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INTEGER-VALUED ENTIRE FUNCTIONS(1)

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Abstract. The theory of integer-valued entire functions is organized in an improved fashion. Detailed results are proved when the indicator diagram is a line segment. For the first time, a method is developed for treating completely integer-valued functions with an unsymmetrical growth pattern.

1. **Introduction.** An entire function $F(t)$ is said to be of exponential type if there are positive constants k and K such that $|F(t)| \leqq Ke^{k|t|}$ for all t. The lower bound of the possible values of k is called the type. Thus e^{az} has type $|a|$. Equivalently, $\alpha^z = \exp(z \log \alpha)$ has type $|\log \alpha|$, where the principal value of the logarithm is to be used.

By an exponential polynomial, we shall mean a finite sum of terms of the form $z^n e^{az}$, where $n \geqq 0$ is an integer, and a is a complex number. The above term may also be written as $z^n \alpha^z$. In most of the cases which we encounter in this paper, the base α of the exponential will be an algebraic integer.

The study of integer-valued entire functions began in 1915 with the discovery by Pólya [9] that an entire function $F(t)$ of type $\tau < \log 2$ which has integer values at the nonnegative integers must be a polynomial, whereas for type $\log 2$ the function $F(t) = 2^t$ is admissible. Thus 0 and $\log 2$ are the first two possible types. In the same paper, Pólya also showed that an entire function $F(t)$ of type $\tau < \log [(3+\sqrt{5})/2]$ which has integer values at all integers must be a polynomial, whereas for type $\log [(3+\sqrt{5})/2]$ the function

$$F(t) = ((3+\sqrt{5})/2)^t + ((3-\sqrt{5})/2)^t$$

is admissible. Thus 0 and $\log [(3+\sqrt{5})/2]$ are the first two possible types.

Improvements of these results were made by various authors. I shall not trace the history of the problem, but shall mention only some of the newer results. For each of the two general problems considered above, Pisot [6], [7], [8] determined the third possible type. He also determined the critical type below which $F(t)$ must be an exponential polynomial. Some results where different rates of growth are

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allowed on the positive real axis, the negative real axis, and the imaginary axis were obtained by Buck [1]. Additional references may be found in this paper. There do not seem to have been any improvements of these results published during the last twenty years.

In the further discussion, it will be convenient to introduce the indicator function $h(\phi)$, which measures the type of $F(t)$ along the ray amp $t=\phi$. It is defined as the lower bound of real constants k such that $|F(Re^{i\phi})| \leqq Ke^{kR}$ for a suitable constant K and all $R \geqq 0$. If $F(t)=0$, then $h(\phi)=-\infty$. We shall exclude this case throughout. Otherwise, it turns out that $h(\phi)$ is continuous, and is indeed the supporting function of a certain nonempty bounded closed convex set, called the indicator diagram of $F(t)$, which is discussed in §2.

It may be noted that the indicator diagram is the same for $F(t)$ as for $F(t+1)$. It follows that the possible indicator diagrams for functions of exponential type are exactly the same whether we assume that $F(t)$ has integer values at the nonnegative integers or at the positive integers. The latter assumption will be more convenient for us. We shall call a function integer-valued if it has integer values at the positive integers, and completely integer-valued if it has integer values at all integers.

In this paper, aside from trying to organize the whole subject in an improved fashion, we shall be concerned primarily with functions whose indicator diagram is a line segment. Considerably more complete results can be given in this case than in general.

Further progress in the theory of integer-valued functions depends largely on an improved knowledge of the distribution of algebraic integers. However, another weakness so far has been that a method has been lacking for treating completely integer-valued functions whose indicator diagram is restricted to a set which is not symmetric to the origin. I have developed such a method, and apply it in §§7–8 to solve the following problem: *If $F(t)$ is an entire function of exponential type which is of type* 0 *on the imaginary axis and assumes integer values at all integers, for what values of λ and μ will the inequalities $h(0)<\lambda$ and $h(\pi)<\mu$ insure that $F(t)$ is an exponential polynomial?*

Some of the theorems in this paper have been known to me for about twenty years, and have been presented in lectures, but as they apparently have not appeared in print, it seems appropriate to include them here with more recent discoveries. I would like to add that, although I have not previously published anything about integer-valued functions, my studies of the distribution of algebraic integers, including [13], [14], [15], [17], and several other papers, grew out of my interest in this subject. Furthermore, my extension [16] of Pólya's theorem on power series with integer coefficients was made specifically for the application in §8, but was carried further than needed for this purpose.

We may note here that the oldest results concerning the distribution of algebraic integers are those of Kronecker [5]; see also the discussion in [13]. The first theorem is that a set of conjugate algebraic integers lying on the unit circle consists of all primitive roots of unity of a certain order. The second theorem is that a set of conjugate algebraic integers lying in the interval $[-2, 2]$ consists of all numbers of the form $2 \cos 2k\pi/m$ with $0 \leqq k \leqq m/2$ and $(k, m) = 1$. These results are used at several places in the present paper.

2. **The Laplace transform.** An essential tool in our study of integer-valued functions is the Laplace transform of functions of exponential type. The required results may be found in Pólya [12, pp. 578–586], or in Doetsch [2, Chapter 5]. A slight variation of the Laplace transform for functions of exponential type was studied by Borel, so the term Borel transform is also used. We shall summarize the principal results here.

Let $F(t)$ be any entire function of exponential type, and write

$$F(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}.$$

Its Laplace transform $f(s)$ is defined by

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}.$$

The series converges for large $|s|$, so $f(s)$ is regular at ∞. Indeed, the above formulas establish a one-to-one correspondence between entire functions $F(t)$ of exponential type, and functions $f(s)$ which are regular at ∞ and vanish there.

The smallest convex set outside of which $f(s)$ is regular is called the singularity hull of $f(s)$. This is a bounded closed set, and is empty only if $f(s) = 0$, that is, only in the excluded case $F(t) = 0$.

Each of the functions can also be expressed in terms of the other by means of an integral. In the first place, the name Laplace transform is explained by the fact that we have

$$f(s) = \int_0^{\infty \exp(i\phi)} F(t)e^{-st}\, dt \quad \text{for } \Re(se^{i\phi}) > h(\phi),$$

where the notation indicates integration along the ray amp $t = \phi$. Conversely, we have the inversion formula

$$F(t) = \frac{1}{2\pi i} \int_C f(s)e^{st}\, ds,$$

where C is a simple closed rectifiable curve with the singularity hull of $f(s)$ in its interior.

