

APPROXIMATIONS AND REPRESENTATIONS FOR FOURIER TRANSFORMS

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Abstract. G is a locally compact abelian group with dual Γ . If $p(\gamma) = \sum_1^N a_n(x_n, \gamma)$ is a trigonometric polynomial, its capacity, by definition is $\sum |a_n|$. The main theorem is: Let φ be a measurable function defined on the measurable subset Λ of Γ . If φ can be approximated on finite sets in Λ by trigonometric polynomials of capacity at most C (constant), then $\varphi = \hat{\mu}$, locally almost everywhere on Λ , where μ is a regular bounded measure on G and $\|\mu\| \leq C$.

In this paper G is a locally compact abelian group with dual Γ . The set of bounded regular measures on G will be denoted $M(G)$. If $\mu \in M(G)$ its transform $\hat{\mu}$ is defined by

$$\hat{\mu}(\gamma) = \int_G (x, -\gamma) d\mu(x), \quad \gamma \in \Gamma.$$

DEFINITION. If $p(\gamma) = \sum_1^N a_n(x_n, \gamma)$ is a trigonometric polynomial on Γ , its *capacity*, by definition, is $\sum_1^N |a_n|$. If $s(\gamma) = \sum_1^\infty a_n(x_n, \gamma)$ with $\sum_1^\infty |a_n| = C < \infty$, then $s(\gamma)$ will be called a trigonometric series with capacity C .

Now if φ is any continuous function on Γ , then φ can be uniformly approximated on compact sets in Γ by trigonometric polynomials (Stone-Weierstrass theorem). In general the capacities of these polynomials will be unbounded. If we demand that these capacities be bounded by a fixed constant C we get a characterization of the transform of a measure. We shall prove

PROPOSITION 1. *Let φ be a function defined on Γ . In order that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ it is necessary and sufficient that there exists a constant C such that φ can be uniformly approximated on any compact set in Γ by trigonometric polynomials of capacity at most C .*

This approximation property can be strengthened to a representation property. In fact,

PROPOSITION 1'. *Let φ be a function defined on Γ . In order that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ it is necessary and sufficient that there exists a constant C such that for any compact set Λ in Γ , φ is equal on Λ to a trigonometric series of capacity at most C .*

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These propositions are not entirely new for they are equivalent to known results, stated differently (see below).

A function φ with the above approximation or representation property is automatically continuous. It is an important theorem of Bochner [2] and Eberlein [5] that the continuity (or even measurability) of φ , combined with the approximation property on *finite* [instead of compact] sets in Γ , implies that $\varphi = \hat{\mu}$ (almost everywhere in case of measurability of φ), where $\mu \in M(G)$. In fact, the Bochner-Eberlein theorem may be given the form:

THEOREM B-E. *Let φ be continuous (resp. measurable) on Γ . If φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C , then $\varphi = \hat{\mu}$ (resp. locally almost everywhere) where $\mu \in M(G)$ and $\|\mu\| \leq C$.*

Our main result is an analogous theorem valid for restrictions to a measurable subset Λ of Γ . Namely,

THEOREM. *Assume φ is measurable on the measurable set Λ in Γ and that φ is approximable on finite sets in Λ by trigonometric polynomials with capacity at most C , then $\varphi = \hat{\mu}$ locally almost everywhere on Λ , where $\mu \in M(G)$ and $\|\mu\| \leq C$.*

Particular cases of this theorem are due to Bochner [2]: $\Gamma = R$, $\Lambda = R$; to Krein (cf. [1, pp. 154–159]): $\Gamma = R$, $\Lambda =$ an interval; to Eberlein [5]: $\Lambda = \Gamma$; and to Rosenthal [7]: $\Gamma = R$ although their statements are expressed somewhat differently but equivalently. (See below.)

In the final part of the paper we restate in a new form, the result appearing in [4], that the transform of an integrable function *lives* mostly on compact sets, while the transform of a singular measure is scattered all over Γ .

The proof of Propositions 1 and 1' is based on the following.

