinite dimensional. We shall need the following remarkable theorem of Kuiper.
THEOREM G (KUIPER [16]). Let $H$ be an infinite-dimensional Hilbert space and $\mathcal{U}$ the unitary group of $H$. Then $\mathcal{U}$ is contractible in the uniform topology.

NOTATION. If $X$ and $Y$ are topological spaces we denote by $[X, Y]$ the set of homotopy classes of maps $f: X \to Y$. The homotopy class of $f$ is denoted by $[f]$.

If $G$ is a topological group then $[X, G]$ may be given the structure of discrete group by pointwise multiplication (of representatives) of homotopy classes.

Recollection from algebraic topology. If $\pi$ is an abelian group and $n$ is a positive integer, an Eilenberg-Mac Lane space of type $(\pi, n)$ is a topological space $K(\pi, n)$ such that

$$\pi_i(K(\pi, n)) = \pi \text{ if } i = n,$$

$$= 0 \text{ if } i \neq n,$$

where $\pi_i(\ )$ denotes the $i$th-homotopy group [23].

For example, the circle, $S^1$, is a $K(Z, 1)$-space, where $Z$ is the additive group of integers.

The spaces $K(\pi, n)$ have a natural abelian group structure and thus for any space $X$, $[X, K(\pi, n)]$ is an abelian group. In fact for any compact space $X$ there is a natural isomorphism [23] of abelian groups,

$$[X, K(\pi, n)] = \check{H}^n(X; \pi),$$

where $\check{H}^n(X, \pi)$ denotes the $n$th Čech cohomology group of $X$ with coefficients in $\pi$. If $X$ is a "nice" space (for example, a cell complex with the weak topology, i.e. a CW-complex) then

$$[X, K(\pi, n)] = H^n(X; \pi),$$

where $H^n(X; \pi)$ denotes the $n$th singular cohomology group of $X$ with coefficients in $\pi$. More precisely, for such nice spaces, Čech and singular cohomologies are naturally isomorphic [28].

If $G$ is a topological group and $p: E \to E/G$ is a principal $G$-bundle then there is an exact homotopy sequence [23, Theorem 7.2.10], [24, Theorem 17.4],

$$\cdots \to \pi_i(G) \to \pi_i(E) \xrightarrow{p_*} \pi_i(E/G) \xrightarrow{\partial} \pi_{i-1}(G) \to \cdots.$$ 

If $E$ is contractible then $\pi_i(E) = 0$ for all $i \geq 0$, and hence

$$\partial: \pi_i(E/G) \to \pi_{i-1}(G)$$

is an isomorphism for all $i > 0$.

We may apply these considerations to the principal $S^1$-bundle $\mathcal{U} \to \mathcal{U}/S^1$. Since we have assumed $H$ to be infinite dimensional $\mathcal{U}$ is contractible by the theorem of Kuiper. Therefore

$$\pi_i(\mathcal{U}/S^1) = \pi_{i-1}(S^1).$$
for all \( i \geq 1 \). Since \( S^1 \) is a \( K(Z, 1) \)-space we obtain
\[
\pi_i(\mathcal{U}/S^1) = \begin{cases} 
Z & \text{if } i = 2, \\
0 & \text{if } i \neq 2.
\end{cases}
\]
Thus \( \mathcal{U}/S^1 \) is an Eilenberg-Mac Lane space of type \((Z, 2)\).

**Notation.** Let \( X \) be a topological space and \( G \) be a topological group. Denote by \( C_0(X; G) \) the subgroup of \( C(X; G) \) consisting of the null-homotopic maps. If \( X \) is compact Hausdorff then \( C_0(X; G) \) is just the identity component of \( C(X; G) \) with the compact open topology.

