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APPROXIMATIONS AND REPRESENTATIONS FOR FOURIER TRANSFORMS

BY
RAOUF DOSS

Abstract. G is a locally compact abelian group with dual Γ. If $p(\gamma)=\sum_1^N a_n(x_n, \gamma)$ is a trigonometric polynomial, its capacity, by definition is $\sum |a_n|$. The main theorem is: Let φ be a measurable function defined on the measurable subset Λ of Γ. If φ can be approximated on finite sets in Λ by trigonometric polynomials of capacity at most C (constant), then $\varphi=\hat{\mu}$, locally almost everywhere on Λ, where μ is a regular bounded measure on G and $\|\mu\| \leqq C$.

In this paper G is a locally compact abelian group with dual Γ. The set of bounded regular measures on G will be denoted $M(G)$. If $\mu \in M(G)$ its transform $\hat{\mu}$ is defined by

$$\hat{\mu}(\gamma) = \int_G (x, -\gamma)\, d\mu(x), \qquad \gamma \in \Gamma.$$

DEFINITION. If $p(\gamma)=\sum_1^N a_n(x_n, \gamma)$ is a trigonometric polynomial on Γ, its *capacity*, by definition, is $\sum_1^N |a_n|$. If $s(\gamma)=\sum_1^\infty a_n(x_n, \gamma)$ with $\sum_1^\infty |a_n|=C<\infty$, then $s(\gamma)$ will be called a trigonometric series with capacity C.

Now if φ is any continuous function on Γ, then φ can be uniformly approximated on compact sets in Γ by trigonometric polynomials (Stone-Weierstrass theorem). In general the capacities of these polynomials will be unbounded. If we demand that these capacities be bounded by a fixed constant C we get a characterization of the transform of a measure. We shall prove

PROPOSITION 1. *Let φ be a function defined on Γ. In order that $\varphi=\hat{\mu}$ for some $\mu \in M(G)$ it is necessary and sufficient that there exists a constant C such that φ can be uniformly approximated on any compact set in Γ by trigonometric polynomials of capacity at most C.*

This approximation property can be strengthened to a representation property. In fact,

PROPOSITION 1′. *Let φ be a function defined on Γ. In order that $\varphi=\hat{\mu}$ for some $\mu \in M(G)$ it is necessary and sufficient that there exists a constant C such that for any compact set Λ in Γ, φ is equal on Λ to a trigonometric series of capacity at most C.*

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These propositions are not entirely new for they are equivalent to known results, stated differently (see below).

A function φ with the above approximation or representation property is automatically continuous. It is an important theorem of Bochner [2] and Eberlein [5] that the continuity (or even measurability) of φ, combined with the approximation property on *finite* [instead of compact] sets in Γ, implies that $\varphi=\hat{\mu}$ (almost everywhere in case of measurability of φ), where $\mu\in M(G)$. In fact, the Bochner-Eberlein theorem may be given the form:

THEOREM B-E. *Let φ be continuous (resp. measurable) on Γ. If φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C, then $\varphi=\hat{\mu}$ (resp. locally almost everywhere) where $\mu\in M(G)$ and $\|\mu\|\leqq C$.*

Our main result is an analogous theorem valid for restrictions to a measurable subset Λ of Γ. Namely,

THEOREM. *Assume φ is measurable on the measurable set Λ in Γ and that φ is approximable on finite sets in Λ by trigonometric polynomials with capacity at most C, then $\varphi=\hat{\mu}$ locally almost everywhere on Λ, where $\mu\in M(G)$ and $\|\mu\|\leqq C$.*

Particular cases of this theorem are due to Bochner [2]: $\Gamma=R$, $\Lambda=R$; to Krein (cf. [1, pp. 154–159]): $\Gamma=R$, $\Lambda=$ an interval; to Eberlein [5]: $\Lambda=\Gamma$; and to Rosenthal [7]: $\Gamma=R$ although their statements are expressed somewhat differently but equivalently. (See below.)

