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ON HIGHER-DIMENSIONAL FIBERED KNOTS

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Abstract. The geometrical properties of a fibration of a knot complement over S^1 are used to develop presentations for the homotopy groups as modules over the fundamental group. Some homotopy groups of spun and twist-spun knots are calculated.

I. **Introduction.** Let K^n be a smooth submanifold of S^{n+2} which is homeomorphic to S^n. The pair (S^{n+2}, K) is a knot, with complement $S=S^{n+2}-K$. A tubular neighborhood of K in S^{n+2} provides us with a map $p\colon S\to S^1$, which in some cases is a fiber map [7]. When the complement fibers, the homotopy exact sequence of the fibration $F^{n+1}\to S \xrightarrow{p} S^1$ tells us that $\Pi_i(F)\cong\Pi_i(S)$, $i\geqq 2$, $\Pi_1(F)\cong[\Pi_1(S),\Pi_1(S)]$ the commutator subgroup of $\Pi_1(S)$, and $\Pi_1(S)\cong\Pi_1(F)\times Z$ the semidirect product of $\Pi_1(F)$ and the integers.

If $\Pi_1(S)=Z$, and $n\geqq 4$, a necessary and sufficient condition that a fibration exists is that the groups $\Pi_i(S)$ are finitely generated as abelian groups for all i [3]. In this case the fiber is simply connected. If $\Pi_1(S)\neq Z$, there are certain obstructions which determine whether or not S fibers over the circle [5].

There are many interesting cases where S fibers when $\Pi_1(S)\neq Z$, and the fact that $\Pi_1(F)\neq 1$ can be used to obtain information about the groups $\Pi_i(S)$ and about the homotopy type of $\tilde{S}$, the universal cover of S. §II of this paper studies the general situation of a fiber bundle $F^{n+1}\to S\to S^1$ where $\Pi_1(F)\neq 1$ and $F\simeq\bar{F}$, a smooth compact bounded manifold with $\partial\bar{F}$ homeomorphic to S^n.

In §III we study the particular case of fibered k-spun knots. If a knot fibers, then the k-spin of the knot fibers by k-spinning the fibration. For a knot of S^1 in S^3, the k-spun knot fibers if and only if the knot of S^1 in S^3 fibers. In this case, no matter which two (nontrivial) Neuwirth knots one k-spins, the universal covers of the resulting knot complements are found to be homotopy equivalent.

In §IV we develop an algebraic result about extending module presentations. If

$$1\longrightarrow H\xrightarrow{\alpha} G\underset{\gamma}{\overset{\beta}{\rightleftarrows}} T\longrightarrow 1$$

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is an exact sequence of groups such that $\beta\gamma$ = identity on T (G is then the semi-direct product of H and T), and A is a finitely presented ZH-module (ZH = integral group ring of H) such that T acts on A, then the given presentation for A as a ZH-module can be extended to a presentation of A as a ZG-module. This theorem is then used to determine the structure of the groups $\{\Pi_i(S)\}$ as $Z\Pi_1(S)$ modules and the groups $\{H_i(S^*; Z)\}$ as Λ-modules, where S^* = infinite cyclic cover of S and Λ = integral group ring of $J(t)$, the infinite cyclic group of covering transformations of S^* generated by t.

In §V we apply the results of §IV to calculate the homotopy structure of simple fibered knots. A fibered knot (S^{n+2}, K^n) is said to be simple if $\Pi_i(S)=0$, $2 \leqq i \leqq n-1$. Any fibered (S^4, K^2) knot is said to be simple. For example, any k-spun Neuwirth knot is simple by a theorem of Epstein [4]. For simple knots, complete results are obtained for $\Pi_n(S)$ as a $Z\Pi_1(S)$-module when the commutator subgroup $[\Pi_1(S), \Pi_1(S)]$ is a finite group.

In §VI we calculate as examples the homotopy structure of the 5-twist-spun trefoil and the k-spun trefoil.

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II. **General theorems on fibered knots.** Consider the bundle $F^{n+1} \to S \to S^1$ where $S = S^{n+2} - K^n$ a knot complement, $n \geqq 2$. Let $\approx$ denote diffeomorphism, $\simeq$ denote homotopy equivalence, and $\bigvee$ denote wedge product of spaces. Let $\tilde{S}$, $\tilde{F}$ denote the universal covers of S and F. Clearly $\tilde{S} \approx \tilde{F} \times R^1 \simeq \tilde{F}$. Let $G = \Pi_1(F)$. and 1_G denote the identity of G. Let $V^+ = \bigvee_{g \in G} K_g^n$ be the wedge product of copies of K indexed by $g \in G$ with the identification topology.

Now the fiber F deformation retracts to $\bar{F}$, a compact bounded $(n+1)$-manifold with $\partial\bar{F} \approx K$. The universal cover $\tilde{F}$ induces $\bar{F}^{\sim}$ over $\bar{F}$, and $\tilde{F}$ deformation retracts to $\bar{F}^{\sim}$. We have $\partial\bar{F}^{\sim} = \bigcup_{g \in G} K_g^n$ the disjoint union of copies of K indexed by G, the group of covering translations of $\bar{F}^{\sim}$.

