

## ON HIGHER-DIMENSIONAL FIBERED KNOTS

BY

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**Abstract.** The geometrical properties of a fibration of a knot complement over  $S^1$  are used to develop presentations for the homotopy groups as modules over the fundamental group. Some homotopy groups of spun and twist-spun knots are calculated.

**I. Introduction.** Let  $K^n$  be a smooth submanifold of  $S^{n+2}$  which is homeomorphic to  $S^n$ . The pair  $(S^{n+2}, K)$  is a knot, with complement  $S = S^{n+2} - K$ . A tubular neighborhood of  $K$  in  $S^{n+2}$  provides us with a map  $p: S \rightarrow S^1$ , which in some cases is a fiber map [7]. When the complement fibers, the homotopy exact sequence of the fibration  $F^{n+1} \rightarrow S \xrightarrow{p} S^1$  tells us that  $\Pi_i(F) \cong \Pi_i(S)$ ,  $i \geq 2$ ,  $\Pi_1(F) \cong [\Pi_1(S), \Pi_1(S)]$  the commutator subgroup of  $\Pi_1(S)$ , and  $\Pi_1(S) \cong \Pi_1(F) \times Z$  the semidirect product of  $\Pi_1(F)$  and the integers.

If  $\Pi_1(S) = Z$ , and  $n \geq 4$ , a necessary and sufficient condition that a fibration exists is that the groups  $\Pi_i(S)$  are finitely generated as abelian groups for all  $i$  [3]. In this case the fiber is simply connected. If  $\Pi_1(S) \neq Z$ , there are certain obstructions which determine whether or not  $S$  fibers over the circle [5].

There are many interesting cases where  $S$  fibers when  $\Pi_1(S) \neq Z$ , and the fact that  $\Pi_1(F) \neq 1$  can be used to obtain information about the groups  $\Pi_i(S)$  and about the homotopy type of  $\tilde{S}$ , the universal cover of  $S$ . §II of this paper studies the general situation of a fiber bundle  $F^{n+1} \rightarrow S \rightarrow S^1$  where  $\Pi_1(F) \neq 1$  and  $F \simeq \bar{F}$ , a smooth compact bounded manifold with  $\partial\bar{F}$  homeomorphic to  $S^n$ .

In §III we study the particular case of fibered  $k$ -spun knots. If a knot fibers, then the  $k$ -spin of the knot fibers by  $k$ -spinning the fibration. For a knot of  $S^1$  in  $S^3$ , the  $k$ -spun knot fibers if and only if the knot of  $S^1$  in  $S^3$  fibers. In this case, no matter which two (nontrivial) Neuwirth knots one  $k$ -spins, the universal covers of the resulting knot complements are found to be homotopy equivalent.

In §IV we develop an algebraic result about extending module presentations. If

$$1 \longrightarrow H \xrightarrow{\alpha} G \begin{array}{c} \xrightarrow{\beta} T \longrightarrow 1 \\ \xleftarrow{\gamma} \end{array}$$

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is an exact sequence of groups such that  $\beta\gamma = \text{identity on } T$  ( $G$  is then the semi-direct product of  $H$  and  $T$ ), and  $A$  is a finitely presented  $ZH$ -module ( $ZH = \text{integral group ring of } H$ ) such that  $T$  acts on  $A$ , then the given presentation for  $A$  as a  $ZH$ -module can be extended to a presentation of  $A$  as a  $ZG$ -module. This theorem is then used to determine the structure of the groups  $\{\Pi_i(S)\}$  as  $Z\Pi_1(S)$  modules and the groups  $\{H_i(S^*; Z)\}$  as  $\Lambda$ -modules, where  $S^* = \text{infinite cyclic cover of } S$  and  $\Lambda = \text{integral group ring of } J(t)$ , the infinite cyclic group of covering transformations of  $S^*$  generated by  $t$ .

In §V we apply the results of §IV to calculate the homotopy structure of simple fibered knots. A fibered knot  $(S^{n+2}, K^n)$  is said to be simple if  $\Pi_i(S) = 0$ ,  $2 \leq i \leq n-1$ . Any fibered  $(S^4, K^2)$  knot is said to be simple. For example, any  $k$ -spun Neuwirth knot is simple by a theorem of Epstein [4]. For simple knots, complete results are obtained for  $\Pi_n(S)$  as a  $Z\Pi_1(S)$ -module when the commutator subgroup  $[\Pi_1(S), \Pi_1(S)]$  is a finite group.

In §VI we calculate as examples the homotopy structure of the 5-twist-spun trefoil and the  $k$ -spun trefoil.

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**II. General theorems on fibered knots.** Consider the bundle  $F^{n+1} \rightarrow S \rightarrow S^1$  where  $S = S^{n+2} - K^n$  a knot complement,  $n \geq 2$ . Let  $\approx$  denote diffeomorphism,  $\simeq$  denote homotopy equivalence, and  $\vee$  denote wedge product of spaces. Let  $\tilde{S}, \tilde{F}$  denote the universal covers of  $S$  and  $F$ . Clearly  $\tilde{S} \approx \tilde{F} \times R^1 \simeq \tilde{F}$ . Let  $G = \Pi_1(F)$ , and  $1_G$  denote the identity of  $G$ . Let  $V^+ = \bigvee_{g \in G} K_g^n$  be the wedge product of copies of  $K$  indexed by  $g \in G$  with the identification topology.

Now the fiber  $F$  deformation retracts to  $\bar{F}$ , a compact bounded  $(n+1)$ -manifold with  $\partial\bar{F} \approx K$ . The universal cover  $\tilde{F}$  induces  $\bar{F}^\sim$  over  $\bar{F}$ , and  $\tilde{F}$  deformation retracts to  $\bar{F}^\sim$ . We have  $\partial\bar{F}^\sim = \bigcup_{g \in G} K_g^n$  the disjoint union of copies of  $K$  indexed by  $G$ , the group of covering translations of  $\bar{F}^\sim$ .