PROPOSITION 2. *Let φ be a function defined on a subset Λ of Γ . Then the following two statements are equivalent:*

- (A) *φ is approximable on finite sets in Λ by polynomials with capacity at most C .*
- (B) *If $q(x) = \sum_1^M b_m(x, \gamma_m)$ with $\gamma_m \in \Lambda$ and $\|q\|_\infty \leq 1$ then $|\sum b_m \varphi(\gamma_m)| \leq C$.*

Proof. Assume (A) holds. Let $q(x) = \sum_1^M b_m(x, \gamma_m)$, with $\gamma_m \in \Lambda$ and $\|q\|_\infty \leq 1$; $\varepsilon > 0$ being given, there is, by hypothesis, a polynomial $p(\gamma) = \sum a_n(x_n, \gamma)$ with $\sum |a_n| \leq C$ such that

$$|\varphi(\gamma_m) - p(\gamma_m)| \leq \varepsilon / \sum_k |b_k|, \quad m = 1, \dots, M.$$

Then

$$\left| \sum b_m \varphi(\gamma_m) - \sum b_m p(\gamma_m) \right| \leq \varepsilon.$$

But

$$\left| \sum_m b_m p(\gamma_m) \right| = \left| \sum_n a_n q(x_n) \right| \leq \sum |a_n| \leq C.$$

Hence

$$\left| \sum b_m \varphi(\gamma_m) \right| \leq C + \varepsilon.$$

ε being arbitrary, statement (B) holds.

Conversely, assume (B) holds.

Going to the Bohr compactification \bar{G} of G , using the Hahn-Banach extension theorem and the Riesz representation theorem, we see that there is a measure $\mu \in M(\bar{G})$, with $\|\mu\| \leq C$, whose transform $\hat{\mu}$ is equal to φ on Λ .

Let $\{\gamma_1, \dots, \gamma_M\}$ be a finite subset of Λ . Let $\varepsilon > 0$ be given. There is a finite, disjoint, not necessarily open covering W_1, \dots, W_N of the compact group \bar{G} such that

$$|(\bar{x}, \gamma_m) - (\bar{x}', \gamma_m)| < \varepsilon, \quad m = 1, \dots, M,$$

provided $\bar{x}, \bar{x}' \in W_n, n=1, \dots, N$.

Choose some $\bar{x}_n \in W_n, n=1, \dots, N$, and put

$$(1) \quad \begin{aligned} a_n &= \mu(W_n), \\ \bar{p}(\gamma) &= \sum_n a_n(\bar{x}_n, -\gamma). \end{aligned}$$

Then

$$\sum |a_n| = \sum |\mu(W_n)| \leq \|\mu\| \leq C.$$

We have

$$(2) \quad \hat{\mu}(\gamma_m) = \sum_n \int_{W_n} (\bar{x}, -\gamma_m) d\mu(\bar{x}), \quad m=1, \dots, M.$$

If $\bar{x} \in W_n$ then $|(\bar{x}, \gamma_m) - (\bar{x}_n, \gamma_m)| < \varepsilon$. Therefore

$$\left| \int_{W_n} (\bar{x}, -\gamma_m) d\mu(\bar{x}) - \int_{W_n} (\bar{x}_n, -\gamma_m) d\mu(\bar{x}) \right| < \varepsilon |\mu|(W_n).$$

This is

$$\left| \int_{W_n} (\bar{x}, -\gamma_m) d\mu(\bar{x}) - a_n(\bar{x}_n, -\gamma_m) \right| < \varepsilon |\mu|(W_n).$$

We conclude, from (1) and (2), for $m=1, \dots, M$,

$$|\hat{\mu}(\gamma_m) - \bar{p}(\gamma_m)| < \varepsilon \sum_n |\mu|(W_n) \leq \varepsilon \|\mu\| \leq \varepsilon C.$$

Finally, since G is dense in \bar{G} we can choose $x_n \in G$ such that

$$|(x_n, \gamma_m) - (\bar{x}_n, \gamma_m)| \leq \varepsilon, \quad m = 1, \dots, M.$$

Put $p(\gamma) = \sum_n a_n(x_n, -\gamma)$. Then

$$|\hat{\mu}(\gamma_m) - p(\gamma_m)| < 2\varepsilon C, \quad m = 1, \dots, M.$$

ε being arbitrary, property (A) holds.

Proposition 2 is now proved.

Proposition 2 shows that our statement of Theorem B-E is equivalent to the original statement of the Bochner-Eberlein theorem.