\( C_0(X; G) \) is a normal subgroup of \( C(X; G) \) and there is a natural isomorphism of discrete groups,
\[
C(X; G)/C_0(X; G) = [X, G] \quad [23].
\]

**Theorem 4.1.** Let \( X \) be a compact Hausdorff space, \( H \) an infinite-dimensional Hilbert space, \( \mathcal{B} = \mathcal{L}(H) \) and \( \mathcal{A} = C(X; \mathcal{B}) \). Then there is a natural isomorphism of groups \( \text{loc-Inn}(\mathcal{A})/\text{Inn}(\mathcal{A}) = H^2(X; Z) \). If moreover \( X \) is separable, we have the isomorphism of groups
\[
\text{loc-Inn}(\mathcal{A})/\text{Inn}(\mathcal{A}) = H^2(X; Z) = \text{CF}(\mathcal{A})/\text{Inn}(\mathcal{A}).
\]

**Proof.** By 3.18, \( \text{loc-Inn}(\mathcal{A}) = C(X; \mathcal{U}/S^1) \). By Proposition 3.16,
\[
\text{Inn}(\mathcal{A}) = p_*C(X; \mathcal{U}).
\]
Now we assert that
\[
p_*C(X; \mathcal{U}) = C_0(X; \mathcal{U}/S^1)
\]
where \( C_0(X; \mathcal{U}/S^1) \) is the subgroup of \( C(X; \mathcal{U}/S^1) \) consisting of the null-homotopic maps.

For suppose \( \varphi: X \to \mathcal{U}/S^1 \) is null homotopic. Choose a homotopy \( \varphi_t: X \to \mathcal{U}/S^1, \ t \in [0, 1], \) with \( \varphi_1 = \varphi \) and \( \varphi_0 = \) the constant map at \( I \in \mathcal{U}/S^1 \). Let \( \bar{\varphi}_0: X \to \mathcal{U} \) be the constant map at \( I \in \mathcal{U} \). Clearly \( p\bar{\varphi}_0 = \varphi_0 \). Therefore by the covering homotopy theorem there exists a homotopy \( \bar{\varphi}_t: X \to \mathcal{U} \) with \( p\bar{\varphi}_t = \varphi_t \) for all \( t \in [0, 1] \). Let \( \bar{\varphi} = \bar{\varphi}_1 \). Then \( p\bar{\varphi} = \varphi \) and hence \( \varphi \in p_*C(X; \mathcal{U}) \). Thus \( p_*C(X; \mathcal{U}) \supset C_0(X; \mathcal{U}/S^1) \).

Conversely if \( \varphi \in p_*C(X; \mathcal{U}) \) then we may choose \( \bar{\varphi} \in C(X; \mathcal{U}) \) with \( p\bar{\varphi} = \varphi \). Since \( \mathcal{U} \) is contractible by Kuiper's theorem, it follows that \( \bar{\varphi} \) is null homotopic. Let \( \bar{\varphi}_t: X \to \mathcal{U} \) be a homotopy with \( \bar{\varphi}_0 = \) constant map at \( I \in \mathcal{U} \) and \( \bar{\varphi}_1 = \bar{\varphi} \). Then \( \bar{\varphi}_t = p\bar{\varphi}_t \) is a homotopy from \( \varphi \) to the constant map at \( I \in \mathcal{U}/S^1 \). Thus \( \varphi \) is null homotopic and hence \( p_*C(X; \mathcal{U}) \supset C_0(X; \mathcal{U}) \). Combining this with 3.16 and 3.17 we obtain
\[
\text{loc-Inn}(\mathcal{A})/\text{Inn}(\mathcal{A}) = C(X; \mathcal{U}/S^1)/C_0(X; \mathcal{U}/S^1)
\]
\[
= [X, \mathcal{U}/S^1].
\]
Since \( \mathcal{U}/S^1 \) is a \( K(Z, 2) \)-space this yields
\[
\text{loc-Inn}(\mathcal{A})/\text{Inn}(\mathcal{A}) = H^2(X; Z)
\]
as claimed. Using 3.18 we obtain the case where \( X \) is separable. \( \square \)
Remark. E. C. Lance has shown [18] that when $X$ is separable $\pi(\mathfrak{A})/\text{Inn}(\mathfrak{A}) = \hat{H}^2(X; \mathbb{Z})$. Thus we obtain