In the final part of the paper we restate in a new form, the result appearing in [4], that the transform of an integrable function *lives* mostly on compact sets, while the transform of a singular measure is scattered all over Γ.

The proof of Propositions 1 and 1′ is based on the following.

PROPOSITION 2. *Let φ be a function defined on a subset Λ of Γ. Then the following two statements are equivalent:*

(A) *φ is approximable on finite sets in Λ by polynomials with capacity at most C.*

(B) *If $q(x)=\sum_1^M b_m(x,\gamma_m)$ with $\gamma_m\in\Lambda$ and $\|q\|_\infty\leqq 1$ then $|\sum b_m\varphi(\gamma_m)|\leqq C$.*

Proof. Assume (A) holds. Let $q(x)=\sum_1^M b_m(x,\gamma_m)$, with $\gamma_m\in\Lambda$ and $\|q\|_\infty\leqq 1$; $\varepsilon>0$ being given, there is, by hypothesis, a polynomial $p(\gamma)=\sum a_n(x_n,\gamma)$ with $\sum|a_n|\leqq C$ such that

$$|\varphi(\gamma_m)-p(\gamma_m)| \leqq \varepsilon/\sum_k |b_k|, \qquad m=1,\ldots,M.$$

Then

$$\left|\sum b_m\varphi(\gamma_m)-\sum b_m p(\gamma_m)\right| \leqq \varepsilon.$$

But

$$\left|\sum_m b_m p(\gamma_m)\right| = \left|\sum_n a_n q(x_n)\right| \leqq \sum |a_n| \leqq C.$$

Hence

$$\left|\sum b_m\varphi(\gamma_m)\right| \leqq C+\varepsilon.$$

ε being arbitrary, statement (B) holds.

Conversely, assume (B) holds.

Going to the Bohr compactification $\bar{G}$ of G, using the Hahn-Banach extension theorem and the Riesz representation theorem, we see that there is a measure $\mu \in M(\bar{G})$, with $\|\mu\| \leqq C$, whose transform $\hat{\mu}$ is equal to φ on Λ.

Let $\{\gamma_1, \ldots, \gamma_M\}$ be a finite subset of Λ. Let $\varepsilon > 0$ be given. There is a finite, disjoint, not necessarily open covering $W_1, \ldots, W_N$ of the compact group $\bar{G}$ such that

$$|(\bar{x}, \gamma_m) - (\bar{x}', \gamma_m)| < \varepsilon, \qquad m = 1, \ldots, M,$$

provided $\bar{x}, \bar{x}' \in W_n$, $n = 1, \ldots, N$.

Choose some $\bar{x}_n \in W_n$, $n = 1, \ldots, N$, and put

$$\text{(1)} \qquad \begin{aligned} a_n &= \mu(W_n), \\ \bar{p}(\gamma) &= \sum_n a_n(\bar{x}_n, -\gamma). \end{aligned}$$

Then

$$\sum |a_n| = \sum |\mu(W_n)| \leqq \|\mu\| \leqq C.$$

We have

$$\text{(2)} \qquad \hat{\mu}(\gamma_m) = \sum_n \int_{W_n} (\bar{x}, -\gamma_m)\, d\mu(\bar{x}), \qquad m = 1, \ldots, M.$$

If $\bar{x} \in W_n$ then $|(\bar{x}, \gamma_m) - (\bar{x}_n, \gamma_m)| < \varepsilon$. Therefore

$$\left| \int_{W_n} (\bar{x}, -\gamma_m)\, d\mu(\bar{x}) - \int_{W_n} (\bar{x}_n, -\gamma_m)\, d\mu(\bar{x}) \right| < \varepsilon |\mu|(W_n).$$

This is

$$\left| \int_{W_n} (\bar{x}, -\gamma_m)\, d\mu(\bar{x}) - a_n(\bar{x}_n, -\gamma_m) \right| < \varepsilon |\mu|(W_n).$$