Also the sufficiency of the conditions appearing in Propositions 1 and 1' is a consequence of the sufficiency of the weaker condition appearing in Theorem B-E.

There remains only to show the necessity of the condition in Proposition 1'. But this is precisely a theorem of K. de Leeuw and C. Herz ([3, Theorem 1]; take $G_1 = G$, $G_2 = \bar{G}$, the Bohr compactification of G).

We now go to the proof of our main theorem.

LEMMA 1. *Let G_1 be a locally compact abelian group of the form $G_1 = R^a \times T^b \times D$ where a, b are nonnegative integers and D a discrete group. Let V_0 be a neighborhood of 0 in G_1 , which is a direct product of compact symmetric neighborhoods of 0 in the factors R, T occurring in G_1 and of the neighborhood $\{0\}$ of 0 in D . Then to any compact set K containing $V_0 + V_0$ we can associate a function u on G_1 such that*

- (1) $u \geq 0$,
- (2) u vanishes outside $V_0 + V_0$,
- (3) $\int_{G_1} u(x) dx = 1$,
- (4) $u(x) = \sum b_n(x, \gamma_n)$ for $x \in K$,
- (5) $\sum |b_n| \leq m_1(V_0)^{-1}$,
- (6) $\int_{K+x_0} |\sum b_n(x, \gamma_n)| dx \leq 1$ for every $x_0 \in G_1$,

where m_1 is Haar measure on G_1 .

Proof. *Case (i).* $G_1 = T$. Let f be the function in $L^2(G_1)$ equal to $m_1(V_0)^{-1} X_{V_0}$ where X_{V_0} is the characteristic function of V_0 . Then in $L^2(T)$

$$f(x) = \sum a_n(x, \gamma_n), \quad \gamma_n \in \mathbb{Z},$$

where $\sum |a_n|^2 = \|f\|_2^2 = m_1(V_0)^{-1}$. Put $u = f * f$. Since f is nonnegative and symmetric we have

$$u(x) = \sum b_n(x, \gamma_n) \geq 0$$

with

$$\sum |b_n| = \sum |a_n|^2 = m_1(V_0)^{-1}$$

and u vanishes outside the set $V_0 + V_0$. Also, by Fubini

$$\int_{G_1} u(x) dx = \int_{G_1} \int_{G_1} f(y) f(x-y) dy dx = 1.$$

Finally

$$\int_{K+x_0} \left| \sum b_n(x, \gamma_n) \right| dx \leq \int_{G_1} u(x) dx = 1.$$

Case (ii). $G_1 = R$. Assume that the compact set K is interior to the interval $(-N, N]$ which may be identified with T . Define the function u as above:

$$\begin{aligned} u(x) &= \sum b_n(x, \gamma_n), & x \in (-N, +N], \\ u(x) &= 0, & x \notin (-N, +N], \end{aligned}$$

where now (x, γ_n) has period $2N$. Then $\sum b_n(x, \gamma_n)$ has period $2N$ and (6) follows.

Case (iii). $G_1 = D$ discrete. Here $V_0 = \{0\}$. Then $m_1(V_0) = 1$ and

$$\int_{K+x_0} \left| \sum b_n(x, \gamma_n) \right| dx = \sum_{x \in K+x_0} \left| \sum b_n(x, \gamma_n) \right|.$$

Let \bar{G}_1 be the Bohr compactification of G_1 , the dual of Γ_1 made discrete.

We can find a neighborhood W_1 of 0 in \bar{G}_1 which meets the finite set $K - K$ in just the point 0. Let W_2 be a neighborhood of 0 in \bar{G}_1 such that $W_2 - W_2 \subset W_1$. Then

(*) for $x_0 \in G_1$, the set $K + x_0$ meets W_2 in one point at most.

For, assume $k_1, k_2 \in K$; $k_1 + x_0, k_2 + x_0 \in W_2$. Then $k_1 - k_2 \in W_2 - W_2 \subset W_1$ and therefore $k_1 - k_2 = 0$.