THEOREM 1. *If $G = \Pi_1(F)$ is not a finite group, then*

(i) $\Pi_i(F) \cong \Pi_i(\bar{F}^{\sim}) \cong \Pi_i(\partial\bar{F}^{\sim}) \oplus \Pi_i(\bar{F}^{\sim}, \partial\bar{F}^{\sim})$ *as ZG-modules*, $i \geqq 2$,

(ii) $\Pi_i(V^+)$ *is a direct summand of* $\Pi_i(S)$ *as an abelian group for all i.*

Proof. There is a retraction $r: \bar{F}^{\sim} \to \partial\bar{F}^{\sim}$ because the obstructions lie in $H^{i+1}(\bar{F}^{\sim}, \partial\bar{F}^{\sim}; \Pi_i(\partial\bar{F}^{\sim}))$ which vanish for $i<n$ because $\Pi_i(\partial\bar{F}^{\sim})=0$ in this range, and for $i=n$ because $\bar{F}^{\sim}$ is not compact if G is not finite. The retraction commutes with covering translations in $\bar{F}^{\sim}$, so (i) is proved.

Choosing a base point $* \in K_{1_g}$, let $*_g \in K_g$ represent points in the same fiber as $*$, and choose arcs $a_g \subset \bar{F}^{\sim}$ from $*$ to $*_g$, such that $a_{g_1} \cap a_{g_2} = *$ if $g_1 \neq g_2$, and such that the projection of a_g in F is in the homotopy class of $g \in G$. Let $\tilde{V}^+ = \partial\bar{F}^{\sim} \cup (\bigcup_{g \in G} a_g)$. Then $\tilde{V}^+ \simeq V^+$ by shrinking the arcs.

There is a retraction from $\bar{F}^{\sim}$ to $\tilde{V}^+$ for the same reasons as before, only this time the splitting works at the abelian group level.

Now, let $V=\bigvee_{g\in\{G-1_G\}} K_g^n$ be the wedge product of copies of K indexed by elements in the set $\{G-1_G\}$. ($V^+\simeq V$ iff G is infinite.)

THEOREM 2. *If G is a finite group then $\tilde{S}\simeq Q\vee V$ where Q is a simply-connected finite similicial complex of dimension less than or equal to n.*

Proof. $\bar{F}^\sim$ is compact since G is finite, and $\bar{F}^\sim=\bigcup_{i=1}^{|G|} K_i$ the finite union of copies of K, where $|G|=$order of G and $K_1=K_{1_G}$. Choose arcs a_i from $*\in K_1$ to $*_i\in K_i$ as in the proof of Theorem 1. We will deformation retract away from K_1 and isolate K_2 in a wedge product decomposition for $\bar{F}^\sim$. That is, we will obtain a homotopy equivalence $\bar{F}^\sim\simeq K_2\vee Q_2$ where Q_2 is a bounded $(n+1)$-manifold with $\partial Q_2\approx K_1\# K_2$ (connected sum).

Collapse away along a small tubular neighborhood of a_2 until the boundary of a small collar neighborhood of K_2 is reached [Figure 1].

K_2 is a homotopy sphere, so the collar now deformation retracts from an n-disc D_2^n on its boundary component to $K_2\cup\text{arc}\cup L_2$ where $L_2\approx K_2-\text{Int } D_2^n$, and the

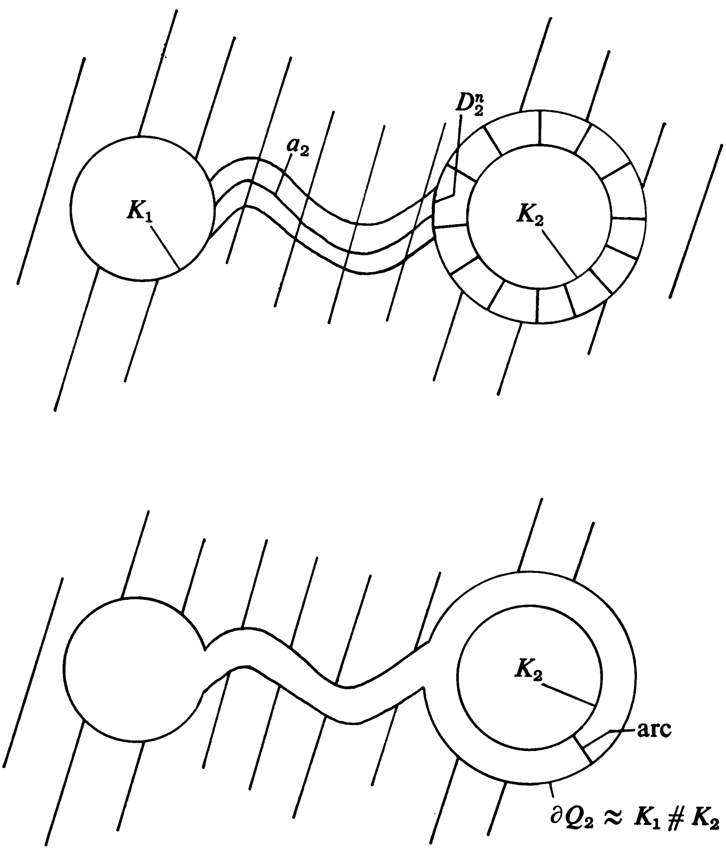

FIGURE 1

arc is "straight" in the product structure of the collar. Continuing this process, we obtain $K \simeq Q' \vee \bigvee_{i=2}^{|G|} K_i = Q' \vee V$ where Q' is a bounded $(n+1)$-manifold with $\partial Q' \approx \#_{i=1}^{|G|} K_i$. Clearly then $K \simeq Q \vee V$, Q a connected simply-connected n-complex obtained from Q' by collapsing out all the top-dimensional cells.