**THEOREM 1.** *If  $G = \Pi_1(F)$  is not a finite group, then*

- (i)  $\Pi_i(F) \simeq \Pi_i(\bar{F}^\sim) \simeq \Pi_i(\partial\bar{F}^\sim) \oplus \Pi_i(\bar{F}^\sim, \partial\bar{F}^\sim)$  as  $ZG$ -modules,  $i \geq 2$ ,
- (ii)  $\Pi_i(V^+)$  is a direct summand of  $\Pi_i(S)$  as an abelian group for all  $i$ .

**Proof.** There is a retraction  $r: \bar{F}^\sim \rightarrow \partial\bar{F}^\sim$  because the obstructions lie in  $H^{i+1}(\bar{F}^\sim, \partial\bar{F}^\sim; \Pi_i(\partial\bar{F}^\sim))$  which vanish for  $i < n$  because  $\Pi_i(\partial\bar{F}^\sim) = 0$  in this range, and for  $i = n$  because  $\bar{F}^\sim$  is not compact if  $G$  is not finite. The retraction commutes with covering translations in  $\bar{F}^\sim$ , so (i) is proved.

Choosing a base point  $* \in K_{1_g}$ , let  $*_g \in K_g$  represent points in the same fiber as  $*$ , and choose arcs  $a_g \subset \bar{F}^\sim$  from  $*$  to  $*_g$ , such that  $a_{g_1} \cap a_{g_2} = *$  if  $g_1 \neq g_2$ , and such that the projection of  $a_g$  in  $F$  is in the homotopy class of  $g \in G$ . Let  $\tilde{V}^+ = \partial\bar{F}^\sim \cup (\bigcup_{g \in G} a_g)$ . Then  $\tilde{V}^+ \simeq V^+$  by shrinking the arcs.

There is a retraction from  $\bar{F}^\sim$  to  $\tilde{V}^+$  for the same reasons as before, only this time the splitting works at the abelian group level.

Now, let  $V = \bigvee_{g \in (G-1_G)} K_g^n$  be the wedge product of copies of  $K$  indexed by elements in the set  $\{G-1_G\}$ . ( $V^+ \simeq V$  iff  $G$  is infinite.)

**THEOREM 2.** *If  $G$  is a finite group then  $\tilde{S} \simeq Q \vee V$  where  $Q$  is a simply-connected finite simplicial complex of dimension less than or equal to  $n$ .*

**Proof.**  $\bar{F}^\sim$  is compact since  $G$  is finite, and  $\bar{F}^\sim = \bigcup_{i=1}^{|G|} K_i$  the finite union of copies of  $K$ , where  $|G| = \text{order of } G$  and  $K_1 = K_{1_G}$ . Choose arcs  $a_i$  from  $* \in K_1$  to  $*_i \in K_i$  as in the proof of Theorem 1. We will deformation retract away from  $K_1$  and isolate  $K_2$  in a wedge product decomposition for  $\bar{F}^\sim$ . That is, we will obtain a homotopy equivalence  $\bar{F}^\sim \simeq K_2 \vee Q_2$  where  $Q_2$  is a bounded  $(n+1)$ -manifold with  $\partial Q_2 \approx K_1 \# K_2$  (connected sum).

Collapse away along a small tubular neighborhood of  $a_2$  until the boundary of a small collar neighborhood of  $K_2$  is reached [Figure 1].

$K_2$  is a homotopy sphere, so the collar now deformation retracts from an  $n$ -disc  $D_2^n$  on its boundary component to  $K_2 \cup \text{arc} \cup L_2$  where  $L_2 \approx K_2 - \text{Int } D_2^n$ , and the

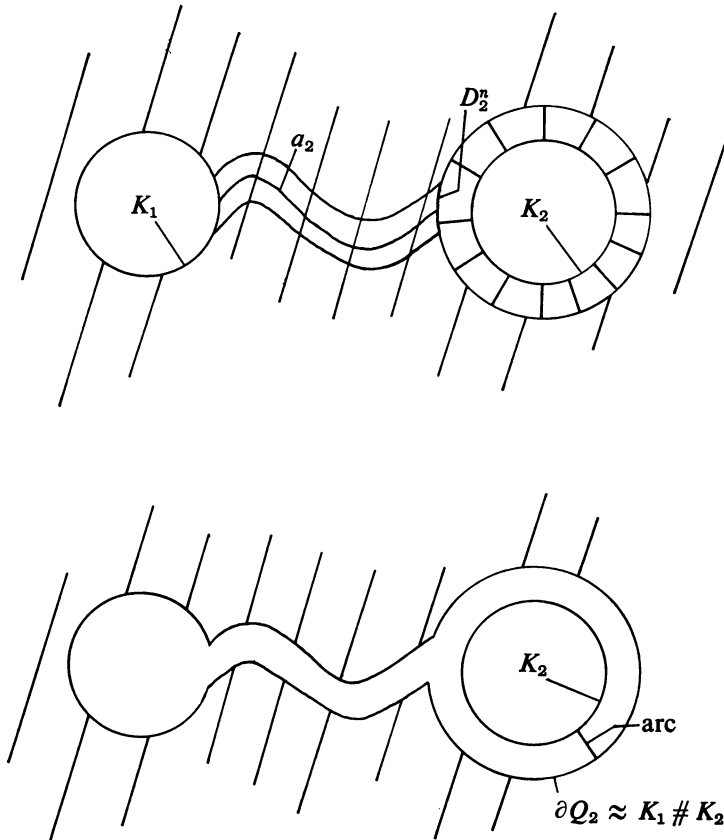


FIGURE 1

arc is “straight” in the product structure of the collar. Continuing this process, we obtain  $K \simeq Q' \vee \bigvee_{i=2}^n K_i = Q' \vee V$  where  $Q'$  is a bounded  $(n + 1)$ -manifold with  $\partial Q' \approx \#_{i=1}^n K_i$ . Clearly then  $K \simeq Q \vee V$ ,  $Q$  a connected simply-connected  $n$ -complex obtained from  $Q'$  by collapsing out all the top-dimensional cells.