Using these integrals, it is not hard to show that the indicator function $h(\phi)$ of $F(t)$ is the supporting function of a certain bounded closed convex set, called the indicator diagram of $F(t)$. Indeed, the indicator diagram of $F(t)$ is the reflection in the real axis of the singularity hull of the Laplace transform $f(s)$.

3. **The generating functions.** Various authors have studied generating functions corresponding to the values of $F(t)$ at the positive integers. The essential properties may be found in Pólya [12, pp. 598–609], where they are obtained as a special case of a more general result. We shall sketch here a different approach.

Let $F(t)$ be an entire function of exponential type for which $h(\pm\pi/2)<\pi$, and let $f(s)$ be its Laplace transform. The singularity hull of $f(s)$ lies in the strip $|\Im s|<\pi$. We wish to show that there is a function $g(s)$ with period $2\pi i$ such that $g(s)-f(s)$ is regular for $|\Im s|\leqq\pi$, and $g(s)$ has the expansions

$$g(s) = \sum_{n=1}^{\infty} F(n)e^{-sn} \quad \text{for } \Re s > h(0),$$

and

$$g(s) = -\sum_{n=0}^{\infty} F(-n)e^{sn} \quad \text{for } \Re s < -h(\pi).$$

We may notice at once that the two series do converge in the indicated half-planes.

We observe next that if $F(t)=c$ then $f(s)=c/s$, and both series for $g(s)$ yield $g(s)=c/(e^s-1)$. Thus $g(s)-f(s)$ is regular at the origin, and all of the stated conclusions follow.

It will therefore be sufficient to prove the desired results under the assumption that $F(0)=0$. Since $2\pi i/(e^{2\pi i t}-1)$ is a function having a simple pole with residue 1 at each integer, and no other finite singularities, we see that

$$\sum_{n=1}^{N} F(n)e^{-sn} = \int_C \frac{F(t)e^{-st}}{e^{2\pi i t}-1}\,dt,$$

where C consists of segments of the two rays amp $t=\pm\alpha$ for some α with $0<\alpha<\pi/2$, and an arc of $|t|=N+1/2$. The integrand is regular at 0 since $F(0)=0$. Letting $N\to\infty$, we find that the integral along the arc approaches 0, and we obtain the function $g(s)$ defined by the first series as a difference of two integrals,

$$g(s) = \int_0^{\infty\exp(-i\alpha)} \frac{F(t)e^{-st}}{e^{2\pi i t}-1}\,dt - \int_0^{\infty\exp(i\alpha)} \frac{F(t)e^{-st}}{e^{2\pi i t}-1}\,dt,$$

at least for large positive s. It follows that

$$g(s)-f(s) = \int_0^{\infty\exp(-i\alpha)} \frac{F(t)e^{-st}}{e^{2\pi i t}-1}\,dt - \int_0^{\infty\exp(i\alpha)} \frac{F(t)e^{-st}e^{2\pi i t}}{e^{2\pi i t}-1}\,dt.$$

It is then easily checked that we may allow α to increase to $\pi/2$, at least for large positive s. But if we start with the second series for $g(s)$, we find that the same formula holds for large negative s. Furthermore, when $\alpha=\pi/2$, the two integrals in, the last formula converge uniformly for $|\Im s|\leqq\pi+\varepsilon$ for some $\varepsilon>0$, and hence $g(s)-f(s)$ is regular for $|\Im s|\leqq\pi$. Thus the two original series do represent the same function $g(s)$ in different half-planes. The condition $F(0)=0$ may now be dropped, and the stated results hold in general.

From the inversion formula for the Laplace transform, it follows at once that

$$F(t) = \frac{1}{2\pi i}\int_C g(s)e^{st}\,ds,$$

where C lies in the strip $|\Im s| < \pi$ but surrounds the singularities of $g(s)$ in the strip. This shows that $F(t)$ is determined by its values at the positive integers, provided that $F(t)$ is an entire function of exponential type with type $<\pi$ on the imaginary axis.

If we make the change of variable $z=e^s$, the function $g(s)$ is transformed into $j(z)=g(\log z)$. Because of the periodicity of $g(s)$, this function is single-valued. It has the expansions

$$j(z) = \sum_{n=1}^{\infty} F(n)z^{-n} \text{ near } \infty, \qquad j(z) = -\sum_{n=0}^{\infty} F(-n)z^{n} \text{ near } 0.$$

Hence $j(z)$ is regular at 0 and ∞, and vanishes at ∞. It is also regular for all negative z, since $g(s)$ is regular for $\Im s=\pi$.

Indeed, the above construction establishes a one-to-one correspondence between entire functions $F(t)$ of exponential type which have type $<\pi$ on the imaginary axis, and functions $j(z)$ which are regular at 0, ∞, and all negative z, and vanish at ∞. In the first place, $j(z)$ determines $g(s)=j(e^s)$ and hence $F(t)$, by the integral formula. It remains only to check that every $j(z)$ of the class described is actually obtained from some $F(t)$. Suppose that $j(z)$ is any function of the above class, and let its expansion at ∞ be

$$j(z) = \sum_{n=1}^{\infty} c_n z^{-n}.$$

Put $g(s)=j(e^s)$, and then define $F(t)$ in terms of $g(s)$ by the integral. We see that its indicator function satisfies

$$h(\phi) \leqq \max_{s\in C} \Re(se^{i\phi}),$$

so that $F(t)$ is admissible. Finally, we must see that it does lead back to the given $j(z)$. Notice that the integral formula for $F(t)$ may be written

$$F(t) = \frac{1}{2\pi i}\int_{C'} j(z)z^{t-1}\,dz,$$

where C' is the map of C by the transformation $z=e^s$, and $z^{t-1}=\exp[(t-1)\log z]$ with the principal value of $\log z$. If t is a positive integer n, we may replace C' by a large circle, and then we find that $F(n)=c_n$. Thus $j(z)$ is indeed the function corresponding to $F(t)$.