Let W_3 be a compact symmetric neighborhood of 0 in \bar{G}_1 such that $W_3 + W_3 \subset W_2$. Let F be the function in $L^2(\bar{G}_1)$ equal to $\bar{m}_1(W_3)^{-1/2} X_{W_3}$ where \bar{m}_1 is Haar measure in \bar{G}_1 . Then, in $L^2(\bar{G}_1)$,

$$F(\bar{x}) = \sum a_n(\bar{x}, \gamma_n), \quad \bar{x} \in \bar{G}_1, \gamma_n \in \Gamma_1,$$

where $\sum |a_n|^2 = \|F\|_2^2 = 1$. Put $U = F * F$. Then

$$U(\bar{x}) = \sum b_n(\bar{x}, \gamma_n) \geq 0, \quad \bar{x} \in \bar{G}_1,$$

with $\sum b_n = \sum |b_n| = \sum |a_n|^2 = 1$ and U vanishes outside $W_3 + W_3$; in particular, U vanishes outside W_2 . Since, by (*), K meets W_2 in just the point 0, then $U(x) = 0$ for $x \in K, x \neq 0$. Also $U(0) = 1$. Put

$$u(0) = 1, \quad u(x) = 0 \quad \text{for } x \in G_1, x \neq 0.$$

Then conditions (1), (2), (3) are satisfied and

$$u(x) = \sum b_n(x, \gamma_n) = U(x) \quad \text{for } x \in K.$$

Finally, by (*), at most one term in the sum

$$\sum_{x \in K+x_0} \left| \sum b_n(x, \gamma_n) \right|$$

is different from 0 and this term is at most 1.

General case. Let G_1 be the finite direct product of the groups G_α . If K is compact in G_1 , then K is contained in a direct product $\prod K_\alpha$, where K_α is compact in G_α . Take u to be the product of the u 's constructed for each G_α .

LEMMA 2. *Let G be any locally compact abelian group with dual Γ . Let φ, f be two bounded measurable functions with compact support Λ_0 in Γ and let $\varepsilon > 0$ be given. Then, for a certain polynomial $p(x) = \sum_1^N b_n(x, \gamma_n)$, we have*

$$(7) \quad \int_{\Lambda_0} \left| \sum_1^N b_n \varphi(\gamma + \gamma_n) - \int_{\Gamma} \varphi \right| d\gamma < \varepsilon,$$

and the function on Λ_0

$$\sup_{\gamma \in G} \left| \sum_1^N b_n f(\gamma + \gamma_n)(\gamma, \gamma + \gamma_n) \right|, \quad \gamma \in \Lambda_0,$$

is majorized by a certain function F such that

$$(8) \quad \int_{\Lambda_0} F(\gamma) d\gamma \leq \nu(\Lambda_0)(\|f\|_\infty + \varepsilon)$$

where ν is Haar measure on Γ .

Proof. Extend Λ_0 to a compact neighborhood Λ_1 of 0 in Γ and let Γ_1 be the locally compact group generated by Λ_1 . Let H_1 be the annihilator of Γ_1 and put $G_1 = G/H_1$. Then G_1 is the dual of Γ_1 . By the structure theorem for compactly generated groups, (see e.g. [6, (9.8)]), G_1 is of the form $G_1 = R^a \times T^b \times D$ where a, b are nonnegative integers and D a discrete group.

Choose a compact symmetric neighborhood V_0 of 0 in G_1 , which is of the form described in Lemma 1, in such a way that, if u is any function on G_1 satisfying conditions (1), (2), (3) of Lemma 1, then

$$(9) \quad \left| \int_{G_1} u(x)(x, \gamma) \hat{\varphi}(x) dx - \hat{\varphi}(0) \right| < \varepsilon \quad \text{for all } \gamma \in \Lambda_0 \text{ (compact)}.$$

Observe that, since φ is concentrated on Γ_1 ; $\hat{\varphi}$ is constant on the cosets of H_1 and therefore $\hat{\varphi}$ is defined on G_1 .

Next choose $k \in L^1(\Gamma_1)$ such that \hat{k} has compact support, say K , in G_1 and such that

$$(10) \quad \int_{G_1} |\varphi * k - \varphi| < \varepsilon m_1(V_0) \leq \varepsilon, \quad V_0 \text{ small,}$$

$$(11) \quad \int_{\Gamma_1} |f * k - f| < \varepsilon m_1(V_0) \leq \varepsilon$$

where m_1 is Haar measure on G_1 . By (10)

$$(10') \quad \|\hat{\varphi}\hat{k} - \hat{\varphi}\|_\infty < \varepsilon \quad (\text{sup over } G_1).$$

Now V_0 and the compact set K (which we may extend to include $V_0 + V_0$) are fixed and we choose u satisfying the six conditions (1)–(6) of Lemma 1.