We conclude, from (1) and (2), for $m = 1, \ldots, M$,

$$|\hat{\mu}(\gamma_m) - \bar{p}(\gamma_m)| < \varepsilon \sum_n |\mu|(W_n) \leqq \varepsilon \|\mu\| \leqq \varepsilon C.$$

Finally, since G is dense in $\bar{G}$ we can choose $x_n \in G$ such that

$$|(x_n, \gamma_m) - (\bar{x}_n, \gamma_m)| \leqq \varepsilon, \qquad m = 1, \ldots, M.$$

Put $p(\gamma) = \sum_n a_n(x_n, -\gamma)$. Then

$$|\hat{\mu}(\gamma_m) - p(\gamma_m)| < 2\varepsilon C, \qquad m = 1, \ldots, M.$$

ε being arbitrary, property (A) holds.

Proposition 2 is now proved.

Proposition 2 shows that our statement of Theorem B-E is equivalent to the original statement of the Bochner-Eberlein theorem.

Also the sufficiency of the conditions appearing in Propositions 1 and 1′ is a consequence of the sufficiency of the weaker condition appearing in Theorem B-E.

There remains only to show the necessity of the condition in Proposition 1′. But this is precisely a theorem of K. de Leeuw and C. Herz ([3, Theorem 1]; take $G_1=G$, $G_2=\bar{G}$, the Bohr compactification of G).

We now go to the proof of our main theorem.

Lemma 1. *Let G_1 be a locally compact abelian group of the form $G_1=R^a\times T^b\times D$ where a, b are nonnegative integers and D a discrete group. Let V_0 be a neighborhood of* 0 *in G_1, which is a direct product of compact symmetric neighborhoods of* 0 *in the factors R, T occurring in G_1 and of the neighborhood* $\{0\}$ *of* 0 *in D. Then to any compact set K containing V_0+V_0 we can associate a function u on G_1 such that*

(1) $u\geqq 0$,

(2) *u vanishes outside V_0+V_0,*

(3) $\int_{G_1} u(x)\,dx=1$,

(4) $u(x)=\sum b_n(x,\gamma_n)$ *for* $x\in K$,

(5) $\sum |b_n|\leqq m_1(V_0)^{-1}$,

(6) $\int_{K+x_0} |\sum b_n(x,\gamma_n)|\,dx\leqq 1$ *for every* $x_0\in G_1$,

where m_1 is Haar measure on G_1.

Proof. *Case* (i). $G_1=T$. Let f be the function in $L^2(G_1)$ equal to $m_1(V_0)^{-1}X_{V_0}$ where X_{V_0} is the characteristic function of V_0. Then in $L^2(T)$

$$f(x) = \sum a_n(x,\gamma_n), \qquad \gamma_n\in Z,$$

where $\sum |a_n|^2=\|f\|_2^2=m_1(V_0)^{-1}$. Put $u=f*f$. Since f is nonnegative and symmetric we have

$$u(x) = \sum b_n(x,\gamma_n) \geqq 0$$

with

$$\sum |b_n| = \sum |a_n|^2 = m_1(V_0)^{-1}$$

and u vanishes outside the set V_0+V_0. Also, by Fubini

$$\int_{G_1} u(x)\,dx = \int_{G_1}\int_{G_1} f(y)f(x-y)\,dy\,dx = 1.$$

Finally

$$\int_{K+x_0} \left|\sum b_n(x,\gamma_n)\right| dx \leqq \int_{G_1} u(x)\,dx = 1.$$

Case (ii). $G_1=R$. Assume that the compact set K is interior to the interval $(-N,N]$ which may be identified with T. Define the function u as above:

$$u(x) = \sum b_n(x,\gamma_n), \qquad x\in(-N,+N],$$

$$u(x) = 0, \qquad x\notin(-N,+N],$$

where now (x, γ_n) has period $2N$. Then $\sum b_n(x, \gamma_n)$ has period $2N$ and (6) follows.