**COROLLARY 3.**  *$G$  a finite group and  $S$  a simple fibered knot then  $\tilde{S} \simeq V$ .*

**Proof.** From the proof of Theorem 2 we have  $\bar{F} \simeq Q' \vee V$ .  $Q'$  is contractible since  $S$  is a simple knot.

We would also note that if  $G$  is a finite group, then from Serre’s theorem [6, p. 509] we have that  $\Pi_i(\bar{F}) \cong \Pi_i(S)$  is a finitely generated abelian group for all  $i \geq 2$ , since  $\bar{F}$  compact.

**III. Fibered spun knots.** Let  $f: B^n \rightarrow B^{n+2}$  be a smooth proper embedding ( $n \geq 1$ ) of balls. We obtain the knot  $(S^{n+2+k}, K^{n+k})$  by  $k$ -spinning  $(B^{n+2}, f(B^n))$  in the following manner [4]:

$$S^{n+2+k} = (S^k \times B^{n+2}) \bigcup_{S^k \times f(S^{n-1})} (B^{k+1} \times S^{n+1}),$$

$$S^{n+k} \approx K^{n+k} = (S^k \times f(B^n)) \bigcup_{S^k \times f(S^{n-1})} (B^{k+1} \times f(S^{n-1})).$$

*Note.* We allow the boundary sphere pair of the ball pair to be knotted.

**LEMMA 4.**  *$(S^{k+3}, K^{k+1})$  a  $k$ -spun  $(B^3, f(B^1))$  fibers iff  $[\Pi_1(B), \Pi_1(B)]$  is finitely generated, where  $B = B^3 - f(B^1)$ .*

**Proof.** In this case  $\Pi_1(S) = \Pi_1(B)$  where  $S = S^{k+3} - K^{k+1}$ . If  $S$  fibers over  $S^1$ , then clearly  $[\Pi_1(B), \Pi_1(B)]$  is finitely generated. Now  $(B^3, f(B^1))$  determines a unique sphere pair  $(S^3, f'(S^1))$ , and by the Neuwirth-Stallings Theorem [7, p. 475],  $S^3 - f'(S^1)$  fibers if  $[\Pi_1(S), \Pi_1(S)] \cong [\Pi_1(B), \Pi_1(B)]$  is finitely generated. One obtains the fibration for  $S$  by  $k$ -spinning the fibration of the ball pair. Twist spinning a fibered knot to obtain a fibration is described in detail by Zeeman [7]. In order to obtain a fibration by spinning only, one neglects to twist as the spinning goes on.

This process is considered in greater detail in the proof of Theorem 6. Lemma 4 says that if  $(S^{k+3}, K^{k+1})$  is a  $k$ -spun fibered knot, then in fact it is a  $k$ -spun Neuwirth knot. As before, let  $V^+ = \bigvee_{g \in G} S^{k+1}$ .

**THEOREM 5.** *Let  $(S^{k+3}, K^{k+1})$  be a  $k$ -spun Neuwirth knot of genus  $g \geq 1$ . Then  $\tilde{S} \simeq V^+ \vee V^+ \vee \dots \vee V^+ (2g\text{-fold wedge products})$ .*

**Proof.** We will study carefully the geometry of the spun fibration. Figure 2 describes the fibration of  $B^3 - f(B^1)$  by  $M_g^2$  the “partially-bounded” punctured torus of genus  $g$  [Figure 2].

$M_{g,\phi}^2$  is the fiber at time  $\phi \in S^1$ .  $\partial M_{g,\phi}^2$  is an open arc lying on  $\partial B^3$ , and is the fiber of the bundle induced on  $S^2 - f(S^0)$ .