We need to know the form of $F(t)$ corresponding to a rational function $j(z)$. If $a \nleq 0$, then the function

$$j(z) = \frac{1}{(z-a)^m} = z^{-m}\sum_{n=0}^{\infty}\binom{-m}{n}\left(-\frac{a}{z}\right)^n$$
$$= \sum_{n=0}^{\infty}\binom{m+n-1}{m-1}\frac{a^n}{z^{m+n}} = a^{-m}\sum_{n=m}^{\infty}\binom{n-1}{m-1}\frac{a^n}{z^n}$$

corresponds to

$$F(t) = a^{-m}\binom{t-1}{m-1}a^t,$$

where $a^t = \exp[t \log a]$ with the principal value of $\log a$. It follows that if $j(z)$ is a rational function having a as its only pole, then $F(t)$ has the form $P(t)a^t$, where $P(t)$ is a polynomial, and conversely. The degree of the polynomial is one less than the order of the pole. Generally speaking, $F(t)$ is an exponential polynomial if and only if $j(z)$ is rational. The bases of the exponentials occurring in $F(t)$ will be the poles of $j(z)$.

4. **Integer-valued functions (general).** We shall make use of a theorem of Pólya which states that if the power series

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

has integer coefficients, and if $\psi(z)$ is regular in a region whose complement is a bounded closed set E with transfinite diameter less than 1, then $\psi(z)$ must be rational. Indeed, $\psi(z)$ will have the form $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials with integer coefficients, the leading coefficient of $Q(z)$ being 1. Thus the poles of $\psi(z)$ can occur only at sets of conjugate algebraic integers lying in E, and all of the conjugates in any set will be poles of the same order. If the transfinite diameter of E is $\geqq 1$, then $\psi(z)$ need not be rational. There are in fact a nondenumerable infinity of possible nonrational functions $\psi(z)$.

The concept of transfinite diameter is explained in Fekete [3]. The positive result and the converse were proved by Pólya [10] for the case in which the complement of E is simply connected. This is sufficient for almost all of the applications to be made here. The positive result was extended by Pólya [11] to the general case. A new proof of this result, and a proof of the converse in the general case, is included in [16].

Let $F(t)$ be an entire function of exponential type which assumes integer values at the positive integers. Assume that $h(\pm\pi/2) < \pi$, so that the indicator diagram D of $F(t)$ lies in the strip $|\Im t| < \pi$. Since $F(t)$ is determined by its values at the positive integers, it follows that D is symmetric to the real axis. Hence the singularity hull of $f(s)$ is also D, and the singularities of $g(s)$ in $|\Im s| \leqq \pi$ lie in D, Let the transformation $z = e^s$ take D into D'. Then $j(z)$ is regular outside of D'. The coefficients of $j(z)$ at ∞ are integers. If the transfinite diameter of D' is less than 1, then $j(z)$ must be rational, hence $F(t)$ is an exponential polynomial. The poles of $j(z)$, hence the bases of the exponentials in $F(t)$, must consist of one or more sets of conjugate algebraic integers lying in D'. Compare Buck [1, Theorem 2.3]. There will be a finite number of such sets by Fekete [3]. Notice that any set of conjugate algebraic integers, $\alpha_1, \alpha_2, \ldots, \alpha_m$ may be used to construct an integer-valued function, for example,

$$F(t) = \alpha_1^t + \alpha_2^t + \cdots + \alpha_m^t.$$

If the transfinite diameter of D' is $\geqq 1$, then $j(z)$ need not be rational, hence $F(t)$ need not be an exponential polynomial.

In particular, suppose that $F(t)$ is of type $\tau < \pi$. Then the singularities of $j(z)$ lie in the set $|\log z| \leqq \tau$, where the principal value of the logarithm is used. Choose τ_0 so that the transfinite diameter of $|\log z| \leqq \tau_0$ is equal to 1. Then for $\tau < \tau_0$, $F(t)$ must be an exponential polynomial. The conclusion will not follow if $\tau \geqq \tau_0$, so τ_0 is the critical type. According to Pisot [7], $\tau_0 = 0.843\cdots$.

For $\tau < \tau_0$, the possible types $\leqq \tau$ may be determined by finding the sets of conjugate algebraic integers in the set $|\log z| \leqq \tau$. Hence there will be only a finite number of types $\leqq \tau$. The first two types were known to be 0 and $\log 2 = 0.693\cdots$. Pisot [6], [7] showed that the only other possible type $\leqq 0.8$ is $|\log [(3+i\sqrt{3})/2]| = 0.758\cdots$. It is not known whether there are other possible types $< \tau_0$.

On the other hand, it is easily seen that every type $\tau \geqq \tau_0$ is possible. If $\tau \geqq \pi$, this is trivial; indeed, $F(t) = \sin \pi t \sin (\tau - \pi)t$ is an exponential polynomial of type τ and vanishes at all integers. Now suppose that $\tau_0 \leqq \tau < \pi$. We may proceed as follows. Start with the set $|\log z| \leqq \tau$, which has transfinite diameter $\geqq 1$. Delete the portion of this set inside of some circle $|z| = c$ in such a way that the remaining set E, defined by $|\log z| \leqq \tau$, $|z| \geqq c$, has transfinite diameter exactly 1. Construct a nonrational function $j(z)$ with integer coefficients in the expansion at ∞ which is regular outside of E. This function must have the boundary of E as a natural boundary, since otherwise $j(z)$ would be regular outside of a set with transfinite diameter < 1. The indicator diagram of the corresponding function $F(t)$ will be defined by $|t| \leqq \tau$, $\Re t \geqq \log c$, and hence $F(t)$ will be of type τ. If $\tau = \tau_0$, then $F(t)$ is indeed of type τ_0 along every ray.

We may also notice that even if $F(t)$ is required to be an exponential polynomial, the possible types τ will be everywhere dense in (τ_0, π). This follows from Fekete and Szegö [4], Theorem K, which implies that there will be infinitely many sets of conjugate algebraic integers in the set $\tau_1 \leqq |\log z| \leqq \tau_2$ if $\tau_0 < \tau_1 < \tau_2 < \pi$.

In the following two sections, we shall try to give as complete information as possible about the type of an integer-valued function $F(t)$ whose indicator diagram is a line segment, which may be either horizontal (§5) or vertical (§6).