COROLLARY 3. *G a finite group and S a simple fibered knot then $\tilde{S} \simeq V$.*

Proof. From the proof of Theorem 2 we have $\bar{F}^{\sim} \simeq Q' \vee V$. Q' is contractible since S is a simple knot.

We would also note that if G is a finite group, then from Serre's theorem [6, p. 509] we have that $\Pi_i(\bar{F}^{\sim}) \cong \Pi_i(S)$ is a finitely generated abelian group for all $i \geqq 2$, since $\bar{F}^{\sim}$ compact.

III. **Fibered spun knots.** Let $f: B^n \to B^{n+2}$ be a smooth proper embedding $(n \geqq 1)$ of balls. We obtain the knot (S^{n+2+k}, K^{n+k}) by k-spinning $(B^{n+2}, f(B^n))$ in the following manner [4]:

$$S^{n+2+k} = (S^k \times B^{n+2}) \bigcup_{S^k \times f(S^{n-1})} (B^{k+1} \times S^{n+1}),$$
$$S^{n+k} \approx K^{n+k} = (S^k \times f(B^n)) \bigcup_{S^k \times f(S^{n-1})} (B^{k+1} \times f(S^{n-1})).$$

Note. We allow the boundary sphere pair of the ball pair to be knotted.

LEMMA 4. *(S^{k+3}, K^{k+1}) a k-spun $(B^3, f(B^1))$ fibers iff $[\Pi_1(B), \Pi_1(B)]$ is finitely generated, where $B = B^3 - f(B^1)$.*

Proof. In this case $\Pi_1(S) = \Pi_1(B)$ where $S = S^{k+3} - K^{k+1}$. If S fibers over S^1, then clearly $[\Pi_1(B), \Pi_1(B)]$ is finitely generated. Now $(B^3, f(B^1))$ determines a unique sphere pair $(S^3, f'(S^1))$, and by the Neuwirth-Stallings Theorem [7, p. 475], $S^3 - f'(S^1)$ fibers if $[\Pi_1(S), \Pi_1(S)] \cong [\Pi_1(B), \Pi_1(B)]$ is finitely generated. One obtains the fibration for S by k-spinning the fibration of the ball pair. Twist spinning a fibered knot to obtain a fibration is described in detail by Zeeman [7]. In order to obtain a fibration by spinning only, one neglects to twist as the spinning goes on.

This process is considered in greater detail in the proof of Theorem 6. Lemma 4 says that if (S^{k+3}, K^{k+1}) is a k-spun fibered knot, then in fact it is a k-spun Neuwirth knot. As before, let $V^+ = \bigvee_{g \in G} S^{k+1}$.

THEOREM 5. *Let (S^{k+3}, K^{k+1}) be a k-spun Neuwirth knot of genus $g \geqq 1$. Then $\tilde{S} \simeq V^+ \vee V^+ \vee \cdots \vee V^+$ ($2g$-fold wedge products).*

Proof. We will study carefully the geometry of the spun fibration. Figure 2 describes the fibration of $B^3 - f(B^1)$ by M_g^2 the "partially-bounded" punctured torus of genus g [Figure 2].

$M_{g,\phi}^2$ is the fiber at time $\phi \in S^1$. $\partial M_{g,\phi}^2$ is an open arc lying on ∂B^3, and is the fiber of the bundle induced on $S^2 - f(S^0)$.