Corollary 4.2. If $X$ is a separable compact Hausdorff space, $B = \mathcal{L}(H)$ for an infinite-dimensional Hilbert space $H$, and $\mathfrak{A} = C(X; B)$, then

$$\pi(\mathfrak{A}) = \text{CF}(\mathfrak{A}) = \text{loc-Inn}(\mathfrak{A}). \quad \square$$

Also,

Theorem 4.3. If $X$ is a separable compact Hausdorff space, $B = \mathcal{L}(H)$ for an infinite-dimensional Hilbert space $H$, and $\mathfrak{A} = C(X; B)$ then $\text{Aut}_0(\mathfrak{A}) = \text{Inn}(\mathfrak{A})$ where $\text{Aut}_0(\mathfrak{A})$ is the identity component of $\text{Aut}(\mathfrak{A})$.

Proof. Let us denote by $C_0(X; \mathfrak{U}/S^1)$ and $\text{CF}_0(\mathfrak{A})$ the identity components of topological groups $C(X; \mathfrak{U}/S^1)$ and $\text{CF}(\mathfrak{A})$, respectively. First we want to show that

$$C_0(X; \mathfrak{U}/S^1) = \text{Inn}(\mathfrak{A}). \quad (1)$$

By 3.16 we know $\text{Inn}(\mathfrak{A}) = p_* C(X; \mathfrak{U})$. Also in the proof of 4.1 we saw $p_* C(X; \mathfrak{U})$ is null homotopic to $C(X; \mathfrak{U}/S^1)$. But it is well known that [23] $f: X \to \mathfrak{U}/S^1$ is null homotopic if and only if $f$ belongs to the identity component of $C(X; \mathfrak{U}/S^1)$, where $C(X; \mathfrak{U}/S^1)$ is equipped with the compact-open topology. Thus $\text{Inn}(\mathfrak{A}) = C_0(X; \mathfrak{U}/S^1)$, obtaining (1).

Next we note

$$\text{CF}_0(\mathfrak{A}) = \text{Aut}_0(\mathfrak{A}). \quad (2)$$

For by [13] and the remark in §II we have

$$\text{Aut}_0(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \text{CF}(\mathfrak{A}) \subseteq \text{Aut}(\mathfrak{A}).$$

So $\text{CF}(\mathfrak{A})$ is open, and $\text{Aut}_0(\mathfrak{A}) = \text{CF}_0(\mathfrak{A})$.

Hence we only have to show $\text{Inn}(\mathfrak{A}) = \text{CF}_0(\mathfrak{A})$ in order to complete the proof of the theorem. But by Corollaries 3.17 and 3.18 we know that

$$C(X; \mathfrak{U}/S^1) = \text{loc-Inn}(\mathfrak{A}) = \text{CF}(\mathfrak{A}).$$

Hence $\text{CF}_0(\mathfrak{A}) = C_0(X; \mathfrak{U}/S^1)$. But by (1), we obtain

$$\text{CF}_0(\mathfrak{A}) = C_0(X; \mathfrak{U}/S^1) = \text{Inn}(\mathfrak{A}). \quad \square$$

Remark. Theorem 4.3 has been obtained by Lance also, see [18].

More generally, when $X$ is not separable we have the following:

If $\mathfrak{A}$ is a $C^*$-algebra recall that $\text{Aut}_0(\mathfrak{A})$ is the identity component of $\text{Aut}(\mathfrak{A})$ in the norm topology.

Proposition 4.4. Let $X$ be a compact Hausdorff space, $H$ an infinite-dimensional Hilbert space, $B = \mathcal{L}(H)$ and $\mathfrak{A} = C(X; B)$. Then $\text{Inn}(\mathfrak{A}) \subseteq \text{Aut}_0(\mathfrak{A})$. 
Proof. By Proposition 3.15 and Proposition 3.14 \( \sim \) induces an isomorphism of topological groups (compare with the proof of Theorem 4.1)

\[ \sim : \text{Inn} (\mathfrak{U}) \cong p_* C(X; \mathfrak{U}). \]

By Kuiper's theorem \( \mathfrak{U} \) is contractible and hence \( C(X; \mathfrak{U}) \) is connected [22]. Therefore \( p_* C(X; \mathfrak{U}) \) is connected and hence \( \text{Inn} (\mathfrak{U}) \) is connected. Thus we must have \( \text{Inn} (\mathfrak{U}) \subseteq \text{Aut}_0 (\mathfrak{U}) \).