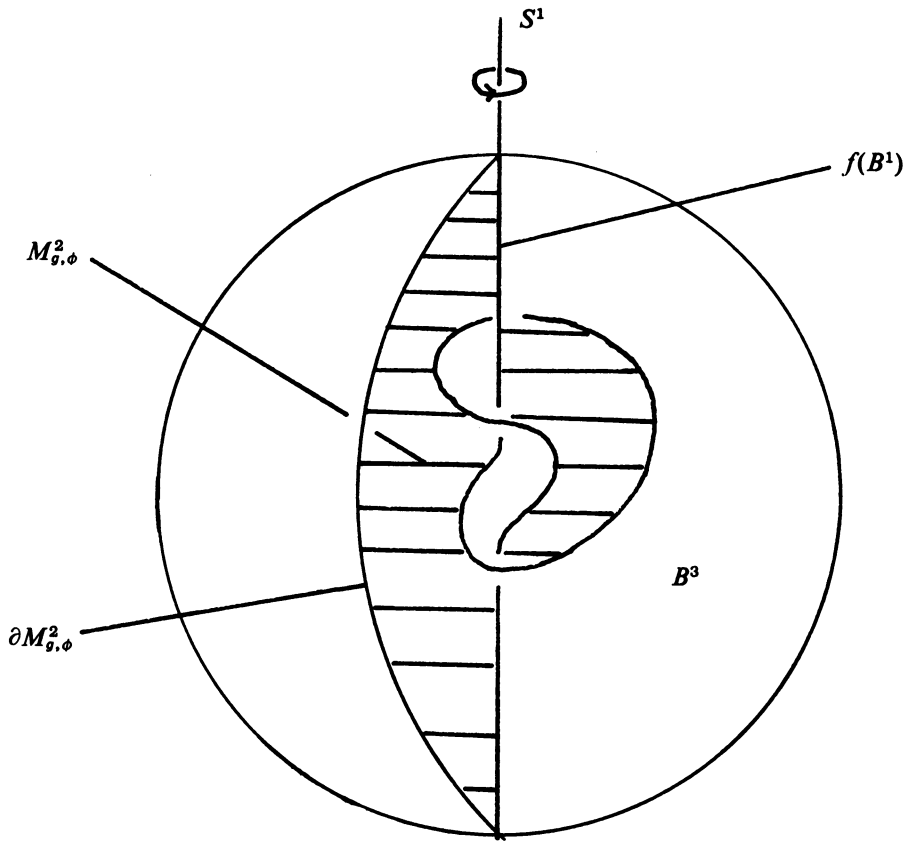


FIGURE 2.  $g=1$ .

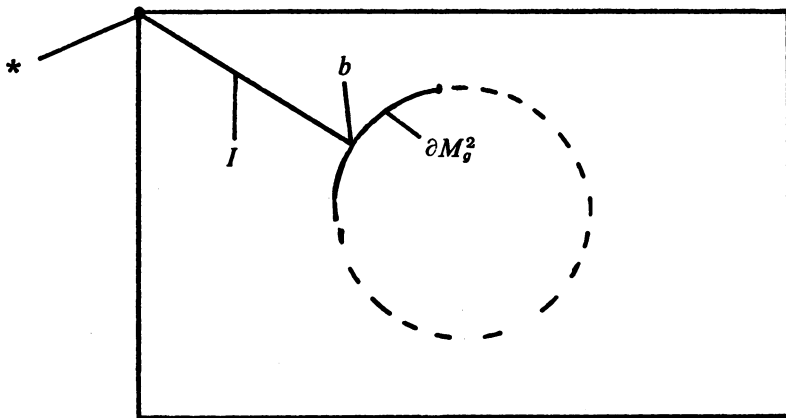


FIGURE 3.  $g=1$ .

The spun fiber  $F_{g,\phi}$  of  $S^{k+3} - K^{k+1}$  at time  $\phi$  is the following:

$$F_{g,\phi} = (S^k \times M_{g,\phi}^2) \cup_{S^k \times \partial M_{g,\phi}^2} (B^{k+1} \times \partial M_{g,\phi}^2).$$

Let  $L_{2g}^p = S^p \vee \dots \vee S^p$  the  $2g$ -fold wedge product of  $S^p$  with itself. Now  $M_g^2$  deformation retracts to  $\partial M_g^2 \cup I \cup L_{2g}^1$  where  $I$  (Figure 2) is an arc in  $M_g^2$  running from a point  $b \in \partial M_g^2$  to  $*$  the wedge point of  $L_{2g}^1$  [Figure 3].

$\partial M_g^2$  deformation retracts to  $b$ . These deformations produce a deformation retraction from  $F_g$  to  $F'_g = (S^k \times L_{2g}^1) \cup_{S^k \times *} (B^{k+1} \times *)$  [Figure 4].

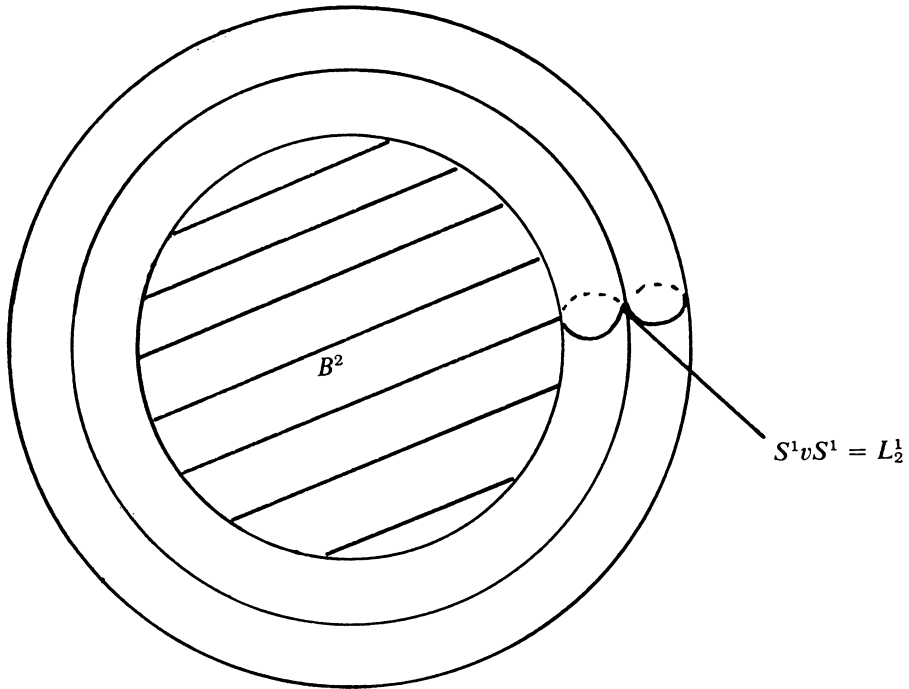


FIGURE 4.  $k=1, g=1$ .

Consider now  $P = (S^k \times S^1) \cup_{S^k \times *} (B^{k+1} \times *)$ . Thinking of  $S^1$  as the union of two arcs  $I_1$  and  $I_2$ , we have by shrinking  $I_1$  to a point that  $P \simeq P'$ , where  $P' = (S^k \times S^1) \cup_{S^k \times I_1} (B^{k+1} \times I_1)$  [Figure 5].