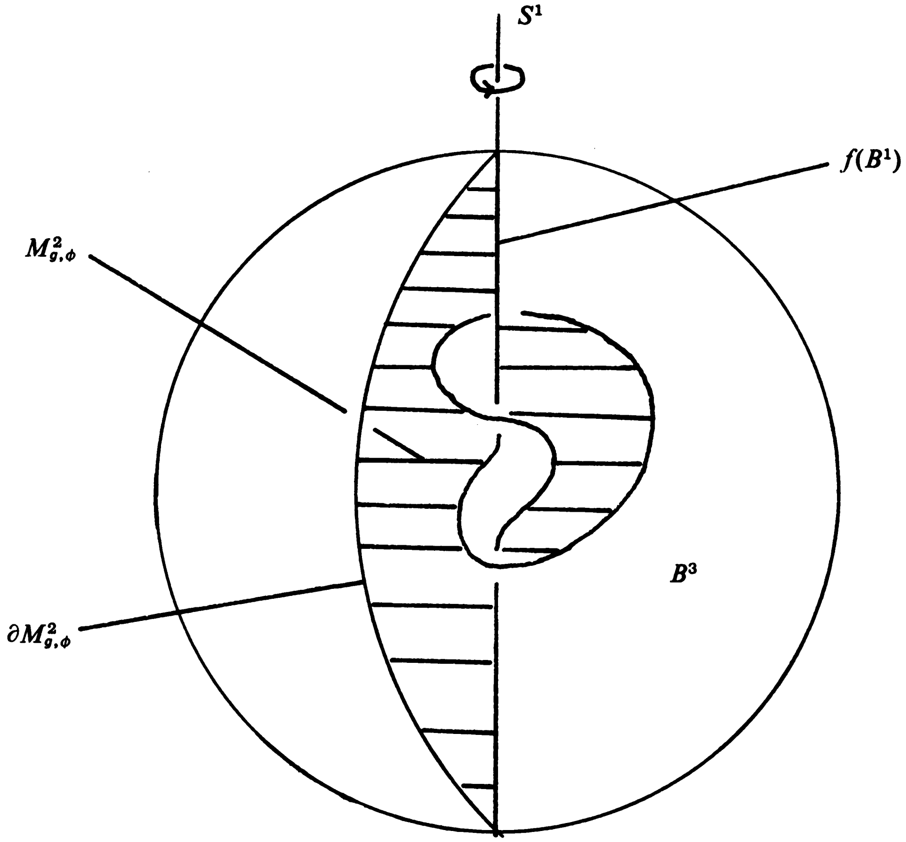

FIGURE 2. $g=1$.

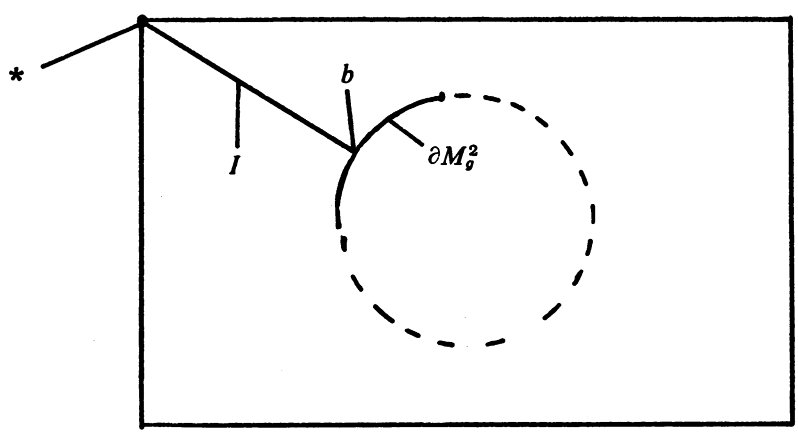

FIGURE 3. $g=1$.

The spun fiber $F_{g,\phi}$ of $S^{k+3}-K^{k+1}$ at time ϕ is the following:

$$F_{g,\phi} = (S^k \times M^2_{g,\phi}) \bigcup_{S^k \times \partial M^2_{g,\phi}} (B^{k+1} \times \partial M^2_{g,\phi}).$$

Let $L^p_{2g} = S^p \vee \cdots \vee S^p$ the $2g$-fold wedge product of S^p with itself. Now M^2_g deformation retracts to $\partial M^2_g \cup I \cup L^1_{2g}$ where I (Figure 2) is an arc in M^2_g running from a point $b \in \partial M^2_g$ to $*$ the wedge point of L^1_{2g} [Figure 3].

∂M^2_g deformation retracts to b. These deformations produce a deformation retraction from F_g to $F'_g = (S^k \times L^1_{2g}) \cup_{S^k \times *} (B^{k+1} \times *)$ [Figure 4].

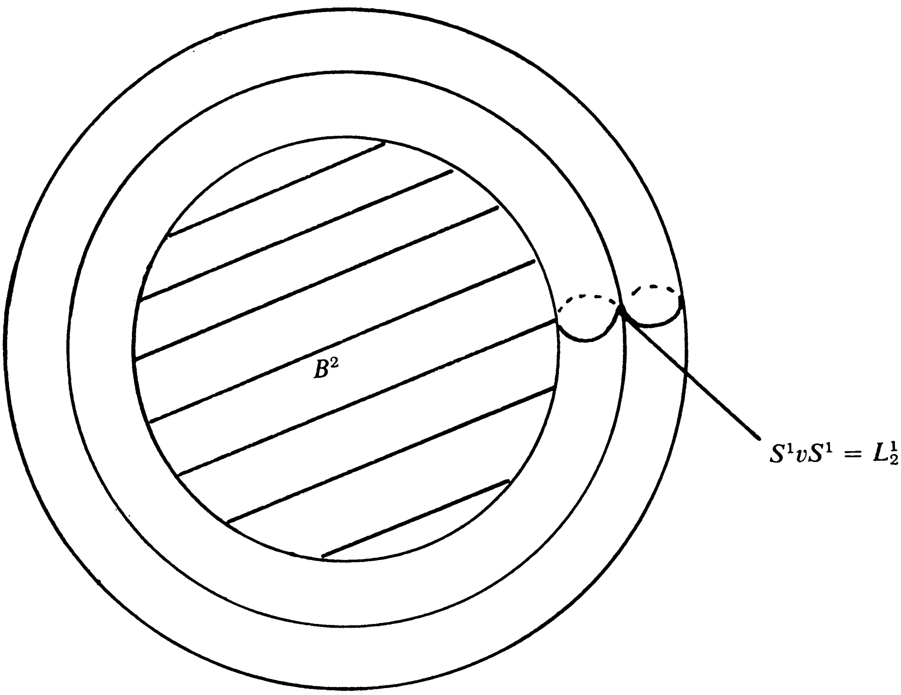

FIGURE 4. $k=1, g=1$.