Case (iii). $G_1 = D$ discrete. Here $V_0 = \{0\}$. Then $m_1(V_0) = 1$ and

$$\int_{K+x_0} \left|\sum b_n(x, \gamma_n)\right| dx = \sum_{x\in K+x_0} \left|\sum b_n(x, \gamma_n)\right|.$$

Let $\bar{G}_1$ be the Bohr compactification of G_1, the dual of Γ_1 made discrete.

We can find a neighborhood W_1 of 0 in $\bar{G}_1$ which meets the finite set $K-K$ in just the point 0. Let W_2 be a neighborhood of 0 in $\bar{G}_1$ such that $W_2 - W_2 \subset W_1$. Then

(*) for $x_0 \in G_1$, the set $K + x_0$ meets W_2 in one point at most.

For, assume $k_1, k_2 \in K$; $k_1 + x_0$, $k_2 + x_0 \in W_2$. Then $k_1 - k_2 \in W_2 - W_2 \subset W_1$ and therefore $k_1 - k_2 = 0$.

Let W_3 be a compact symmetric neighborhood of 0 in $\bar{G}_1$ such that $W_3 + W_3 \subset W_2$. Let F be the function in $L^2(\bar{G}_1)$ equal to $\bar{m}_1(W_3)^{-1/2}X_{W_3}$ where $\bar{m}_1$ is Haar measure in $\bar{G}_1$. Then, in $L^2(\bar{G}_1)$,

$$F(\bar{x}) = \sum a_n(\bar{x}, \gamma_n), \qquad \bar{x} \in \bar{G}_1,\ \gamma_n \in \Gamma_1,$$

where $\sum |a_n|^2 = \|F\|_2^2 = 1$. Put $U = F * F$. Then

$$U(\bar{x}) = \sum b_n(\bar{x}, \gamma_n) \geqq 0, \qquad \bar{x} \in \bar{G}_1,$$

with $\sum b_n = \sum |b_n| = \sum |a_n|^2 = 1$ and U vanishes outside $W_3 + W_3$; in particular, U vanishes outside W_2. Since, by (*), K meets W_2 in just the point 0, then $U(x) = 0$ for $x \in K$, $x \neq 0$. Also $U(0) = 1$. Put

$$u(0) = 1, \quad u(x) = 0 \quad \text{for } x \in G_1,\ x \neq 0.$$

Then conditions (1), (2), (3) are satisfied and

$$u(x) = \sum b_n(x, \gamma_n) = U(x) \quad \text{for } x \in K.$$

Finally, by (*), at most one term in the sum

$$\sum_{x\in K+x_0} \left|\sum b_n(x, \gamma_n)\right|$$

is different from 0 and this term is at most 1.

General case. Let G_1 be the finite direct product of the groups G_α. If K is compact in G_1, then K is contained in a direct product $\prod K_\alpha$, where K_α is compact in G_α. Take u to be the product of the u's constructed for each G_α.

LEMMA 2. *Let G be any locally compact abelian group with dual Γ. Let φ, f be two bounded measurable functions with compact support Λ_0 in Γ and let $\varepsilon > 0$ be given. Then, for a certain polynomial $p(x) = \sum_1^N b_n(x, \gamma_n)$, we have*

$$\int_{\Lambda_0} \left|\sum_1^N b_n\varphi(\gamma+\gamma_n) - \int_\Gamma \varphi\right| d\gamma < \varepsilon, \tag{7}$$

and the function on Λ_0

$$\sup_{y\in G}\left|\sum_1^N b_n f(\gamma+\gamma_n)(y,\gamma+\gamma_n)\right|, \qquad \gamma\in\Lambda_0,$$

is majorized by a certain function F such that

$$\int_{\Lambda_0} F(\gamma)\,d\gamma \leqq \nu(\Lambda_0)(\|\hat{f}\|_\infty+\varepsilon) \tag{8}$$

where ν *is Haar measure on* Γ.

Proof. Extend Λ_0 to a compact neighborhood Λ_1 of 0 in Γ and let Γ_1 be the locally compact group generated by Λ_1. Let H_1 be the annihilator of Γ_1 and put $G_1=G/H_1$. Then G_1 is the dual of Γ_1. By the structure theorem for compactly generated groups, (see e.g. [6, (9.8)]), G_1 is of the form $G_1=R^a\times T^b\times D$ where a, b are nonnegative integers and D a discrete group.