Now  $P' \simeq S^1 \vee S^{k+1}$  by collapsing  $B^{k+1} \times I_1$  to  $I_1$ , and then shrinking  $I_2$  to a point. Hence  $F'_g \simeq L_{2g}^1 \vee L_{2g}^{k+1}$ . Since a homotopy equivalence of CW complexes induces a homotopy equivalence of their universal covers, we have that  $\tilde{F}'_g \simeq (L_{2g}^1 \vee L_{2g}^{k+1})^\sim$ , the latter obtained (up to homotopy type) by taking the universal cover of  $L_{2g}^1$  (a snowflake with four  $g$  arcs emanating from each vertex) and attaching a copy of  $L_{2g}^{k+1}$  to each vertex. Now  $[\Pi_1(S), \Pi_1(S)] \cong \Pi_1(M_g^2) \cong \Pi_1(L_{2g}^1)$

=free group on  $2g$  generators. By squeezing the 1-skeleton to a point, we have  $\tilde{S} \simeq V^+ \vee \dots \vee V^+$  the  $2g$ -fold wedge product of  $V^+$  with itself. This completes the proof of Theorem 6.

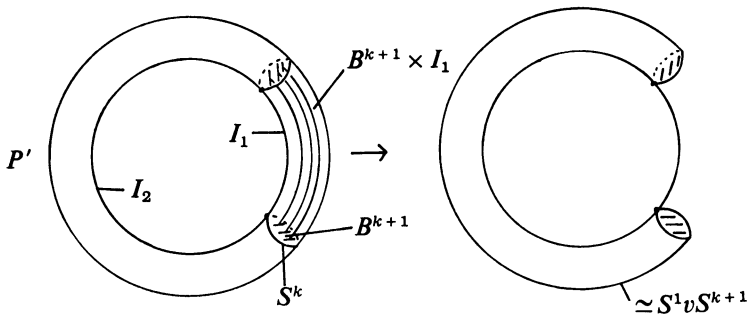


FIGURE 5

In [1], Andrews and Curtis ask the following question: If  $(S^4, K^2)$  is the 1-spun trefoil, and  $S = S^4 - K^2$ , is  $\Pi_3(S) = 0$ ? Theorem 5 answers this question in the negative.

**COROLLARY 6.** *Let  $(S^{k+3}, K^{k+1})$  be a  $k$ -spun Neuwirth knot of genus  $g \geq 1$ . Then  $\tilde{S} \simeq \bigvee_{i=1}^{\infty} S_i^{k+1}$  the infinite wedge of  $k+1$ -spheres.*

**Proof.**  $V^+ \vee \dots \vee V^+$  ( $2g$  times)  $= \bigvee_{i=1}^{\infty} S_i^{k+1}$  for all  $g \geq 1$ .

This means that if we  $k$ -spin any two nontrivial Neuwirth knots, the universal covers of the resulting knot complements are homotopy equivalent. In fact, this also is true for iterated  $k$ -spinning ( $k$  allowed to vary) of Neuwirth knots, and will be dealt with in a future paper.

**COROLLARY.** *Let  $(S^{k+3}, K^{k+1})$  be a  $k$ -spun Neuwirth knot of genus  $g$ . Then  $\Pi_{k+1}(S)$  is a free  $ZG$ -module on  $2g$  generators.*

**Proof.** Select a lift of  $L_{2g}^{k+1}$  to  $\tilde{F}_g$ . Each of the spheres in the lift is a free generator of

$$H_{k+1}(\tilde{F}_g) \cong_{ZG} \Pi_{k+1}(F_g) \cong \Pi_{k+1}(S).$$

The above shows that if  $(S^{k+3}, K^{k+1})$  is a  $k$ -spun Neuwirth knot of genus  $g$ , then  $\Pi_{k+1}(S)$  is the free abelian group on  $2g$  copies of the symbols in  $\{[\Pi_1(S), \Pi_1(S)]\}$ . Compare the result of Epstein [4], which says that in this case  $\Pi_{k+1}(S)$  is the free abelian group on the symbols  $\{[\Pi_1(S), \Pi_1(S)] - \text{id}\}$ . (There is in this case a 1-1 correspondence between  $\{[\Pi_1(S), \Pi_1(S)] - \text{id}\}$  and  $2g$  copies of  $[\Pi_1(S), \Pi_1(S)]$ .)

**IV. Change of rings.** Let

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightleftharpoons[\gamma]{\beta} T \longrightarrow 1$$

be an exact sequence of groups such that  $\beta\gamma = 1_T$ . Then  $G$  is the semidirect product of  $H$  and  $T$  so that each element of  $G$  can be written uniquely as  $ht$  where  $h \in \alpha(H) = H$  and  $t \in \gamma(T) = T$ . Now given a  $ZH$ -module  $A$  the group  $ZG \otimes_{ZH} A$  can be considered as a  $ZG$ -module under the action  $g \cdot (g' \otimes a) = gg' \otimes a$  where  $g, g' \in ZG$  and  $a \in A$ . Also if we assume  $T$  acts on  $A$  as a group of automorphisms then  $A$  can also be considered as a  $ZG$ -module under the action  $ga = (ht)a = h(ta)$  where  $t \in T, h \in H, a \in A$  and  $g = ht \in G$ . If  $T$  is finitely generated by, say,  $t_1, \dots, t_k$  and  $A$  is finitely generated as a  $ZH$ -module by  $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$  then the action of  $T$  on  $A$  may be given by a matrix  $\tau$  coming from  $t_p(\bar{X}_i) = \sum \lambda_{ij}^p \bar{X}_j$ . Let  $\tau$  be the  $(nk+n)$  matrix

$$(1) \quad \begin{matrix} t_1 I - [\lambda_{ij}^1] \\ \vdots \\ t_k I - [\lambda_{ij}^k] \end{matrix}$$

where  $I$  is the  $(n \times n)$  identity matrix, and  $[\lambda_{ij}^p]$  is the  $(n \times n)$  matrix of the coefficients.