Consider now $P = (S^k \times S^1) \cup_{S^k \times *} (B^{k+1} \times *)$. Thinking of S^1 as the union of two arcs I_1 and I_2, we have by shrinking I_1 to a point that $P \simeq P'$, where $P' = (S^k \times S^1) \cup_{S^k \times I_1} (B^{k+1} \times I_1)$ [Figure 5].

Now $P' \simeq S^1 \vee S^{k+1}$ by collapsing $B^{k+1} \times I_1$ to I_1, and then shrinking I_2 to a point. Hence $F'_g \simeq L^1_{2g} \vee L^{k+1}_{2g}$. Since a homotopy equivalence of CW complexes induces a homotopy equivalence of their universal covers, we have that $\tilde{F}_g \simeq (L^1_{2g} \vee L^{k+1}_{2g})^{\sim}$, the latter obtained (up to homotopy type) by taking the universal cover of L^1_{2g} (a snowflake with four g arcs emanating from each vertex) and attaching a copy of L^{k+1}_{2g} to each vertex. Now $[\Pi_1(S), \Pi_1(S)] \cong \Pi_1(M^2_g) \cong \Pi_1(L^1_{2g})$

=free group on $2g$ generators. By squeezing the 1-skeleton to a point, we have $\tilde{S} \simeq V^+ \vee \cdots \vee V^+$ the $2g$-fold wedge product of V^+ with itself. This completes the proof of Theorem 6.

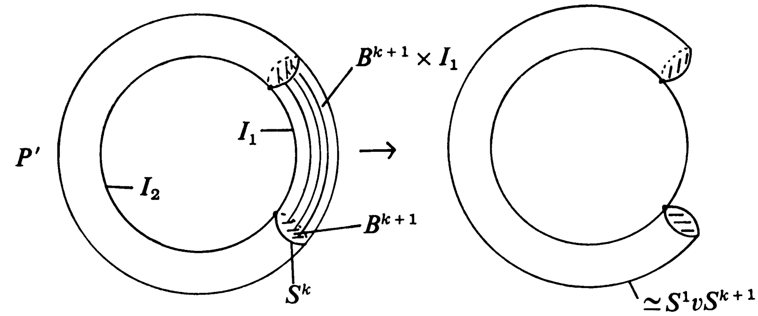

FIGURE 5

In [1], Andrews and Curtis ask the following question: If (S^4, K^2) is the 1-spun trefoil, and $S = S^4 - K^2$, is $\Pi_3(S) = 0$? Theorem 5 answers this question in the negative.

COROLLARY 6. *Let (S^{k+3}, K^{k+1}) be a k-spun Neuwirth knot of genus $g \geqq 1$. Then $\tilde{S} \simeq \bigvee_{i=1}^{\infty} S_i^{k+1}$ the infinite wedge of $k+1$-spheres.*

Proof. $V^+ \vee \cdots \vee V^+$ ($2g$ times) $= \bigvee_{i=1}^{\infty} S_i^{k+1}$ for all $g \geqq 1$.

This means that if we k-spin any two nontrivial Neuwirth knots, the universal covers of the resulting knot complements are homotopy equivalent. In fact, this also is true for iterated k-spinning (k allowed to vary) of Neuwirth knots, and will be dealt with in a future paper.

COROLLARY. *Let (S^{k+3}, K^{k+1}) be a k-spun Neuwirth knot of genus g. Then $\Pi_{k+1}(S)$ is a free ZG-module on $2g$ generators.*

Proof. Select a lift of L_{2g}^{k+1} to $\tilde{F}_g$. Each of the spheres in the lift is a free generator of

$$H_{k+1}(\tilde{F}_g) \cong_{ZG} \Pi_{k+1}(F_g) \cong \Pi_{k+1}(S).$$

The above shows that if (S^{k+3}, K^{k+1}) is a k-spun Neuwirth knot of genus g, then $\Pi_{k+1}(S)$ is the free abelian group on $2g$ copies of the symbols in $\{[\Pi_1(S), \Pi_1(S)]\}$. Compare the result of Epstein [4], which says that in this case $\Pi_{k+1}(S)$ is the free abelian group on the symbols $\{[\Pi_1(S), \Pi_1(S)] - \mathrm{id}\}$. (There is in this case a 1-1 correspondence between $\{[\Pi_1(S), \Pi_1(S)] - \mathrm{id}\}$ and $2g$ copies of $[\Pi_1(S), \Pi_1(S)]$.)