Then, by the L^1 -inversion theorem, we have, for $\gamma \in \Gamma_1$,

$$\begin{aligned} \sum_1^\infty b_n(\varphi * k)(\gamma + \gamma_n) &= \int_K \hat{\varphi}\hat{k}(x) \sum_1^\infty b_n(x, \gamma + \gamma_n) dx \\ &= \int_K (\hat{\varphi}\hat{k})(x)(x, \gamma)u(x) dx. \end{aligned}$$

Hence, for large N and any $\gamma \in \Gamma_1$,

$$(12) \quad \left| \sum_1^N b_n(\varphi * k)(\gamma + \gamma_n) - \int_K (\hat{\phi}k)(x)(x, \gamma)u(x) dx \right| < \varepsilon.$$

By (10') and (1)-(3),

$$\left| \int_K (\hat{\phi}k)(x)(x, \gamma)u(x) dx - \int_K \hat{\phi}(x)(x, \gamma)u(x) dx \right| < \varepsilon.$$

Whence, by (12) and (9),

$$\left| \sum_1^N b_n(\varphi * k)(\gamma + \gamma_n) - \hat{\phi}(0) \right| < 3\varepsilon \quad \text{for } \gamma \in \Lambda_0.$$

We conclude

$$\int_{\Lambda_0} \left| \sum_1^N b_n(\varphi * k)(\gamma + \gamma_n) - \hat{\phi}(0) \right| d\gamma < 3\varepsilon\nu(\Lambda_0).$$

(Since Γ_1 is open in Γ we may take Haar measure on Γ_1 to be the restriction of Haar measure ν on Γ .)

Finally, by (10)

$$\begin{aligned} \int_{\Lambda_0} \left| \sum_1^N b_n\varphi(\gamma + \gamma_n) - \hat{\phi}(0) \right| d\gamma &\leq \sum_1^N |b_n| \varepsilon m_1(V_0) + 3\varepsilon\nu(\Lambda_0) \\ &\leq \varepsilon + 3\varepsilon\nu(\Lambda_0), \end{aligned}$$

which is the property (7) required for φ .

Again, by the L^1 -inversion theorem, we have, for any $y \in G_1$ and any $\gamma \in \Gamma_1$,

$$\begin{aligned} \left| \sum_1^\infty b_n(f * k)(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| &= \left| \int_K (\hat{f}k)(x) \sum b_n(x + y, \gamma + \gamma_n) \right| \\ &\leq \|\hat{f}k\|_\infty \int_{K+y} \left| \sum b_n(x, \gamma + \gamma_n) \right| \\ &\leq \|\hat{f}k\|_\infty \leq \|f\|_\infty + \varepsilon. \end{aligned}$$

Hence, for large N , and any $y \in G_1, \gamma \in \Gamma_1$,

$$\left| \sum_1^N b_n(f * k)(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| \leq \|f\|_\infty + 2\varepsilon.$$

We deduce, for $\gamma \in \Gamma_1$, since $\gamma_n \in \Gamma_1$ and the characters $(y, \gamma + \gamma_n)$ are constant on the cosets of H_1 :

$$\begin{aligned} \sup_{y \in G} \left| \sum_1^N b_n f(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| &= \sup_{y \in G_1} \left| \sum_1^N b_n f(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| \\ &\leq \sup_{y \in G_1} \left| \sum_1^N b_n (f - f * k)(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| + \|f\|_\infty + 2\varepsilon \\ &\leq \sum_1^N |b_n (f - f * k)(\gamma + \gamma_n)| + \|f\|_\infty + 2\varepsilon \\ &= F(\gamma) \quad \text{say} \end{aligned}$$

where by (11)

$$\begin{aligned} \int_{\Lambda_0} F(\gamma) \, d\gamma &\leq \sum_1^N |b_n| \varepsilon m_1(V_0) + \nu(\Lambda_0)(\|f^\sharp\|_\infty + 2\varepsilon) \\ &\leq \varepsilon + \nu(\Lambda_0)(\|f^\sharp\|_\infty + 2\varepsilon). \end{aligned}$$

ε being arbitrary, (8) is now proved.