We shall in this section use the same symbols for a map and the associated matrix given by some selection of generators. Content will make the meaning clear.

**LEMMA 8.** *Suppose  $K \xrightarrow{\tau} M \xrightarrow{\phi} A \rightarrow 0$  is an exact sequence of  $R$ -modules with  $M$  finitely presented and  $K$  finitely generated. Let  $\psi$  be a presentation matrix for  $M$ . Then a presentation matrix for  $A$  is  $(\begin{smallmatrix} \psi \\ \tau \end{smallmatrix})$ .*

**Proof.** Let  $F_2 \xrightarrow{\psi} F_1 \xrightarrow{\sigma'} A \rightarrow 0$  be a presentation of  $M$  by free finitely generated  $R$ -modules  $F_1, F_2$  and let  $F \xrightarrow{\sigma} K \rightarrow 0$  be a map of a free finitely generated  $R$ -module  $F$  onto  $K$ . We have the following diagram

$$\begin{array}{ccccccc} & & & & F_2 & & \\ & & & & \downarrow \psi & & \\ & & & & F_1 & & \\ F & \xrightarrow{\tilde{\tau}} & & & & & \\ \downarrow \sigma & & & & \downarrow \sigma' & & \\ K & \xrightarrow{\tau} & M & \xrightarrow{\phi} & A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

Since  $F$  is projective there is a map  $\tilde{\tau}: F \rightarrow F_1$  such that  $\sigma' \tilde{\tau} = \tau \sigma$ . It follows that  $\text{Ker } \phi \sigma' = \text{Im } \tilde{\tau} + \text{Im } \psi$  so that matrix associated with this presentation of  $A$  is  $(\begin{smallmatrix} \psi \\ \tau \end{smallmatrix})$ . But the matrix  $\tilde{\tau} = \tau$ .

**THEOREM 9.** *Given*

$$0 \longrightarrow H \xrightarrow{\alpha} G \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{matrix} T \longrightarrow 0$$



an exact sequence of groups with  $T$  finitely generated and  $\beta\gamma=1_T$ . Let  $\psi$  be a presentation matrix for a finitely presented  $ZH$ -module  $A$  on which  $T$  acts as a group of automorphism. Then  $A$  is a  $ZG$ -module with presentation matrix  $(\psi)$  where  $\tau$  is as in (1).

**Proof.** Let

$$F_2 \xrightarrow{\psi} F_1 \xrightarrow{\phi} A \longrightarrow 0$$

be an appropriate presentation of  $A$ . We have the exact sequence of  $ZG$ -modules

$$ZG \otimes_{ZH} F_2 \xrightarrow{1 \otimes \psi} ZG \otimes_{ZH} F_1 \xrightarrow{1 \otimes \phi} ZG \otimes_{ZH} A \longrightarrow 0.$$

Let  $A'$  be the set  $A$  with the  $ZG$ -structure obtained from the action of  $ZH$  and  $T$  on  $A$ . Then there is a natural map  $\alpha: ZG \otimes_{ZH} A \rightarrow A'$  given by  $\alpha(\lambda \otimes X) = \lambda X$ . Clearly the kernel  $K$  of  $\alpha$  is generated by all elements of the form  $\{\lambda \otimes X - 1 \otimes (\lambda X)\}$  where  $(\lambda X)$  is the element of  $A$  corresponding to  $\lambda X$  in  $A'$ . ( $A' = A$ , setwise.) Now  $K$  is generated by  $\{t_p \otimes \phi(X_i) - 1 \otimes \sum_j \phi_{ij}^p X_j\}$  for all  $p, i$ . That is,  $K$  is generated by the images of  $\{t_p \otimes X_i - 1 \otimes \sum \lambda_{ij} X_j\}$  under  $1 \otimes \phi$ . The desired presentation now follows from Lemma 8.

**V. Applications.** Let  $Q^{n+2}$  be an  $(n+2)$ -manifold which fibers over  $S^1$  with fiber an  $(n+1)$ -manifold  $F^{n+1}$ . Let  $p: Q \rightarrow S^1$  be the fiber map. We think of  $Q$  as the product  $F \times I$  with  $F \times 0$  and  $F \times 1$  identified by a homeomorphism  $h: F \times 1 \rightarrow F \times 0$  given by  $h(X, 1) = (h(X), 0)$ . We can without loss of generality assume  $h(q) = q$  for some  $q \in F$ . Let  $t': I \rightarrow q \times I \subset F \times I$  be the product path from  $q \times 1$  to  $q \times 0$  in  $F \times I$ . Then (under  $h$ )  $t'$  corresponds to an element  $t \in \Pi_1(Q, q \times 1)$  which maps onto a generator of  $\Pi_1(S^1)$  under the fiber map  $p_*$ . From the homotopy exact sequence of the fibration, this gives us a semidirect splitting  $\Pi_1(S) \cong \Pi_1(F) \times Z$ . If  $i$  denotes the inclusion  $i: F \rightarrow Q$  by  $i(x) = (x, 1)$  and  $\beta \in \Pi_k(F, q)$  we then have  $t \cdot i_*(\beta) = i_*(h_*\beta)$ .