IV. **Change of rings.** Let

$$1 \longrightarrow H \xrightarrow{\alpha} G \underset{\gamma}{\overset{\beta}{\rightleftarrows}} T \longrightarrow 1$$

be an exact sequence of groups such that $\beta\gamma = 1_T$. Then G is the semidirect product of H and T so that each element of G can be written uniquely as ht where $h \in \alpha(H) = H$ and $t \in \gamma(T) = T$. Now given a ZH-module A the group $ZG \otimes_{ZH} A$ can be considered as a ZG-module under the action $g \cdot (g' \otimes a) = gg' \otimes a$ where $g, g' \in ZG$ and $a \in A$. Also if we assume T acts on A as a group of automorphisms then A can also be considered as a ZG-module under the action $ga = (ht)a = h(ta)$ where $t \in T$, $h \in H$, $a \in A$ and $g = ht \in G$. If T is finitely generated by, say, $t_1, \ldots, t_k$ and A is finitely generated as a ZH-module by $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$ then the action of T on A may be given by a matrix τ coming from $t_p(\bar{X}_i) = \sum \lambda_i^p \bar{X}_j$. Let τ be the $(nk+n)$ matrix

$$\begin{matrix} t_1 I - [\lambda_{ij}^1] \\ \vdots \\ t_k I - [\lambda_{ij}^k] \end{matrix} \tag{1}$$

where I is the $(n \times n)$ identity matrix, and $[\lambda_{ij}^p]$ is the $(n \times n)$ matrix of the coefficients.

We shall in this section use the same symbols for a map and the associated matrix given by some selection of generators. Content will make the meaning clear.

LEMMA 8. *Suppose $K \xrightarrow{\tau} M \xrightarrow{\phi} A \to 0$ is an exact sequence of R-modules with M finitely presented and K finitely generated. Let ψ be a presentation matrix for M. Then a presentation matrix for A is $\binom{\psi}{\tau}$.*

Proof. Let $F_2 \xrightarrow{\psi} F_1 \xrightarrow{\sigma'} A \to 0$ be a presentation of M by free finitely generated R-modules F_1, F_2 and let $F \xrightarrow{\sigma} K \to 0$ be a map of a free finitely generated R-module F onto K. We have the following diagram

$$\begin{tikzcd} & F_2 \arrow[d, "\psi"] & & \\ F \arrow[r, "\tilde{\tau}"] \arrow[d, "\sigma"] & F_1 \arrow[d, "\sigma'"] & & \\ K \arrow[r, "\tau"] \arrow[d] & M \arrow[r, "\phi"] \arrow[d] & A \arrow[r] & 0 \\ 0 & 0 & & \end{tikzcd}$$

Since F is projective there is a map $\tilde{\tau}: F \to F_1$ such that $\sigma'\tilde{\tau} = \tau\sigma$. It follows that $\operatorname{Ker} \phi\sigma' = \operatorname{Im} \tilde{\tau} + \operatorname{Im} \psi$ so that matrix associated with this presentation of A is $\binom{\psi}{\tilde{\tau}}$. But the matrix $\tilde{\tau} = \tau$.

THEOREM 9. *Given*

$$0 \longrightarrow H \xrightarrow{\alpha} G \underset{\gamma}{\overset{\beta}{\rightleftarrows}} T \longrightarrow 0$$

an exact sequence of groups with T finitely generated and $\beta\gamma = 1_T$. Let ψ be a presentation matrix for a finitely presented ZH-module A on which T acts as a group of automorphism. Then A is a ZG-module with presentation matrix $\binom{\psi}{\tau}$ where τ is as in (1).

Proof. Let

$$F_2 \xrightarrow{\psi} F_1 \xrightarrow{\phi} A \longrightarrow 0$$

be an appropriate presentation of A. We have the exact sequence of ZG-modules

$$ZG \otimes_{ZH} F_2 \xrightarrow{1 \otimes \psi} ZG \otimes_{ZH} F_1 \xrightarrow{1 \otimes \phi} ZG \otimes_{ZH} A \longrightarrow 0.$$

Let A' be the set A with the ZG-structure obtained from the action of ZH and T on A. Then there is a natural map $\alpha\colon ZG \otimes_{ZH} A \to A'$ given by $\alpha(\lambda \otimes X) = \lambda X$. Clearly the kernel K of α is generated by all elements of the form $\{\lambda \otimes X - 1 \otimes (\lambda X)\}$ where (λX) is the element of A corresponding to λX in A'. ($A' = A$, setwise.) Now K is generated by $\{t_p \otimes \phi(X_i) - 1 \otimes \sum_j \phi_{ij}^p X_j\}$ for all p, i. That is, K is generated by the images of $\{t_p \otimes X_i - 1 \otimes \sum \lambda_{ij} X_j\}$ under $1 \otimes \phi$. The desired presentation now follows from Lemma 8.

V. **Applications.** Let Q^{n+2} be an $(n+2)$-manifold which fibers over S^1 with fiber an $(n+1)$-manifold F^{n+1}. Let $p\colon Q \to S^1$ be the fiber map. We think of Q as the product $F \times I$ with $F \times 0$ and $F \times 1$ identified by a homeomorphism $\hat{h}\colon F \times 1 \to F \times 0$ given by $\hat{h}(X, 1) = (h(X), 0)$. We can without loss of generality assume $h(q) = q$ for some $q \in F$. Let $t'\colon I \to q \times I \subset F \times I$ be the product path from $q \times 1$ to $q \times 0$ in $F \times I$. Then (under h) t' corresponds to an element $t \in \Pi_1(Q, q \times 1)$ which maps onto a generator of $\Pi_1(S^1)$ under the fiber map p_*. From the homotopy exact sequence of the fibration, this gives us a semidirect splitting $\Pi_1(S) \cong \Pi_1(F) \times Z$. If i denotes the inclusion $i\colon F \to Q$ by $i(x) = (x, 1)$ and $\beta \in \Pi_k(F, q)$ we then have $t \cdot i_*(\beta) = i_*(h_*\beta)$.