Choose a compact symmetric neighborhood V_0 of 0 in G_1, which is of the form described in Lemma 1, in such a way that, if u is any function on G_1 satisfying conditions (1), (2), (3) of Lemma 1, then

$$\left|\int_{G_1} u(x)(x,\gamma)\hat{\varphi}(x)\,dx-\hat{\varphi}(0)\right| < \varepsilon \quad \text{for all } \gamma\in\Lambda_0 \text{ (compact).} \tag{9}$$

Observe that, since φ is concentrated on Γ_1; $\hat{\varphi}$ is constant on the cosets of H_1 and therefore $\hat{\varphi}$ is defined on G_1.

Next choose $k\in L^1(\Gamma_1)$ such that $\hat{k}$ has compact support, say K, in G_1 and such that

$$\int_{\Gamma_1} |\varphi * k-\varphi| < \varepsilon m_1(V_0) \leqq \varepsilon, \qquad V_0 \text{ small,} \tag{10}$$

$$\int_{\Gamma_1} |f * k-f| < \varepsilon m_1(V_0) \leqq \varepsilon \tag{11}$$

where m_1 is Haar measure on G_1. By (10)

$$\|\hat{\varphi}\hat{k}-\hat{\varphi}\|_\infty < \varepsilon \qquad \text{(sup over } G_1). \tag{10'}$$

Now V_0 and the compact set K (which we may extend to include V_0+V_0) are fixed and we choose u satisfying the six conditions (1)–(6) of Lemma 1.

Then, by the L^1-inversion theorem, we have, for $\gamma\in\Gamma_1$,

$$\begin{aligned}\sum_1^\infty b_n(\varphi * k)(\gamma+\gamma_n) &= \int_K \hat{\varphi}\hat{k}(x)\sum_1^\infty b_n(x,\gamma+\gamma_n)\,dx\\ &= \int_K (\hat{\varphi}\hat{k})(x)(x,\gamma)u(x)\,dx.\end{aligned}$$

Hence, for large N and any $\gamma \in \Gamma_1$,

$$\left|\sum_1^N b_n(\varphi * k)(\gamma+\gamma_n) - \int_K (\hat{\varphi}\hat{k})(x)(x, \gamma)u(x)\,dx\right| < \varepsilon. \tag{12}$$

By (10′) and (1)–(3),

$$\left|\int_K (\hat{\varphi}\hat{k})(x)(x, \gamma)u(x)\,dx - \int_K \hat{\varphi}(x)(x, \gamma)u(x)\,dx\right| < \varepsilon.$$

Whence, by (12) and (9),

$$\left|\sum_1^N b_n(\varphi * k)(\gamma+\gamma_n) - \hat{\varphi}(0)\right| < 3\varepsilon \quad \text{for } \gamma \in \Lambda_0.$$

We conclude

$$\int_{\Lambda_0} \left|\sum_1^N b_n(\varphi * k)(\gamma+\gamma_n) - \hat{\varphi}(0)\right| d\gamma < 3\varepsilon\nu(\Lambda_0).$$

(Since Γ_1 is open in Γ we may take Haar measure on Γ_1 to be the restriction of Haar measure ν on Γ.)

Finally, by (10)

$$\begin{aligned}\int_{\Lambda_0} \left|\sum_1^N b_n\varphi(\gamma+\gamma_n) - \hat{\varphi}(0)\right| d\gamma &\leqq \sum_1^N |b_n|\varepsilon m_1(V_0) + 3\varepsilon\nu(\Lambda_0)\\ &\leqq \varepsilon + 3\varepsilon\nu(\Lambda_0),\end{aligned}$$

which is the property (7) required for φ.