Remark. Proposition 4.4 is in striking contrast with the case when \( \dim H \) is finite. In [13, Example d] it is shown that \( \text{Aut}_0 (\mathfrak{U}) \subseteq \text{Inn} (\mathfrak{U}) \) when \( \dim H \) is finite.

Conjecture. If \( X \) is a compact Hausdorff space, \( H \) an infinite-dimensional Hilbert space, \( B = L (H) \) and \( \mathfrak{U} = C(X; B) \) then \( \text{Inn} (\mathfrak{U}) = \text{Aut}_0 (\mathfrak{U}) \).

This conjecture is closely related to (and would follow from) the continuity of \( \bar{a} \).

Corollary 4.5. Let \( H \) be an infinite-dimensional Hilbert space, \( B = L(H) \), and \( G \) a finitely generated abelian group. Then there exists a separable compact Hausdorff space \( X \) such that \( G = CF (\mathfrak{U})/\text{Inn} (\mathfrak{U}) = \text{loc-Inn} (\mathfrak{U})/\text{Inn} (\mathfrak{U}) \), where \( \mathfrak{U} = C(X; B) \).

Proof. According to Theorem 4.1 we need only construct a separable compact Hausdorff space \( X \) with \( \hat{A}^2 (X; Z) = G \). However this is well-known algebraic topology [23, Example C.6, p. 206].

Corollary 4.6. Let \( X \) be a compact Hausdorff space, \( H \) an infinite-dimensional Hilbert space, \( B = L (H) \) and \( \mathfrak{U} = C(X; B) \). Then \( \text{loc-Inn} (\mathfrak{U})/\text{Inn} (\mathfrak{U}) \) is abelian. If moreover \( X \) is separable then \( CF (\mathfrak{U})/\text{Inn} (\mathfrak{U}) \) is always an abelian group.

Proof. \( \hat{A}^2 (X; Z) \) is always abelian.

Remark. Corollary 4.6 is in striking contrast to the situation when \( H \) is finite dimensional. For in the finite-dimensional case \( \text{loc-Inn} (\mathfrak{U}) = CF (\mathfrak{U}) \) and the examples constructed in [13] show that \( CF (\mathfrak{U})/\text{Inn} (\mathfrak{U}) \) need not be abelian.

V. Miscellaneous results. In this section we collect various miscellaneous results concerning automorphisms of the \( C^* \)-algebras \( C(X; B) \).

The \( C(X) \)-module structure. Let \( X \) be a compact Hausdorff space and \( B \) a \( C^* \)-algebra. Denote by \( C(X) \) the \( C^* \)-algebra \( C(X; C) \). There is a natural map

\[ \mu : C(X) \otimes^* C(X; B) \to C(X; B) \]

given by the \( * \)-linear continuous extension of the mapping

\[ [\mu (f \otimes g)](x) = f(x) \cdot g(x) \in B \]

where \( f \in C(X) \) and \( g \in C(X; B) \). We denote \( \mu (f \otimes g) \) by \( f \circ g \). This provides \( C(X; B) \) with the structure of an (continuous) algebra over the algebra \( C(X) \).

Proposition 5.1. Let \( X \) be a compact Hausdorff space, \( B \) a \( C^* \)-algebra with \( Z(B) = C \cdot I \). An automorphism of \( C(X; B) \) is center-fixing iff it is an automorphism of \( C(X) \)-algebras.
Proof. Suppose \( \alpha \in \text{CF} (C(X; B)) \). Let \( f \in C(X) \), \( g \in C(X; B) \). Then
\[
\alpha(f \circ g) = \alpha((f \circ g) g).
\]
Since \( f \circ I \in Z(C(X; B)) \) we obtain in addition
\[
\alpha(f \circ g) = \alpha((f \circ I) g) = \alpha((f \circ I) g) = f \circ \alpha(g)
\]
and thus \( \alpha \) is an automorphism of \( C(X) \)-modules. Since it is also a \( C^* \)-algebra isomorphism it is an isomorphism of \( C(X) \)-algebras.