5. **Integer-valued functions (horizontal).** Let $F(t)$ be an entire function of exponential type which assumes integer values at the positive integers. Suppose that $h(\pm\pi/2) = 0$, $h(0) \leqq \lambda$, and $h(\pi) \leqq \mu$. Here λ and μ denote real constants, with $\lambda + \mu \geqq 0$. Then the indicator diagram of $F(t)$ is included in the horizontal segment $-\mu \leqq t \leqq \lambda$. The singularities of the corresponding function $j(z)$ will be included in the segment $e^{-\mu} \leqq z \leqq e^{\lambda}$. It will follow that $j(z)$ is rational, and hence $F(t)$ is an exponential polynomial, if this segment has transfinite diameter < 1, that is, if $e^{\lambda} - e^{-\mu} < 4$. This was proved by Buck [1, Theorem 4.1]. The possible bases for the exponentials will be the sets of conjugate algebraic integers in the interval $[e^{-\mu}, e^{\lambda}]$. There will be only a finite number of such sets. Hence there will be only a finite number of possible pairs of types $h(0) \leqq \lambda$ and $h(\pi) \leqq \mu$.

The critical case is $e^\lambda - e^{-\mu} = 4$. Indeed, if $e^\lambda - e^{-\mu} \geqq 4$, then we can construct an admissible function $F(t)$ which is not an exponential polynomial and for which we have exactly $h(0) = \lambda$ and $h(\pi) = \mu$. This may be done as follows. To each of the intervals $[e^\lambda - 4, e^\lambda]$ and $[e^{-\mu}, e^{-\mu}+4]$, there will exist a nondenumerable infinity of nonrational functions with integer coefficients at ∞ and vanishing there which are regular outside of the interval. The end points of the interval must be singular points. Let $j(z)$ be the sum of two functions, one corresponding to each interval. We may assume that $j(z)$ is not rational. It will be regular outside of the interval $[e^{-\mu}, e^\lambda]$, and have the end points as singularities. The corresponding $F(t)$ will satisfy the stated conditions. Even if we restrict ourselves to exponential polynomials, we see using [13] that the possible pairs $(h(0), h(\pi))$ are everywhere dense in the region $e^\lambda - e^{-\mu} > 4$.

We shall discuss three special cases in some detail, namely the cases $\mu = 0$, $\mu < \infty$, and $\lambda = \mu$.

The case $\mu = 0$. Here $F(t)$ is of type 0 in the left half-plane, and $\tau = h(0)$. If $\lambda < \log 5$, then this case is subcritical, and $F(t)$ is an exponential polynomial. The poles of $j(z)$ lie in the interval $1 \leqq z < 5$. Now the sets of conjugate algebraic integers in this interval have the form

$$z = 3 + 2\cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, (k, m) = 1],$$

where $m \geqq 2$. (Recall the theorems of Kronecker which were mentioned at the end of §1.) If one of these points is a pole of $j(z)$ of a certain order, then so are all the others. The most general function $F(t)$ of type 0 in the left half-plane and type $\tau < \log 5$ which assumes integer values at the positive integers is of the form

$$F(t) = \sum_{m=2}^{N} \sum_{0 \leqq k \leqq m/2; (k,m)=1} P_{mk}(t)(3 + 2\cos 2k\pi/m)^t,$$

where $P_{mk}(t)$ is a polynomial. Also, for each m, all of the polynomials $P_{mk}(t)$ have the same degree, or all vanish identically. If N is chosen as small as possible, then this function has the type

$$\tau = \log(3 + 2\cos 2\pi/N) \qquad (N = 2, 3, 4, 5, \ldots).$$

These are the only possible types $< \log 5$. The type $\log 5$ is possible not only for $F(t) = 5^t$ and other exponential polynomials, but also for functions which are not exponential polynomials. Every type $> \log 5$ is possible. Even for exponential polynomials, the possible types are everywhere dense in $(\log 5, \infty)$.

The case $\mu < \infty$. In this case, no restriction is placed on the rate of growth of $F(t)$ on the negative real axis, beyond the assumption that it is of exponential type. If $\lambda \leqq \log 4$, then this case is subcritical, and $F(t)$ is an exponential polynomial. The most general such function of type $< \log 4$ on the positive real axis has the form

$$F(t) = \sum_{m=3}^{N} \sum_{0 \leqq k \leqq m/2; (k,m)=1} P_{mk}(t)(2 + 2\cos 2k\pi/m)^t,$$

where $P_{mk}(t)$ is a polynomial. The possible types $h(0) < \log 4$ are given by

$$h(0) = \log(2+2\cos 2\pi/N) \qquad (N = 3, 4, 5, 6, \ldots).$$

If $F(t)$ has type $\log 4$ on the positive real axis, then a term $P(t) \cdot 4^t$ may be added, where $P(t)$ is a polynomial. Every type $h(0) > \log 4$ is possible, and the types $h(0)$ for which $F(t)$ is an exponential polynomial are everywhere dense in $(\log 4, \infty)$.

The case $\lambda = \mu$. Here we are concerned with the type $\tau = \max(h(0), h(\pi))$. This case is subcritical if $\lambda = \mu < \log(\sqrt{5}+2)$. The poles of $j(z)$ will lie in the interval $(\sqrt{5}-2, \sqrt{5}+2)$. We cannot give as complete results in this case as in the other two, since we do not know all of the sets of conjugate algebraic integers in this interval. We can, however, find the first eleven possible types τ and certain additional ones less than the critical type $\tau = \log(\sqrt{5}+2)$.

We start by finding the possible types $\tau < \log 4$. For this purpose, we need only find all sets of conjugate algebraic integers in the interval $(1/4, 4)$. These are easily seen to be

$$z = 2-2\cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, (k, m) = 1],$$

for $3 \leqq m \leqq 12$. These may also be written in the form

$$z = 2+2\cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, (k, m) = 1],$$

for $m = 6, 4, 10, 3, 14, 8, 18, 5, 22, 12$, respectively. The type on the positive real axis is determined from the largest conjugate, corresponding to $k=1$. The type on the negative real axis is determined from the smallest conjugate, corresponding to the maximum k. We find by examination of the cases that the type on the negative real axis is never greater than the type on the positive real axis. Thus the first ten possible types are

$$\tau = \log(2+2\cos 2\pi/N) \qquad (N = 3, 4, 5, 6, 8, 10, 12, 14, 18, 22).$$

In each case, $h(0) = \tau$, and for $N = 3, 5, 12$, we also have $h(\pi) = \tau$, but in the other seven cases, $h(\pi) < \tau$.