We have the following corollary to Theorem 9:

COROLLARY 10. *Suppose $\Pi_k(F)$ is finitely presented as a $Z\Pi_1(F)$-module by a matrix M. Then $\Pi_k(Q) \cong \Pi_k(F) k \geqq 2$ is presented as a $Z\Pi_1(Q)$-module by $\binom{M}{t - h_*}$, where $t - h_*$ denotes the appropriate matrix.*

Consider the infinite cyclic cover Q^* of Q. We think of Q^* in two different ways. First as the product $Q^* \approx F \times R$, and second as the union of $X_i = (F \times I)_i$ $(i \in Z)$ with $(F \times 1)_i$ identified with $(F \times 0)_{i+1}$ by the homeomorphism h. Now t acts on $H_k(Q^*) \cong H_k(F)$ by $t(u) = h_*(u)$ where $u \in H_k(F)$ and $h_*\colon H_k(F) \to H_k(F)$ is the automorphism induced by h. Let Λ = the group ring of the infinite cyclic group generated by t.

COROLLARY 11. *Suppose $H_k(F)$ is finitely presented as an abelian group by a matrix M. Then $H_k(Q^*) \cong H_k(F)$ is presented as a Λ-module by $\binom{M}{t - h_*}$.*

Now assume S and F are as in §I. That is, S is the complement of K^n in S^{n+2} which fibers over S^1 with fiber F^{n+1}.

THEOREM 12. *Suppose that $G=\Pi_1(F)$ is finite, and that S is a simple fibered knot. Then $\Pi_n(F)$ is presented as a ZG-module by the (1×1) matrix $(\sum_{g\in G} g)$.*

Proof. Take F to be compact and bounded with $\partial F\approx K$. Then the universal cover $\tilde{F}$ is a compact contractible manifold with $\partial\tilde{F}=\bigcup_{g\in G} K_g$, as in the proof of Corollary 3. By Lefschetz Duality, $H_n(\tilde{F},\partial\tilde{F})\cong H^1(\tilde{F})=0$, and the homology exact sequence of the pair $(\tilde{F},\partial\tilde{F})$ yields

$$H_{n+1}(\tilde{F},\partial\tilde{F})\xrightarrow{\partial} H_n(\partial\tilde{F})\longrightarrow H_n(\tilde{F})\longrightarrow 0.$$

$H_{n+1}(\tilde{F},\partial\tilde{F})$ is free abelian on 1 generator ξ, and $H_n(\partial\tilde{F})$ is free ZG-module on 1 generator α, and $\partial\xi=\sum_{g\in G} g\alpha$. By Lemma 8, the result follows.

THEOREM 13. *Suppose (S^{n+2},K^n) is a simple fibered knot, and that $G=\Pi_1(F)=[\Pi_1(S),\Pi_1(S)]$ is finite, then $\Pi_n(S)$ is presented as a $Z\,\Pi_1(S)$-module by the (2×1)-matrix*

$$\begin{pmatrix}\sum_{g\in G} g\\ t-1\end{pmatrix}.$$

Proof. We can without loss of generality replace the complement $S^{n+2}-K$ by the closed complement S, a manifold-with-boundary in which the entire manifold S fibers over S^1 with fiber F, and the fibration induced on ∂S has fiber $\partial F\approx K$. Moreover, up to homotopy, the fibering on the boundary is the product bundle, because any orientation preserving homeomorphism $h\colon K\to K$ is homotopic to the identity. We can take the homotopy class of the embedded sphere K as the generator for $\Pi_n(F)$ as a ZG-module. Clearly then h_* is the identity automorphism on $\Pi_n(F)$.

We are left with the case of simple fibered knots whose commutator subgroups are not finite. The general case, unfortunately, seems not so clear as the special case of k-spun Neuwirth knots treated earlier in §III. We do, however, have the following corollary to Theorem 1:

COROLLARY 14. *(S^{n+2},K^n) a simple fibered knot, and $G=\Pi_1(S)$ is infinite. Then $\Pi_n(S)$ is free abelian, and $\Pi_n(S)\cong ZG\oplus H_n(\tilde{F},\partial\tilde{F})$.*

Proof. We have

$$\Pi_n(\tilde{S})\cong\Pi_n(F)\cong_{ZG} H_n(\tilde{F})\cong_{ZG} H_n(\partial\tilde{F})\oplus H_n(\tilde{F},\partial\tilde{F})\cong_{ZG} ZG\oplus H_n(\tilde{F},\partial\tilde{F}).$$

$H_n(\tilde{F})$ is free abelian, because F^{n+1} is a manifold-with-boundary, hence $F\simeq X$ an n-dimensional finite simplicial complex. Now $H_n(\tilde{X})\cong Z_n(\tilde{X})\subset C_n(\tilde{X})$, so $H_n(\tilde{X})$ is free abelian. Note that $H_n(\tilde{F},\partial\tilde{F})\cong H_n(F,\partial F;\{ZG\})\cong H^1(F;\{ZG\})\cong H^1(G;ZG)$ where $\{ZG\}$ denotes local coefficients and the last isomorphism is due to the fact that $\tilde{F}$ is simply connected.