Next suppose that \( \alpha \) is an automorphism of \( C(X) \)-algebras. Then \( \alpha \) is an automorphism of the \( C^* \)-algebra \( C(X; B) \). Let \( f \in Z(C(X; B)) \), then there exists \( g : X \to C \) such that \( f = g \circ I \) and hence
\[
\alpha(f) = \alpha(g \circ I) = g \circ \alpha(I) = g \circ I = f
\]
and \( \alpha \) is center-fixing as required. □

Carefully ideal preserving automorphisms.

Definition. Let \( \mathfrak{A} \) be a \( C^* \)-algebra. An automorphism \( \alpha \) of \( \mathfrak{A} \) is said to be ideal preserving iff \( \alpha(J) \subseteq J \) for every closed two sided ideal \( J \) of \( \mathfrak{A} \).

An automorphism \( \alpha \) of \( \mathfrak{A} \) is said to be carefully ideal preserving iff \( \alpha(J) = J \) for each closed two sided ideal \( J \) in \( \mathfrak{A} \).

In this subsection we study the relation between center-fixing and ideal preserving automorphisms of the \( C^* \)-algebras \( C(X; B) \).

We will use portions of the theory of §111.

Notation. Let \( \mathfrak{A} \) be a \( C^* \)-algebra. Denote by \( \tau(\mathfrak{A}) \) the set of all ideal preserving automorphisms of \( \mathfrak{A} \). Note that \( \tau(\mathfrak{A}) \) is only a subsemigroup of \( \text{Aut} (\mathfrak{A}) \).

Denote by \( \tau_0(\mathfrak{A}) \) the set of all carefully ideal preserving automorphisms of \( \mathfrak{A} \). Note that \( \tau_0(\mathfrak{A}) \) is a subgroup of \( \text{Aut} (\mathfrak{A}) \) [17].

Let \( X \) be a compact Hausdorff space and \( B \) a \( C^* \)-algebra. Let \( J \subseteq B \) be a closed two sided ideal in \( B \) and \( x \in X \) a fixed point. Define \( J(x) \) in \( C(X; B) \) by
\[
J(x) = \{ f \in C(X; B) \mid f(x) \in J \}.
\]
Then one readily checks that \( J(x) \) is a closed two sided ideal in \( C(X; B) \). These ideals may be used to describe the general form of the closed two sided ideals in \( C(X; B) \). For the proof of the next theorem we refer to [5], [15].

Theorem H (I. Kaplansky). Let \( X \) be a compact Hausdorff space and \( B \) a \( C^* \)-algebra. If \( J \subseteq C(X; B) \) is a closed two sided ideal then there exists a closed subset \( S \subseteq X \) and for each \( x \in S \) a closed two sided ideal \( J_x \subseteq B \) such that
\[
J = \bigcap_{x \in S} J_x(x).
\]

Proposition 5.2. Let \( X \) be a compact Hausdorff space, \( B \) a \( C^* \)-algebra, and \( \mathfrak{A} = C(X; B) \). If \( \alpha \in \text{CF} (\mathfrak{A}) \) and \( x \in X \) then
\[
[\alpha(f)](x) = [\alpha(f(x))]^{-1}(x)
\]
for all \( f \in \mathfrak{A} \).
Proof. Let $f \in \mathcal{A}$. Then $\epsilon_x(f-(f(x))^\sim) = 0$. Thus by Lemma 3.2

$$\epsilon_x[\alpha(f-(f(x))^\sim)] = 0.$$ 

But this is by definition

$$[\alpha(f)](x) - [\alpha(f(x))]^\sim(x) = 0,$$

and the result follows. \qed

Remark. This result has been used at several key points in §III and points out the "local" nature of center-fixing automorphisms of $C(X; B)$.

Lemma 5.3. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra and $\mathcal{A} = C(X; B)$. Let $J \subseteq B$ be a closed two sided ideal and $x \in X$ a fixed point of $X$. Assume in addition that $\text{CF}(B) \subseteq \tau_0(B)$. Then for any $\alpha \in \text{CF}(\mathcal{A})$, $\alpha(J(x)) = J(x)$.