TABLE 1

N	e^τ	S	e^τ	S	e^τ
3	1.0000	*	4.0000	4*a*	4.1935
4	2.0000	5*b*	4.0431	7*f*	4.2018
5	2.6180	4*b*	4.0953	7*l*	4.2101
6	3.0000	5*c*	4.1064	3*d*	4.2143
8	3.4142	4*d*	4.1268	8*o*	4.2215
10	3.6180	5*f*	4.1388	5*d*	4.2242
12	3.7321	7*e*	4.1532	8*w*	4.2306
14	3.8019	6*d*	4.1588	8*i*	4.2321
18	3.8794	3*c*	4.1701	8*m*	4.2356
22	3.9190	7*h*	4.1867	**	4.2361

The eleventh possible type is $\tau=\log 4$. In this case, again $h(0)=\tau$ and $h(\pi)<\tau$. By examining the tables in [14], we can read off 18 additional sets of conjugate algebraic integers in the interval $(\sqrt{5}-2, \sqrt{5}+2)$. All of these are roots of equations of degree at most 8. There are probably no other admissible equations of degree $\leqq 8$, and certainly none of degree $\leqq 6$. In this way, we find 18 additional possible types less than the critical type. In each case, we have $h(0)=\tau$ and $h(\pi)<\tau$.

For all of the known types up to the critical type, the value of e^τ is shown in Table 1, rounded off to four decimals. For the first 10 types, the corresponding value of N is shown. The other 20 types are marked with a symbol S. The symbol $*$ marks the 11th type, $e^\tau=4$, and the symbol $**$ marks the critical type, $e^\tau=\sqrt{5}+2$. The 18 known types between $*$ and $**$ are marked with a symbol indicating the corresponding equation in [14]. The digit indicates the degree of the equation. The roots of this equation must be increased by 2 to give a set of conjugate algebraic integers in $(\sqrt{5}-2, \sqrt{5}+2)$. The largest of these conjugates is e^τ. It is not known how many other possible types there are less than the critical type. The type $\tau=\log 4.04314\cdots$ is probably the 12th possible type, but it is not easy to show this.

Considering only functions with $h(0)=h(\pi)=\tau$, we can show that every type $\tau \geqq \log(\sqrt{5}+2)$ is possible. Even for exponential polynomials, the possible types are everywhere dense in $(\log(\sqrt{5}+2), \infty)$, and we can choose at which end of the real axis the type is to be attained.

6. **Integer-valued functions (vertical).** Let $F(t)$ be an entire function of exponential type which assumes integer values at the positive integers. As usual, we shall assume that $h(\pm\pi/2)<\pi$. The indicator diagram of $F(t)$ will be a vertical segment if $h(\pi)=-h(0)$. If $h(0)<0$, then $F(n)=0$ for large positive integers n, and hence $F(t)=0$ identically. Thus we may assume that $h(0)\geqq 0$. We shall put $h(0) =\log R$ and $h(\pi)=-\log R$, where $R\geqq 1$. In other words, we assume that $F(t)R^{-t}$ is of type 0 on the real axis. The indicator diagram of $F(t)$ will be a segment of the line $\Re t=\log R$ which is symmetric to the real axis. Hence the singularities of $j(z)$ will lie on an arc of the circle $|z|=R$ having its midpoint at R. The critical type will be determined by making the transfinite diameter of this arc be equal to 1.

It is not hard to see that the required arc of $|z|=R$ having transfinite diameter 1 is defined by $|z-R|\leqq 2$, or by $|\text{amp } z|\leqq 2 \arcsin 1/R$. See, for example, [17, §4]. Thus in the critical case, the indicator diagram of $F(t)$ is the segment $\Re t=\log R$, $|\Im t|\leqq 2 \arcsin 1/R$, so that the critical type on the imaginary axis is $2 \arcsin 1/R$.

The cases $R=1$ and $R>1$ are very different from each other. When $R=1$, the critical case is $h(\pm\pi/2)=\pi$, which is on the boundary of the permissible values. Thus the assumption $h(\pm\pi/2)<\pi$, which we made, insures that $F(t)$ is an exponential polynomial. The function $j(z)$ will have all of its poles on the unit circle, but different from -1. The poles of $j(z)$, and hence the bases of the exponentials in $F(t)$, must be at sets of conjugate roots of unity. Compare Buck [1, Theorem 3.2].

The type will be determined by the root nearest to -1. Thus it will be convenient to write the sets of conjugates in the form

$$z = -e^{2k\pi i/m} \qquad [1 \leqq k \leqq m-1, (k, m) = 1],$$

for $m \geqq 2$. We see that $F(t)$ must have the form

$$F(t) = \sum_{m=2}^{N} \sum_{1 \leqq k \leqq m-1; (k,m)=1} P_{mk}(t) e^{(2k-m)\pi i t/m},$$

where $P_{mk}(t)$ is a polynomial. If N is minimal, then the type on the imaginary axis is

$$h(\pm\pi/2) = (1-2/N)\pi \qquad (N = 2, 3, 4, 5, \ldots).$$

These are the only possible types with $h(\pm\pi/2) < \pi$. (A particularly simple function realizing the smallest positive type is $F(t) = 2\cos \pi t/3$.) On the other hand, for any $\tau \geqq \pi$, we can find an exponential polynomial $F(t)$ with $h(0) = h(\pi) = 0$, $h(\pm\pi/2) = \tau$, and $F(t) = 0$ at every integer.

Now suppose that $R > 1$. The first observation is that in general there are no admissible functions which are exponential polynomials. For all the poles of $j(z)$ must lie on $|z| = R$, and by [17, Theorem 2.1], unless some power of R is an integer, no set of conjugate algebraic integers lies on the circle. Furthermore, even for values of R for which exponential polynomials are possible, there are very few admissible types below the critical type, indeed, by [17, Theorem 4.1], at most three.