VI. **Examples.** Many examples of simple knots may be generated by Zeeman's twist-spinning methods [7]. For example, one can obtain fibered knots of S^2 in S^4 by twist-spinning the bridge knots. The fiber obtained is the punctured lens space $L(p, g)$ (p odd) [7, Corollary 5].

EXAMPLE 1. 5-twist-spun trefoil.

By 5-twist-spinning the trefoil, we obtain the knot (S^4, K^2) and the fibration $F^3 \to S = S^4 - K^2 \to S^1$, F^3=punctured dodecahedral space. From Zeeman [7], we have that

$$\Pi_1(S) = (x, y, t \mid x^5 = (xy)^3 = (xyx)^2, t^{-1}xt = y, t^{-1}yt = yx^{-1}),$$
$$G = \Pi_1(F) = (x, y \mid x^5 = (xy)^3 = (xyx)^2).$$

In the presentation for $\Pi_1(S)$, the generator t corresponds to the generator of $\Pi_1(S^1)$. Now G is the binary dodecahedral group of order 120. By Corollary 4, the universal cover $\tilde{F}$ is homotopy equivalent to the wedge product of 119 2-spheres. ($\tilde{F}$ is a 3-sphere punctured 120 times.) By Theorem 14, $\Pi_2(S)$ is presented as a $Z\Pi_1(S)$-module by the (2×1)-matrix

$$\begin{pmatrix} \sum_{g\in G} g \\ t-1 \end{pmatrix}.$$

Since F is a homology 3-ball, all the Alexander invariants of the infinite cyclic cover S^* are trivial by Corollary 11.

EXAMPLE 2. 2-twist-spun trefoil.

This knot fibers with fiber punctured $L(3, 1)$. In this case $\Pi_1(S) = (a, b \mid aba = bab\; b = a^2ba^{-2}) \cong (u, t \mid u^3 = 1\; tut^{-1} = u^2)$. $G = \Pi_1(F) = (u \mid u^3 = 1)$. The isomorphism can be realized by setting $ab^{-1} = u$, $a = t$. In the second presentation for $\Pi_1(S)$, t represents the generator of $\Pi_1(S^1)$. Exactly as before, $\Pi_2(S)$ is presented as a $Z\Pi_1(S)$-module by the (2×1)-matrix $\binom{1+u+u^2}{t-1}$. The universal cover F is homotopy equivalent to $S^2 \vee S^2$. $H_1(S^*)$ is presented as a Λ-module by the (2×1)-matrix $\binom{3}{t-2}$.

EXAMPLE 3. k-spun trefoil.

See [2], [4]. The fundamental group of the trefoil is $(t, u, v\colon tut^{-1} = vu, tvt^{-1} = vu^{-1}v^{-1})$ where t goes onto the homology generator, and u, v generate the homology of the Seifert surface.

Now if Z is the Seifert surface and p the base point we choose $\tilde{p}$ in the universal cover $\tilde{Z}$ of Z so that $\tilde{p}$ projects onto p. Any other point over p say $\tilde{p}_g$ is a translation of $\tilde{p}$ by an element $g \in \pi_1(Z)$. If $u, v \in \pi_1(Z)$ are represented by maps $u, v\colon (I, \dot{I}) \to (Z, p)$ we obtain paths $\tilde{u}, \tilde{v}\colon I \to \tilde{Z}$ covering u, v and such that $\tilde{u}(0) = \tilde{v}(0) = \tilde{p}$. Under the action of $\pi_1(Z)$ we obtain paths $\tilde{u}_g$, $\tilde{v}_g$ for each $g \in \pi_1(Z)$. When Z is k-spun in order to obtain F we may also spin $\tilde{Z}$ and obtain $\tilde{F}$. Now $\tilde{u}_g$ and $\tilde{v}_g$ give rise to $k+1$-spheres $\tilde{U}_g^*$ and $\tilde{V}_g^*$ which map to gU^* and gV^* in $\pi_k(F)$.

In order to discover the action of t on these $(k+1)$-spheres we need only note that t gives rise to a homeomorphism $h: Z \to Z$ which induces $\tilde{h}: \tilde{Z} \to \tilde{Z}$ so that $\tilde{u} \to \tilde{u}\tilde{v}_u$ and $\tilde{v} \to \tilde{v}\tilde{u}_{u^{-1}}\tilde{v}_{u^{-1}v^{-1}}$. These spin and project to $U^* + uV^*$ and $(1-u^{-1}v)V^* - u^{-1}U^*$ in $\pi_{k+1}(F)$. So that $t(U^*) = U^* + uV^*$ and $t(V^*) = (1-u^{-1}v)V^* - u^{-1}U^*$. Hence we have a presentation matrix

$$\begin{pmatrix} t-1 & -u \\ u^{-1} & t+vuv^{-1}-1 \end{pmatrix}$$

for $\pi_{k+1}(S)$. This is the same as that obtained by Andrews-Lomonaco [2].

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