Again, by the L^1-inversion theorem, we have, for any $y \in G_1$ and any $\gamma \in \Gamma_1$,

$$\begin{aligned}\left|\sum_1^\infty b_n(f * k)(\gamma+\gamma_n)(y, \gamma+\gamma_n)\right| &= \left|\int_K (\hat{f}\hat{k})\,(x) \sum b_n(x+y, \gamma+\gamma_n)\right|\\ &\leqq \|\hat{f}\hat{k}\|_\infty \int_{K+y} \left|\sum b_n(x, \gamma+\gamma_n)\right|\\ &\leqq \|\hat{f}\hat{k}\|_\infty \leqq \|\hat{f}\|_\infty + \varepsilon.\end{aligned}$$

Hence, for large N, and any $y \in G_1$, $\gamma \in \Gamma_1$,

$$\left|\sum_1^N b_n(f * k)(\gamma+\gamma_n)(y, \gamma+\gamma_n)\right| \leqq \|\hat{f}\|_\infty + 2\varepsilon.$$

We deduce, for $\gamma \in \Gamma_1$, since $\gamma_n \in \Gamma_1$ and the characters $(y, \gamma+\gamma_n)$ are constant on the cosets of H_1:

$$\begin{aligned}\sup_{y\in G} \left|\sum_1^N b_n f(\gamma+\gamma_n)(y, \gamma+\gamma_n)\right| &= \sup_{y\in G_1} \left|\sum_1^N b_n f(\gamma+\gamma_n)(y, \gamma+\gamma_n)\right|\\ &\leqq \sup_{y\in G_1} \left|\sum_1^N b_n(f - f * k)(\gamma+\gamma_n)(y, \gamma+\gamma_n)\right| + \|\hat{f}\|_\infty + 2\varepsilon\\ &\leqq \sum_1^N |b_n(f - f * k)(\gamma+\gamma_n)| + \|\hat{f}\|_\infty + 2\varepsilon\\ &= F(\gamma) \quad \text{say}\end{aligned}$$

where by (11)

$$\int_{\Lambda_0} F(\gamma)\,d\gamma \leqq \sum_1^N |b_n|\varepsilon m_1(V_0)+\nu(\Lambda_0)(\|\hat{f}\|_\infty+2\varepsilon)$$
$$\leqq \varepsilon+\nu(\Lambda_0)(\|\hat{f}\|_\infty+2\varepsilon).$$

ε being arbitrary, (8) is now proved.

LEMMA 3. *Assume φ is measurable on $\Lambda\subset\Gamma$, φ is zero outside Λ and φ is approximable on finite sets in Λ with capacity at most C. For any f, bounded, measurable, vanishing outside Λ, with compact support, put $T(f)=\int_\Gamma f\varphi\,d\gamma$. Then $|T(f)|\leqq C\|\hat{f}\|_\infty$.*

Proof. Let $\varepsilon>0$ be given and let Λ_0 be the compact support of f. By Lemma 2, applied to the two functions $f\varphi$ and f, both with compact support Λ_0, there is a polynomial $p(x)=\sum_1^N b_n(x,\gamma_n)$ such that

(1) $$\int_{\Lambda_0}\left|\sum_1^N b_n f(\gamma+\gamma_n)\varphi(\gamma+\gamma_n)-\int f\varphi\right|d\gamma < \varepsilon^2\nu(\Lambda_0)$$

and

$$\sup_y\left|\sum_1^N b_n f(\gamma+\gamma_n)(y,\gamma+\gamma_n)\right| \leqq F(\gamma), \qquad \gamma\in\Lambda_0,$$

where

(2) $$\int_{\Lambda_0} F(\gamma)\,d\gamma \leqq \nu(\Lambda_0)(\|\hat{f}\|_\infty+\varepsilon).$$

Put

$$E_1 = \left\{\gamma\in\Lambda_0 : \left|\sum_1^N b_n f(\gamma+\gamma_n)\varphi(\gamma+\gamma_n)-\int f\varphi\right| > \varepsilon\right\}.$$