This is the first case which we have found where we can prove that there are only a finite number of types less than the critical type. The critical type itself is always possible. For most values of R, it is indeed the smallest possible type.

TABLE 2

R^2	Critical type	Earlier types		
$\sqrt{2}$	114° 28′	112° 30′		
2	90°	45°	69° 18′	82° 14′
3	70° 32′	30°	54° 44′	
4	60°	0°	41° 25′	
5	53° 8′	26° 34′	47° 52′	
6	48° 11′	35° 16′		
7	44° 25′	19° 6′	40° 54′	
8	41° 25′	27° 53′		
9	38° 57′	0°	33° 33′	
k^2+l	2 arc sin $1/R$	arc sin $\sqrt{l}/R$	($k=3$, $1 \leqq l \leqq 3$; $k \geqq 4$, $0 \leqq l \leqq 3$)	
k^2+k+l	2 arc sin $1/R$	arc sin $\sqrt{(4l-1)}/2R$	($k=3$, $1 \leqq l \leqq 3$; $k \geqq 4$, $1 \leqq l \leqq 4$)	

The possible values of R for which there are any exponential polynomials below the critical type are shown in Table 2. All of the cases with more than one type below the critical type are listed individually. The critical type on the imaginary axis is shown, where for convenience we have used degrees and minutes (rounded off to the nearest minute) instead of radian measure. All possible earlier types are also shown. The entries here correspond to the entries in the table in [17, §4]. For example, when $R=2$, the critical type on the imaginary axis is $\pi/3$. The two possible earlier types are realized by the functions $F(t)=2^t$ and

$$F(t) = ((3+i\sqrt{7})/2)^t+((3-i\sqrt{7})/2)^t.$$

In this particular case, the critical type can be realized by

$$F(t) = (1+i\sqrt{3})^t+(1-i\sqrt{3})^t,$$

as well as by a function which is not an exponential polynomial.

We shall now show that any type greater than the critical type is possible. Let β be the desired type on the imaginary axis, where $\beta>2 \arcsin 1/R$. The result is trivial if $\beta \geqq \pi$. So suppose $\beta<\pi$, and choose α so that the set defined by $|z|=R$, $\alpha \leqq |\operatorname{amp} z| \leqq \beta$, which consists of two arcs, has transfinite diameter equal to 1. Then we can find $j(z)$ with integer coefficients at ∞ which is regular outside of these arcs but is not a rational function. Since the exterior region is not simply connected, we need the strong form of the converse of Pólya's theorem proved in [16]. The points $Re^{i\beta}$ and $Re^{-i\beta}$ must be singularities of $j(z)$. It follows that $h(\pm\pi/2)=\beta$, that is, $F(t)$ has the desired type β on the imaginary axis.

On the other hand, considering only exponential polynomials, the possible types will not be dense above the critical type. Only if some power of R is an integer is there any integer-valued exponential polynomial with type $<\pi$ on the imaginary axis. In this case, there will be a smallest type β_0 above which the types of such functions on the imaginary axis are everywhere dense. This type β_0 will be less than π, but larger than the critical type on the imaginary axis.

7. **Completely integer-valued functions (general).** Suppose that $F(t)$ is an entire function of exponential type which assumes integer values at all integers. Assume that $h(\pm\pi/2)<\pi$, so that the indicator diagram D of $F(t)$ lies in the strip $|\Im t|<\pi$. As in §4, the singularities of $j(z)$ will lie in D', the transform of D by $z=e^s$. In the present case, the expansions of $j(z)$ at both 0 and ∞ will have integer coefficients. Pólya's theorem, used in §4, does not provide a direct method of exploiting both facts. We can, however, make use of this theorem in an indirect way.

Let D'' be obtained from D' by the transformation $w=z+1/z$. In other words, D'' is obtained from D by the transformation $w=e^s+e^{-s}$. If D is symmetric with respect to the origin, then each point of D'' will correspond to two points of D or D'. Otherwise, replacing D by its symmetric hull, obtained as the union of D and its reflection in the origin, will increase D' but leave D'' unchanged.

First suppose that $F(t)$ is odd. Then $g(s)$ is even, hence $j(1/z)=j(z)$. If we put $w=z+1/z$, then $j(z)$ will be transformed into a single-valued function $q(w)$, regular outside of D''. Since $j(z)$ has integer coefficients at ∞, the same will be true for $q(w)$. Hence, by Pólya's theorem, $q(w)$ will be rational if the transfinite diameter of D'' is less than 1. Then $j(z)=q(z+1/z)$ will be rational as well, hence $F(t)$ will be an exponential polynomial.

If $F(t)$ is even, then $g(s)$ is odd, and so $j(1/z)=-j(z)$. In this case, $j(z)^2$ will be transformed into a single-valued function $q^*(w)$, regular outside of D''. If the transfinite diameter of D'' is <1, then $q^*(w)$ will be rational. Hence $j(z)^2$ will also be rational. To conclude that $j(z)$ is rational, an additional argument is needed. Notice that $F_1(t)=tF(t)$ is odd and has integer values at all integers, hence the corresponding function $j_1(z)=-zj'(z)$ is rational, that is, $j'(z)$ is rational. From the fact that both $j(z)^2$ and $j'(z)$ are rational, we can conclude that $j(z)$ is rational, and hence that $F(t)$ is an exponential polynomial.

In the general case, $F(t)-F(-t)$ is odd and $F(t)+F(-t)$ is even. Both have integer values at all integers. The singular points of their Laplace transforms will be included in the symmetric hull of D. The corresponding $q(w)$ and $q^*(w)$ will be regular outside of D'', as before. Hence $F(t)$ will be an exponential polynomial if the transfinite diameter of D'' is less than 1. Compare Buck [1, Theorem 5.1]. The poles of $q(w)$ or $q^*(w)$ will consist of one or more sets of conjugate algebraic integers in D''. Hence the poles of $j(z)$ will consist of sets of conjugate algebraic units in D', and these will be the bases of the exponentials in $F(t)$. Notice that any set of conjugate algebraic units $\alpha_1, \alpha_2, \ldots, \alpha_m$ may be used to construct a function $F(t)$ which has integer values at all integers, for example,

$$F(t) = \alpha_1^t + \alpha_2^t + \cdots + \alpha_m^t.$$

The weakness of this method is that if D does not have symmetry with respect to the origin, then we do not fully utilize the given information on the growth of $F(t)$. On the other hand, if D is symmetric with respect to the origin, then we have obtained a satisfactory conclusion. Indeed, if D is symmetric and the transfinite diameter of D'' is $\geqq 1$, then $F(t)$ need not be an exponential polynomial. For we can construct a nondenumerable infinity of functions $q(w)$ regular outside of D'' which have integer coefficients at ∞ and vanish there. The corresponding functions $j(z)=q(z+1/z)$ are regular outside of D', have integer coefficients at both 0 and ∞, and cannot all be rational. Thus we will obtain odd functions $F(t)$ which assume integer values at all integers, which are not exponential polynomials, and whose indicator diagram lies in D.