We have the following corollary to Theorem 9:

**COROLLARY 10.** *Suppose  $\Pi_k(F)$  is finitely presented as a  $Z\Pi_1(F)$ -module by a matrix  $M$ . Then  $\Pi_k(Q) \cong \Pi_k(F)k \geq 2$  is presented as a  $Z\Pi_1(Q)$ -module by  $(t \begin{smallmatrix} M \\ h_* \end{smallmatrix})$ , where  $t-h_*$  denotes the appropriate matrix.*

Consider the infinite cyclic cover  $Q^*$  of  $Q$ . We think of  $Q^*$  in two different ways. First as the product  $Q^* \approx F \times R$ , and second as the union of  $X_i = (F \times I)_i$  ( $i \in Z$ ) with  $(F \times 1)_i$  identified with  $(F \times 0)_{i+1}$  by the homeomorphism  $h$ . Now  $t$  acts on  $H_k(Q^*) \cong H_k(F)$  by  $t(u) = h_*(u)$  where  $u \in H_k(F)$  and  $h_*: H_k(F) \rightarrow H_k(F)$  is the automorphism induced by  $h$ . Let  $\Lambda =$  the group ring of the infinite cyclic group generated by  $t$ .

**COROLLARY 11.** *Suppose  $H_k(F)$  is finitely presented as an abelian group by a matrix  $M$ . Then  $H_k(Q^*) \cong H_k(F)$  is presented as a  $\Lambda$ -module by  $(t \begin{smallmatrix} M \\ h_* \end{smallmatrix})$ .*

Now assume  $S$  and  $F$  are as in §I. That is,  $S$  is the complement of  $K^n$  in  $S^{n+2}$  which fibers over  $S^1$  with fiber  $F^{n+1}$ .

**THEOREM 12.** *Suppose that  $G = \Pi_1(F)$  is finite, and that  $S$  is a simple fibered knot. Then  $\Pi_n(F)$  is presented as a  $ZG$ -module by the  $(1 \times 1)$  matrix  $(\sum_{g \in G} g)$ .*

**Proof.** Take  $F$  to be compact and bounded with  $\partial F \approx K$ . Then the universal cover  $\tilde{F}$  is a compact contractible manifold with  $\partial \tilde{F} = \bigcup_{g \in G} K_g$ , as in the proof of Corollary 3. By Lefschetz Duality,  $H_n(\tilde{F}, \partial \tilde{F}) \cong H^1(\tilde{F}) = 0$ , and the homology exact sequence of the pair  $(\tilde{F}, \partial \tilde{F})$  yields

$$H_{n+1}(\tilde{F}, \partial \tilde{F}) \xrightarrow{\partial} H_n(\partial \tilde{F}) \longrightarrow H_n(\tilde{F}) \longrightarrow 0.$$

$H_{n+1}(\tilde{F}, \partial \tilde{F})$  is free abelian on 1 generator  $\xi$ , and  $H_n(\partial \tilde{F})$  is free  $ZG$ -module on 1 generator  $\alpha$ , and  $\partial \xi = \sum_{g \in G} g\alpha$ . By Lemma 8, the result follows.

**THEOREM 13.** *Suppose  $(S^{n+2}, K^n)$  is a simple fibered knot, and that  $G = \Pi_1(F) = [\Pi_1(S), \Pi_1(S)]$  is finite, then  $\Pi_n(S)$  is presented as a  $Z \Pi_1(S)$ -module by the  $(2 \times 1)$ -matrix*

$$\begin{pmatrix} \sum_{g \in G} g \\ t-1 \end{pmatrix}.$$

**Proof.** We can without loss of generality replace the complement  $S^{n+2} - K$  by the closed complement  $S$ , a manifold-with-boundary in which the entire manifold  $S$  fibers over  $S^1$  with fiber  $F$ , and the fibration induced on  $\partial S$  has fiber  $\partial F \approx K$ . Moreover, up to homotopy, the fibering on the boundary is the product bundle, because any orientation preserving homeomorphism  $h: K \rightarrow K$  is homotopic to the identity. We can take the homotopy class of the embedded sphere  $K$  as the generator for  $\Pi_n(F)$  as a  $ZG$ -module. Clearly then  $h_*$  is the identity automorphism on  $\Pi_n(F)$ .

We are left with the case of simple fibered knots whose commutator subgroups are not finite. The general case, unfortunately, seems not so clear as the special case of  $k$ -spun Neuwirth knots treated earlier in §III. We do, however, have the following corollary to Theorem 1:

**COROLLARY 14.**  *$(S^{n+2}, K^n)$  a simple fibered knot, and  $G = \Pi_1(S)$  is infinite. Then  $\Pi_n(S)$  is free abelian, and  $\Pi_n(S) \cong ZG \oplus H_n(\tilde{F}, \partial \tilde{F})$ .*

**Proof.** We have

$$\Pi_n(\tilde{S}) \cong \Pi_n(F) \cong_{ZG} H_n(\tilde{F}) \cong_{ZG} H_n(\partial \tilde{F}) \oplus H_n(\tilde{F}, \partial \tilde{F}) \cong_{ZG} ZG \oplus H_n(\tilde{F}, \partial \tilde{F}).$$

$H_n(\tilde{F})$  is free abelian, because  $F^{n+1}$  is a manifold-with-boundary, hence  $F \simeq X$  an  $n$ -dimensional finite simplicial complex. Now  $H_n(\tilde{X}) \cong Z_n(\tilde{X}) \subset C_n(\tilde{X})$ , so  $H_n(\tilde{X})$  is free abelian. Note that  $H_n(\tilde{F}, \partial \tilde{F}) \cong H_n(F, \partial F; \{ZG\}) \cong H^1(F; \{ZG\}) \cong H^1(G; ZG)$  where  $\{ZG\}$  denotes local coefficients and the last isomorphism is due to the fact that  $\tilde{F}$  is simply connected.