LEMMA 3. *Assume φ is measurable on $\Lambda \subset \Gamma$, φ is zero outside Λ and φ is approximable on finite sets in Λ with capacity at most C . For any f , bounded, measurable, vanishing outside Λ , with compact support, put $T(f) = \int_\Gamma f\varphi \, d\gamma$. Then $|T(f)| \leq C\|f^\sharp\|_\infty$.*

Proof. Let $\varepsilon > 0$ be given and let Λ_0 be the compact support of f . By Lemma 2, applied to the two functions $f\varphi$ and f , both with compact support Λ_0 , there is a polynomial $p(x) = \sum_1^N b_n(x, \gamma_n)$ such that

$$(1) \quad \int_{\Lambda_0} \left| \sum_1^N b_n f(\gamma + \gamma_n) \varphi(\gamma + \gamma_n) - \int f\varphi \right| d\gamma < \varepsilon^2 \nu(\Lambda_0)$$

and

$$\sup_y \left| \sum_1^N b_n f(\gamma + \gamma_n)(y, \gamma + \gamma_n) \right| \leq F(\gamma), \quad \gamma \in \Lambda_0,$$

where

$$(2) \quad \int_{\Lambda_0} F(\gamma) \, d\gamma \leq \nu(\Lambda_0)(\|f^\sharp\|_\infty + \varepsilon).$$

Put

$$E_1 = \left\{ \gamma \in \Lambda_0 : \left| \sum_1^N b_n f(\gamma + \gamma_n) \varphi(\gamma + \gamma_n) - \int f\varphi \right| > \varepsilon \right\}.$$

Then, by (1)

$$\varepsilon \cdot \nu(E_1) < \varepsilon^2 \nu(\Lambda_0); \quad \nu(E_1) < \varepsilon \nu(\Lambda_0).$$

Put

$$E_2 = \{ \gamma \in \Lambda_0 : F(\gamma) > (1 - \varepsilon)^{-1}(\|f^\sharp\|_\infty + \varepsilon) \}.$$

By (2)

$$\begin{aligned} (1 - \varepsilon)^{-1}(\|f^\sharp\|_\infty + \varepsilon) \nu(E_2) &\leq \nu(\Lambda_0)(\|f^\sharp\|_\infty + \varepsilon), \\ \nu(E_2) &\leq (1 - \varepsilon) \nu(\Lambda_0). \end{aligned}$$

We conclude $\nu(E_1 \cup E_2) < \nu(\Lambda_0)$. Hence there is $\lambda_0 \in \Lambda_0$ such that $\lambda_0 \notin E_1$, $\lambda_0 \notin E_2$, that is

$$(3) \quad \left| \sum_1^N b_n f(\lambda_0 + \gamma_n) \varphi(\lambda_0 + \gamma_n) - T(f) \right| \leq \varepsilon,$$

$$(4) \quad \sup_y \left| \sum_1^N b_n f(\lambda_0 + \gamma_n)(y, \lambda_0 + \gamma_n) \right| \leq (1 - \varepsilon)^{-1}(\|f^\sharp\|_\infty + \varepsilon).$$

Let A be the finite set of elements of the form $\lambda_0 + \gamma_n, n = 1, \dots, N$, which belong to $\Lambda_0 \cap \Lambda$. By hypothesis there is a polynomial $q(\gamma) = \sum_m c_m(y_m, \gamma)$ with $\sum |c_m| \leq C$ such that

$$(5) \quad |q(\gamma) - \varphi(\gamma)| < \varepsilon / \sum_1^N |b_n| \|f\|_\infty \quad \text{for } \gamma \in A.$$

Observing that $f(\lambda_0 + \gamma_n) = 0$ if $\lambda_0 + \gamma_n \notin A$ we get from (3) and (5)

$$\left| T(f) - \sum_1^N b_n f(\lambda_0 + \gamma_n) q(\lambda_0 + \gamma_n) \right| \leq \varepsilon + \varepsilon.$$

This is

$$\left| T(f) - \sum_m c_m \sum_{n=1}^N b_n f(\lambda_0 + \gamma_n)(y_m, \lambda_0 + \gamma_n) \right| \leq 2\varepsilon.$$

By (4) the coefficient of c_m has modulus $\leq (1 - \varepsilon)^{-1}(\|f\|_\infty + \varepsilon)$. Hence

$$\begin{aligned} |T(f)| &\leq \sum_m |c_m| (1 - \varepsilon)^{-1} (\|f\|_\infty + \varepsilon) + 2\varepsilon \\ &\leq C(1 - \varepsilon)^{-1} (\|f\|_\infty + \varepsilon) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the lemma is proved.