In particular, suppose that $F(t)$ is of type $\tau<\pi$. Then D is included in the circle $|t|\leqq\tau$, D' in the set $|\log z|\leqq\tau$, and D'' in the set

$$\left|\log \frac{w+\sqrt{(w^2-4)}}{2}\right| \leqq \tau.$$

Choose $\tau=\tau_0$ so that this set has transfinite diameter equal to 1. Then τ_0 is the critical type, that is, $F(t)$ must be an exponential polynomial if $\tau<\tau_0$ but not necessarily if $\tau\geqq\tau_0$. Pisot [8] gave the value $\tau_0=0.9934\cdots$.

The first two possible types for $F(t)$ were known to be 0 and $\log[(3+\sqrt{5})/2]$. Pisot [8] also determined the third type. It is not known whether there are other possible types $<\tau_0$.

One can show by methods similar to those used in §4 that every type $\tau \geqq \tau_0$ is possible, and that even for exponential polynomials, the possible types are dense in (τ_0, π) and include all types $\tau \geqq \pi$.

An alternative procedure to the one described above would be to apply the following extension of Pólya's theorem:

Let E be a bounded closed set symmetric to the real axis and not containing the origin. Let the complement of E be a region G. Suppose that $\psi(z)$ is regular in G and has the expansions

$$\psi(z) = \sum_{\nu=0}^{\infty} a_\nu z^{-\nu} \text{ near } \infty, \qquad \psi(z) = \sum_{\nu=0}^{\infty} b_\nu z^{\nu} \text{ near } 0,$$

where both series have integer coefficients. If there exists a function

$$H(z) = z^p + A_{p-1}z^{p-1} + \cdots + A_{-n+1}z^{-n+1} + z^{-n},$$

with $p > 0$ and $n > 0$, such that $|H(z)| < 1$ on E, then $\psi(z)$ is rational.

This is a special case of the main theorem of [16], and is stated just before the main theorem itself. Furthermore, if there is any such $H(z)$, then there is also one with integer coefficients. In this case, we can say that the poles of $\psi(z)$ occur at the zeros of $H(z)$, and hence lie at sets of conjugate algebraic units contained in E.

With the same notation used previously, if the set D'' has transfinite diameter < 1, then by Fekete [3] there is a polynomial $P(w) = w^n + \cdots$ such that $|P(w)| < 1$ on D''. Hence $H(z) = P(z + 1/z) = z^n + \cdots + z^{-n}$ satisfies $|H(z)| < 1$ on D'. By the above theorem, $j(z)$ is rational. Thus we are led to exactly the same conclusion as before. We did not need to consider odd functions and even functions separately.

The new method, unlike the old one, can also make use of the assumption that $F(t)$ has an indicator diagram which is not symmetrical with respect to the origin. The method is applied in §8 to find the critical rates of growth at the two ends of the real axis for a function of type 0 on the imaginary axis. Thus §8 concerns completely integer-valued functions whose indicator diagram is a horizontal segment.

The case where the indicator diagram is a vertical segment can be settled briefly, hence does not require a separate section. Assuming as usual that $h(\pm\pi/2) < \pi$, we see that $F(t) = 0$ identically unless the segment is on the imaginary axis. This corresponds to the case $R = 1$ in §6, except that the functions there were only assumed to have integer values at the positive integers. However, the poles of $j(z)$ were found to be at sets of conjugate algebraic units, so that the expansion of $j(z)$ had integer coefficients at 0 as well as at ∞. Thus the functions $F(t)$ obtained there do in fact have integer values at all integers, so that the solution to the present problem is exactly the same.

8. **Completely integer-valued functions (horizontal).** Suppose that $F(t)$ is an entire function of exponential type which assumes integer values at all integers. Suppose also that $F(t)$ is of type 0 on the imaginary axis, that is, $h(\pm\pi/2) = 0$. If $h(0) \leqq \lambda$ and $h(\pi) \leqq \mu$, then for what values of λ and μ can we conclude that $F(t)$

must be an exponential polynomial? Here λ and μ denote real constants. It will be sufficient to consider the case where $\lambda>0$ and $\mu>0$, since otherwise $F(t)$ will be a polynomial.

Under the above assumptions, the indicator diagram of $F(t)$ is included in the segment $-\mu\leqq t\leqq\lambda$. It follows that the singularities of $j(z)$ are included in the interval $e^{-\mu}\leqq z\leqq e^{\lambda}$. Also, the expansions of $j(z)$ at 0 and ∞ have integer coeffiicients. Can we conclude that $j(z)$ is rational?

We shall show that the critical case is the one in which $[e^{-\mu}, e^{\lambda}]$ is a critical interval for the problem of algebraic units, that is, an interval which can be approximated both by intervals containing infinitely many sets of conjugate algebraic units, and by intervals containing only finitely many such sets. By [15, §5], these intervals are defined by $L_1L_2=L^2$, where

$$L_1 = \log\frac{e^{\lambda}-e^{-\mu}}{4} = \log\frac{e^{\lambda+\mu}-1}{4e^{\mu}},$$

$$L_2 = \log\frac{e^{\lambda}-e^{-\mu}}{4e^{\lambda}e^{-\mu}} = \log\frac{e^{\lambda+\mu}-1}{4e^{\lambda}},$$

and

$$L = \log\frac{e^{\lambda/2}+e^{-\mu/2}}{e^{\lambda/2}-e^{-\mu/2}} = \log\frac{e^{(\lambda+\mu)/2}+1}{e^{(\lambda+\mu)/2}-1}.$$

In contrast to this, we found in §5 above that the critical case for the corresponding problem where $F(t)$ has integer values at the positive integers only is $e^{\lambda}-e^{-\mu}=4$, or $L_1=0$.