Then, by (1)

$$\varepsilon\cdot\nu(E_1) < \varepsilon^2\nu(\Lambda_0); \qquad \nu(E_1) < \varepsilon\nu(\Lambda_0).$$

Put

$$E_2 = \{\gamma\in\Lambda_0 : F(\gamma) > (1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon)\}.$$

By (2)

$$(1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon)\nu(E_2) \leqq \nu(\Lambda_0)(\|\hat{f}\|_\infty+\varepsilon),$$
$$\nu(E_2) \leqq (1-\varepsilon)\nu(\Lambda_0).$$

We conclude $\nu(E_1\cup E_2)<\nu(\Lambda_0)$. Hence there is $\lambda_0\in\Lambda_0$ such that $\lambda_0\notin E_1$, $\lambda_0\notin E_2$, that is

(3) $$\left|\sum_1^N b_n f(\lambda_0+\gamma_n)\varphi(\lambda_0+\gamma_n)-T(f)\right| \leqq \varepsilon,$$

(4) $$\sup_y\left|\sum_1^N b_n f(\lambda_0+\gamma_n)(y,\lambda_0+\gamma_n)\right| \leqq (1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon).$$

Let A be the finite set of elements of the form $\lambda_0+\gamma_n$, $n=1,\ldots,N$, which belong to $\Lambda_0\cap\Lambda$. By hypothesis there is a polynomial $q(\gamma)=\sum_m c_m(y_m,\gamma)$ with $\sum|c_m|\leqq C$ such that

$$(5)\qquad |q(\gamma)-\varphi(\gamma)| < \varepsilon\Big/\sum_1^N |b_n|\,\|f\|_\infty \quad \text{for } \gamma\in A.$$

Observing that $f(\lambda_0+\gamma_n)=0$ if $\lambda_0+\gamma_n\notin A$ we get from (3) and (5)

$$\left|T(f)-\sum_1^N b_n f(\lambda_0+\gamma_n)q(\lambda_0+\gamma_n)\right| \leqq \varepsilon+\varepsilon.$$

This is

$$\left|T(f)-\sum_m c_m \sum_{n=1}^N b_n f(\lambda_0+\gamma_n)(y_m,\lambda_0+\gamma_n)\right| \leqq 2\varepsilon.$$

By (4) the coefficient of c_m has modulus $\leqq(1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon)$. Hence

$$\begin{aligned}|T(f)| &\leqq \sum_m |c_m|(1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon)+2\varepsilon\\ &\leqq C(1-\varepsilon)^{-1}(\|\hat{f}\|_\infty+\varepsilon)+2\varepsilon.\end{aligned}$$

Since ε is arbitrary, the lemma is proved.

Main Theorem. *Assume that φ is measurable on the measurable set Λ and that φ is approximable on finite sets in Λ with capacity at most C. Then $\varphi(\gamma)=\hat{\mu}(\gamma)$ locally almost everywhere on Λ, where $\mu\in M(G)$ and $\|\mu\|\leqq C$.*

Proof. The linear functional S given by $S(\hat{f})=T(f)=\int_\Gamma f\varphi\,d\gamma$ is defined on the linear space of the transforms $\hat{f}$ of the bounded measurable functions f vanishing outside Λ, with compact support (a subspace of $C_0(G)$) and satisfies the inequality $|S(\hat{f})|\leqq C\|\hat{f}\|_\infty$.

By the Hahn-Banach theorem, S can be extended to the whole of $C_0(G)$, with norm not exceeding C. By the Riesz-Kakutani representation theorem there is a $\mu\in M(G)$ such that $\|\mu\|\leqq C$ and $S(\hat{f})=\int_G\hat{f}(x)\,d\mu(x)$. Then, by Fubini's theorem

$$S(\hat{f}) = \int_\Gamma f(\gamma)\hat{\mu}(\gamma)\,d\gamma,$$

that is, $\int_\Gamma f(\gamma)\varphi(\gamma)\,d\gamma=\int_\Gamma f(\gamma)\hat{\mu}(\gamma)\,d\gamma$ for every f, bounded, vanishing outside Λ, with compact support. We conclude $\varphi(\gamma)=\hat{\mu}(\gamma)$ locally a.e. on Λ and the theorem is proved.