**VI. Examples.** Many examples of simple knots may be generated by Zeeman's twist-spinning methods [7]. For example, one can obtain fibered knots of  $S^2$  in  $S^4$  by twist-spinning the bridge knots. The fiber obtained is the punctured lens space  $L(p, g)$  ( $p$  odd) [7, Corollary 5].

**EXAMPLE 1.** 5-twist-spun trefoil.

By 5-twist-spinning the trefoil, we obtain the knot  $(S^4, K^2)$  and the fibration  $F^3 \rightarrow S = S^4 - K^2 \rightarrow S^1$ ,  $F^3 =$  punctured dodecahedral space. From Zeeman [7], we have that

$$\begin{aligned} \Pi_1(S) &= (x, y, t \mid x^5 = (xy)^3 = (xyx)^2, t^{-1}xt = y, t^{-1}yt = yx^{-1}), \\ G = \Pi_1(F) &= (x, y \mid x^5 = (xy)^3 = (xyx)^2). \end{aligned}$$

In the presentation for  $\Pi_1(S)$ , the generator  $t$  corresponds to the generator of  $\Pi_1(S^1)$ . Now  $G$  is the binary dodecahedral group of order 120. By Corollary 4, the universal cover  $\tilde{F}$  is homotopy equivalent to the wedge product of 119 2-spheres. ( $\tilde{F}$  is a 3-sphere punctured 120 times.) By Theorem 14,  $\Pi_2(S)$  is presented as a  $Z\Pi_1(S)$ -module by the  $(2 \times 1)$ -matrix

$$\begin{pmatrix} \sum_{g \in G} g \\ t - 1 \end{pmatrix}.$$

Since  $F$  is a homology 3-ball, all the Alexander invariants of the infinite cyclic cover  $S^*$  are trivial by Corollary 11.

**EXAMPLE 2.** 2-twist-spun trefoil.

This knot fibers with fiber punctured  $L(3, 1)$ . In this case  $\Pi_1(S) = (a, b \mid aba = bab, b = a^2ba^{-2}) \cong (u, t \mid u^3 = 1, tut^{-1} = u^2)$ .  $G = \Pi_1(F) = (u \mid u^3 = 1)$ . The isomorphism can be realized by setting  $ab^{-1} = u, a = t$ . In the second presentation for  $\Pi_1(S)$ ,  $t$  represents the generator of  $\Pi_1(S^1)$ . Exactly as before,  $\Pi_2(S)$  is presented as a  $Z\Pi_1(S)$ -module by the  $(2 \times 1)$ -matrix  $\begin{pmatrix} 1 + u + u^2 \\ t - 1 \end{pmatrix}$ . The universal cover  $F$  is homotopy equivalent to  $S^2 \vee S^2$ .  $H_1(S^*)$  is presented as a  $\Lambda$ -module by the  $(2 \times 1)$ -matrix  $\begin{pmatrix} t - 1 \\ t^2 - 1 \end{pmatrix}$ .

**EXAMPLE 3.**  $k$ -spun trefoil.

See [2], [4]. The fundamental group of the trefoil is  $(t, u, v \mid tut^{-1} = vu, tvt^{-1} = vu^{-1}v^{-1})$  where  $t$  goes onto the homology generator, and  $u, v$  generate the homology of the Seifert surface.

Now if  $Z$  is the Seifert surface and  $p$  the base point we choose  $\tilde{p}$  in the universal cover  $\tilde{Z}$  of  $Z$  so that  $\tilde{p}$  projects onto  $p$ . Any other point over  $p$  say  $\tilde{p}_g$  is a translation of  $\tilde{p}$  by an element  $g \in \pi_1(Z)$ . If  $u, v \in \pi_1(Z)$  are represented by maps  $u, v: (I, \partial) \rightarrow (Z, p)$  we obtain paths  $\tilde{u}, \tilde{v}: I \rightarrow \tilde{Z}$  covering  $u, v$  and such that  $\tilde{u}(0) = \tilde{v}(0) = \tilde{p}$ . Under the action of  $\pi_1(Z)$  we obtain paths  $\tilde{u}_g, \tilde{v}_g$  for each  $g \in \pi_1(Z)$ . When  $Z$  is  $k$ -spun in order to obtain  $F$  we may also spin  $\tilde{Z}$  and obtain  $\tilde{F}$ . Now  $\tilde{u}_g$  and  $\tilde{v}_g$  give rise to  $k + 1$ -spheres  $\tilde{U}_g^*$  and  $\tilde{V}_g^*$  which map to  $gU^*$  and  $gV^*$  in  $\pi_k(F)$ .

In order to discover the action of  $t$  on these  $(k+1)$ -spheres we need only note that  $t$  gives rise to a homeomorphism  $h: Z \rightarrow Z$  which induces  $\tilde{h}: \tilde{Z} \rightarrow \tilde{Z}$  so that  $\tilde{u} \rightarrow \tilde{u}\tilde{v}_u$  and  $\tilde{v} \rightarrow \tilde{v}\tilde{u}_u^{-1}\tilde{v}_u^{-1}\tilde{v}^{-1}$ . These spin and project to  $U^* + uV^*$  and  $(1 - u^{-1}v)V^* - u^{-1}U^*$  in  $\pi_{k+1}(F)$ . So that  $t(U^*) = U^* + uV^*$  and  $t(V^*) = (1 - u^{-1}v)V^* - u^{-1}U^*$ . Hence we have a presentation matrix

$$\begin{pmatrix} t-1 & -u \\ u^{-1} & t + uvv^{-1} - 1 \end{pmatrix}$$

for  $\pi_{k+1}(S)$ . This is the same as that obtained by Andrews-Lomonaco [2].

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