Proof. Let $f \in J(x)$. By Proposition 5.2

$$(\alpha f)(x) = \alpha(f(x))^\sim(x) = \tilde{\alpha}_x(f(x)).$$

It follows directly from the definition of the function $\sim$ given in §III that $\tilde{\alpha}_x$ is a center-fixing automorphism of $B$. Since we have assumed that $\text{CF}(B) \subseteq \tau_0(B)$ it follows that $\tilde{\alpha}_x(f(x)) \in J$. Therefore $(\alpha f)(x) \in J$ and hence $f \in J(x)$. Thus $\alpha(J(x)) \subseteq J(x)$. But $\alpha^{-1}$ is also in $\text{CF}(\mathcal{A})$ and hence $\alpha^{-1}(J(x)) \subseteq J(x)$. Applying $\alpha$ to both sides of this latter inclusion gives $J(x) \subseteq \alpha(J(x))$, and hence $\alpha(J(x)) = J(x)$ as required. \qed

Remarks. (1) The hypothesis $\text{CF}(B) \subseteq \tau_0(B)$ is often satisfied. For example if $B = L(H)$, where $H$ is a Hilbert space, then every automorphism of $B$ is inner, and hence carefully ideal preserving. If $B$ has only one closed two sided proper ideal, e.g. $B = \text{the } C^*$-algebra of all compact operators on a Hilbert space $H$ with identity adjoined, then every automorphism of $B$ is carefully ideal preserving.

(2) Setting $X =$ point we see that the assumption $\text{CF}(B) \subseteq \tau_0(B)$ is clearly necessary for the conclusion of Lemma 5.3 to hold.

The next proposition provides numerous additional examples of $C^*$-algebras such that $\text{CF}(\mathcal{A}) \subseteq \tau_0(\mathcal{A})$.

Proposition 5.4. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra and $\mathcal{A} = C(X; B)$. Assume in addition that $\text{CF}(B) \subseteq \tau_0(B)$. Then $\text{CF}(\mathcal{A}) \subseteq \tau_0(\mathcal{A})$.

Proof. It is immediate from Theorem H and Lemma 5.3. \qed

Lemma 5.5. Let $X$ be a compact Hausdorff space, $B$ a $C^*$-algebra, $\mathcal{A} = C(X; B)$, and $\alpha \in \tau(\mathcal{A})$. Then for any $f \in \mathcal{A}$

$$(\alpha f)(x) = \alpha(f(x))^\sim(x),$$

for any $x \in X$.

Proof. Let $x \in X$ and set $J = \ker \epsilon_x$. Then $J$ is a closed two sided ideal in $\mathcal{A}$. Let $f \in \mathcal{A}$. Then $f-(f(x))^\sim \in J$ and since $\alpha$ is ideal preserving $\alpha(f-(f(x))^\sim) \in J$, i.e.

$$\alpha(f)(x) = \alpha(f(x))^\sim(x),$$

as required. \qed
Lemma 5.6. Let $X$ be a compact Hausdorff space, $B$ a C*-algebra and $\mathcal{A} = C(X; B)$. Let $\alpha \in \tau(\mathcal{A})$. Then for each $x \in X$, $\bar{\alpha}_x \in \tau(B)$.

Proof. Let $x \in X$ and $J \subseteq B$ a closed two sided ideal. Let $B \in J$. Then $\bar{B} \in J(x)$. By definition of $\bar{\alpha}_x$, $\bar{\alpha}_x(B) = \alpha(B)(x)$. Since $\alpha \in \tau(\mathcal{A})$, $\alpha(J(x)) \subseteq J(x)$. Therefore $\alpha(B) \subseteq J(x)$ and hence $\alpha(B)(x) \in J$. Hence $\bar{\alpha}_x(J) \subseteq J$ as required. □

By applying the identical argument to $\bar{\alpha}_x^{-1}$ we obtain

Lemma 5.7. Let $X$ be a compact Hausdorff space, $B$ a C*-algebra and $\mathcal{A} = C(X; B)$. Let $\alpha \in \tau_0(\mathcal{A})$. Then for each $x \in X$, $\bar{\alpha}_x \in \tau_0(B)$. □

Proposition 5.8. Let $X$ be a compact Hausdorff space, $B$ a C*-algebra and $\mathcal{A} = C(X; B)$. Assume in addition that $\tau(B) \subseteq \text{CF}(B)$. Then $\tau(\mathcal{A}) \subseteq \text{CF}(\mathcal{A})$.