MAIN THEOREM. *Assume that φ is measurable on the measurable set Λ and that φ is approximable on finite sets in Λ with capacity at most C . Then $\varphi(\gamma) = \hat{\mu}(\gamma)$ locally almost everywhere on Λ , where $\mu \in M(G)$ and $\|\mu\| \leq C$.*

Proof. The linear functional S given by $S(\hat{f}) = T(f) = \int_\Gamma f \varphi \, d\gamma$ is defined on the linear space of the transforms \hat{f} of the bounded measurable functions f vanishing outside Λ , with compact support (a subspace of $C_0(G)$) and satisfies the inequality $|S(\hat{f})| \leq C \|\hat{f}\|_\infty$.

By the Hahn-Banach theorem, S can be extended to the whole of $C_0(G)$, with norm not exceeding C . By the Riesz-Kakutani representation theorem there is a $\mu \in M(G)$ such that $\|\mu\| \leq C$ and $S(\hat{f}) = \int_G \hat{f}(x) \, d\mu(x)$. Then, by Fubini's theorem

$$S(\hat{f}) = \int_\Gamma f(\gamma) \hat{\mu}(\gamma) \, d\gamma,$$

that is, $\int_\Gamma f(\gamma) \varphi(\gamma) \, d\gamma = \int_\Gamma f(\gamma) \hat{\mu}(\gamma) \, d\gamma$ for every f , bounded, vanishing outside Λ , with compact support. We conclude $\varphi(\gamma) = \hat{\mu}(\gamma)$ locally a.e. on Λ and the theorem is proved.

REMARK. This theorem contrasts with the situation where instead of the transform of a measure we consider positive definite functions (the transforms of positive measures).

Suppose Λ_0 is a measurable subset of a locally compact abelian group Γ . Define $PD(\Lambda_0)$ to be the class of all continuous complex valued functions φ on $\Lambda_0 - \Lambda_0$ which satisfy the inequality

$$\sum_{i,j=1}^N c_i \bar{c}_j \varphi(\gamma_i - \gamma_j) \geq 0$$

for every positive integer N , for every choice of complex numbers c_1, \dots, c_N and for every choice of points $\gamma_1, \dots, \gamma_N$ in Λ_0 .

If $G = R$, $\Lambda_0 =$ an interval I , then every $\varphi \in PD(I)$ is the restriction on $I - I$ of the transform $\hat{\mu}$ of some positive measure μ on G (Krein). But if $G = R^2$, $\Lambda_0 =$ a closed square S in R^2 , then there is $\varphi \in PD(S)$ which is not the restriction to $S - S$ of the transform of a positive measure on G (see Rudin [8]).

Before ending our paper we want to state, in a new form, the two theorems appearing in [4].

THEOREM. *A continuous function φ defined on Γ is the Fourier-Stieltjes transform of a singular measure on G if and only if there is a constant C such that*

(i) *φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C .*

(ii) *Whatever be $\varepsilon > 0$ and the compact set Λ in Γ , φ is not approximable on finite sets F , not meeting Λ , by polynomials of capacity $\leq C - \varepsilon$.*

THEOREM. *A continuous function φ defined on Γ is the Fourier transform of an integrable function on G if and only if*

(i) *there is a constant C such that φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C .*

(ii) *to every $\varepsilon > 0$ corresponds a compact set Λ in Γ such that φ can be approximated on any finite set in Γ , not meeting Λ , by trigonometric polynomials of capacity at most ε .*

To prove these theorems we just make use of the equivalence stated in Proposition 2 and combine this equivalence with Theorems 1 and 2 of [4]. Observe also that no form of any of these two theorems is readily available for restrictions, since the transform $\hat{\mu}_s$ of a singular measure can be equal to the transform \hat{f} of an integrable function on very large sets (see e.g. Rudin [9]).

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