Remark. This theorem contrasts with the situation where instead of the transform of a measure we consider positive definite functions (the transforms of positive measures).

Suppose Λ_0 is a measurable subset of a locally compact abelian group Γ. Define $PD(\Lambda_0)$ to be the class of all continuous complex valued functions φ on $\Lambda_0-\Lambda_0$ which satisfy the inequality

$$\sum_{i,j=1}^N c_i\bar{c}_j\varphi(\gamma_i-\gamma_j) \geqq 0$$

for every positive integer N, for every choice of complex numbers $c_1, \ldots, c_N$ and for every choice of points $\gamma_1, \ldots, \gamma_N$ in Λ_0.

If $G=R$, Λ_0=an interval I, then every $\varphi \in PD(I)$ is the restriction on $I-I$ of the transform $\hat{\mu}$ of some positive measure μ on G (Krein). But if $G=R^2$, $\Lambda_0=$ a closed square S in R^2, then there is $\varphi \in PD(S)$ which is not the restriction to $S-S$ of the transform of a positive measure on G (see Rudin [8]).

Before ending our paper we want to state, in a new form, the two theorems appearing in [4].

THEOREM. *A continuous function φ defined on Γ is the Fourier-Stieltjes transform of a singular measure on G if and only if there is a constant C such that*

(i) *φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C.*

(ii s) *Whatever be $\varepsilon>0$ and the compact set Λ in Γ, φ is not approximable on finite sets F, not meeting Λ, by polynomials of capacity $\leqq C-\varepsilon$.*

THEOREM. *A continuous function φ defined on Γ is the Fourier transform of an integrable function on G if and only if*

(i) *there is a constant C such that φ can be approximated on any finite set in Γ by trigonometric polynomials of capacity at most C.*

(ii a) *To every $\varepsilon>0$ corresponds a compact set Λ in Γ such that φ can be approximated on any finite set in Γ, not meeting Λ, by trigonometric polynomials of capacity at most ε.*

To prove these theorems we just make use of the equivalence stated in Proposition 2 and combine this equivalence with Theorems 1 and 2 of [4]. Observe also that no form of any of these two theorems is readily available for restrictions, since the transform $\hat{\mu}_s$ of a singular measure can be equal to the transform $\hat{f}$ of an integrable function on very large sets (see e.g. Rudin [9]).

BIBLIOGRAPHY

1. N. I. Ahiezer, *Lectures on the theory of approximation*, OGIZ, Moscow, 1947; English transl., Ungar, New York, 1956. MR **10**, 33; MR **20** #1872.

2. S. Bochner, *A theorem on Fourier-Stieltjes integrals*, Bull. Amer. Math. Soc. **40** (1934), 271–276.

3. K. de Leeuw and C. Herz, *An invariance property of spectral synthesis*, Illinois J. Math. **9** (1965), 220–229. MR **31** #3787.

4. R. Doss, *On the transform of a singular or an absolutely continuous measure*, Proc. Amer. Math. Soc. **19** (1968), 361–363. MR **36** #5619.

5. W. F. Eberlein, *Characterizations of Fourier-Stieltjes transforms*, Duke Math. J. **22** (1955), 465–468. MR **17**, 281.

6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York, and Springer-Verlag, Berlin, 1963. MR **28** #158.

7. H. P. Rosenthal, *A characterization of restrictions of Fourier-Stieltjes transforms*, Pacific J. Math. **23** (1967), 403–418. MR **36** #3065.

8. W. Rudin, *The extension problem for positive-definite functions*, Illinois J. of Math. **7** (1963), 532–539. MR **27** #1779.

9. ———, *Modifications of Fourier transforms*, Proc. Amer. Math. Soc. **19** (1968), 1069–1074. MR **39** #1903.

STATE UNIVERSITY OF NEW YORK,
STONY BROOK, NEW YORK 11790