Proof. Let $\alpha \in \tau(\mathcal{A})$ and $f \in Z(\mathcal{A})$. Then (since $Z(\mathcal{A}) = C(X; Z(B))$) $f(x) \in Z(B)$ for all $x \in X$. Thus for each $x \in X$ we have by Lemma 5.5 $(\alpha f)(x) = \alpha(f(x)) = \alpha_x(f(x))$. By Lemma 5.6 $\alpha_x \in \tau(B)$ and hence by our hypothesis on $B$, $\bar{\alpha}_x \in \text{CF}(B)$. Therefore $\bar{\alpha}_x(f(x)) = f(x)$. Thus $(\alpha f)(x) = f(x)$ and hence $\alpha$ is center-fixing. Thus $\tau(\mathcal{A}) \subseteq \text{CF}(\mathcal{A})$. □

Remarks. (1) If $Z(B) = C \cdot 1$ then $\text{CF}(B) = \text{Aut}(B)$ and thus clearly $\tau(B) \subseteq \text{CF}(B)$.
(2) $Z(B) = C \cdot 1$ for $B = \mathcal{L}(H)$, $H$ a Hilbert space, or $B = \mathfrak{B}$, the C*-algebra of all compact operators on a Hilbert space with identity adjoined. Thus our hypotheses are satisfied in these cases.
(3) Setting $X = \text{point}$ shows that the hypotheses on $B$ in Proposition 5.8 are necessary.

Theorem 5.9. Let $X$ be a compact Hausdorff space, $B$ a C*-algebra and $\mathcal{A} = C(X; B)$. Assume in addition that $\tau_0(B) = \text{CF}(B) = \tau(B)$. Then $\tau_0(\mathcal{A}) = \text{CF}(\mathcal{A}) = \tau(\mathcal{A})$.

Proof. Since $\tau_0(B) = \text{CF}(B)$ it follows from Proposition 5.4 that $\text{CF}(\mathcal{A}) \subseteq \tau_0(\mathcal{A})$. Since $\tau(B) = \text{CF}(B)$ it follows from Proposition 5.8 that $\tau(\mathcal{A}) \subseteq \text{CF}(\mathcal{A})$. Thus we have inclusions $\tau_0(\mathcal{A}) \subseteq \text{CF}(\mathcal{A}) \subseteq \tau_0(\mathcal{A})$.

Since $\tau_0(\mathcal{A}) \subseteq \tau(\mathcal{A})$ by definition, the result follows. □

Remarks. (1) The assumption that $\tau_0(B) = \text{CF}(B)$ is clearly redundant, for $\tau(B) = \text{CF}(B)$ then $\tau(B)$ is a group and as noted previously this implies $\tau(B) = \tau_0(B)$.
(2) Note that the hypotheses on $B$ are satisfied when $B = \mathcal{L}(H)$, $H$ a Hilbert space, or $B = \mathfrak{B}$, the C*-algebra of compact operators with identity adjoined.

Corollary 5.10. Let $\mathcal{A}$, $B$ be as in Theorem 5.9. Then $\tau(\mathcal{A})$ is a subgroup of $\text{Aut}(\mathcal{A})$. □
COROLLARY 5.11. Let $X$ be a separable compact Hausdorff space, $B=\mathcal{L}(H)$ and $\mathfrak{A}=C(X; B)$. Then

$$\tau_0(\mathfrak{A}) = \text{CF} (\mathfrak{A}) = \tau(\mathfrak{A}) = \pi(\mathfrak{A}).$$

**Proof.** Immediate from Corollary 4.2, Theorem 5.9 and Remark (1) after Lemma 5.3. \qed

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