By [15, §5], if $L_1L_2=L^2$, then $L_1>0$ and $L_2>0$. Hence $\lambda>\log 4$ and $\mu>\log 4$. As λ increases from $\log 4$ to ∞, μ decreases from ∞ to $\log 4$. They are equal when $\lambda=\mu=\log(3+2\sqrt{2})$. Also, if $L_1L_2>L^2$, then $[e^{-\mu}, e^{\lambda}]$ contains a critical interval for the problem of algebraic units in its interior, whereas if $L_1L_2<L^2$, then $[e^{-\mu}, e^{\lambda}]$ is contained in the interior of a critical interval.

Suppose first that $L_1L_2>L^2$, so that $[e^{-\mu}, e^{\lambda}]$ is an enlargment of a critical interval. By [15, §8], there are positive integers p and n, and a function of the form

$$H(z) = z^{p}+A_{p-1}z^{p-1}+\cdots+A_{-n+1}z^{-n+1}+z^{-n},$$

with integer coefficients, which has $p+n$ arbitrarily large oscillations in the interval $[e^{-\mu}, e^{\lambda}]$. Choose a fixed $H(z)$ which has $p+n$ oscillations at least between ± 2 in the interval. Then $H(z)$ cannot assume any values in $[-2, 2]$ elsewhere. We can then choose a nondenumerable infinity of functions $Q(w)$ which are regular outside of the interval $[-2, 2]$, have integer coefficients in the expansions at ∞, and vanish there. Consider the functions $j(z)=Q(H(z))$. These functions will have integer coefficients at both 0 and ∞, and will vanish at both points. Furthermore, $j(z)$ will be regular outside of the interval $[e^{-\mu}, e^{\lambda}]$, hence the corresponding $F(t)$ will satisfy the required conditions. Only a denumerable infinity of the functions $j(z)$ can be rational. In the remaining cases, $F(t)$ will not be an exponential polynomial.

Now suppose that $L_1L_2<L^2$, so that $[e^{-\mu}, e^{\lambda}]$ is a proper part of a critical interval. In this case, by [15, §9], we can find a function $H(z)$ of the same form considered previously for which

$$|H(z)| < 1 \quad \text{for } e^{-\mu} \leqq z \leqq e^{\lambda}.$$

If $F(t)$ is an admissible function, then $j(z)$ will be regular outside of the interval $[e^{-\mu}, e^{\lambda}]$. We can then apply the main theorem of [16] to $j(z)$, as indicated in §7 above, and draw the conclusion that $j(z)$ is rational, and hence that $F(t)$ is an exponential polynomial. Also, the poles of $j(z)$, and hence the bases of the exponentials in $F(t)$, lie at zeros of $H(z)$, and therefore at sets of conjugate algebraic units in the interval $[e^{-\mu}, e^{\lambda}]$.

In summary, the conditions $h(0)\leqq\lambda$ and $h(\pi)\leqq\mu$ insure that $F(t)$ is an exponential polynomial when $L_1L_2<L^2$, but not when $L_1L_2>L^2$. Presumably the conclusion also fails when $L_1L_2=L^2$, but I do not see how to prove this. (The answer was proved to be negative in the critical case for the problems considered previously.) We can, however, completely solve a slightly modified problem, as promised in §1. Indeed, from the above results it follows easily that the conditions $h(0)<\lambda$ and $h(\pi)<\mu$ insure that $F(t)$ is an exponential polynomial if and only if $L_1L_2\leqq L^2$.

It seems likely that we could find an admissible function $F(t)$ for which $h(0)=\lambda$ and $h(\pi)=\mu$ whenever $L_1L_2\geqq L^2$, but I do not see how to prove this, even for the case $L_1L_2>L^2$. However, we can see that the values of λ and μ for which we can satisfy $h(0)=\lambda$ and $h(\pi)=\mu$ are everywhere dense in the region $L_1L_2>L^2$, and this remains true when we insist that $F(t)$ should be an exponential polynomial.

There are two special cases where we can give quite complete results. These are the cases where the growth restriction is dropped on one end of the real axis ($\mu<\infty$), and the symmetrical case ($\lambda=\mu$).

The case $\mu<\infty$. This is subcritical if $\lambda\leqq\log 4$, so all admissible functions are exponential polynomials. The poles of $j(z)$ lie in the interval $0<z\leqq 4$, and must be algebraic units. By [13, §4], these will have the form

$$z = 2+2\cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, (k, m) = 1],$$

where $m\geqq 3$ is not twice a prime power. Hence the possible types on the positive real axis which are less than log 4 are

$$h(0) = \log(2+2\cos 2\pi/N) \qquad (N = 3, 5, 7, 9, 11, 12, 13, \ldots),$$

where N is not twice a prime power. Although these approach $\log 4$, the type $h(0)=\log 4$ itself is not possible. However, the possible types $h(0)$ are everywhere dense in $(\log 4, \infty)$, even for exponential polynomials. I do not see how to prove that all types $h(0)>\log 4$ are possible.

The case $\lambda=\mu$. Here we are concerned with the type $\tau=\max(h(0), h(\pi))$. This case is subcritical if $\lambda=\mu<\log(3+2\sqrt{2})$. Because of the symmetry, it could be treated as indicated in §7, without the use of [15] and [16]. Indeed, it was treated by

Buck [1, Theorem 5.3], but we shall give a more complete answer. The poles of $j(z)$ will lie at sets of conjugate algebraic units in the interval $(3-2\sqrt{2}, 3+2\sqrt{2})$. Such a set of conjugates is taken by the transformation $w=z+1/z$ into a set of conjugate algebraic integers in $2 \leqq w < 6$. These have the form

$$w = 4+2\cos 2k\pi/m \qquad [0 \leqq k \leqq m/2, (k, m) = 1],$$

where $m \geqq 2$. The corresponding values of z either form a single set of conjugates, or fall into two equally numerous sets, each containing one of the two reciprocal values of z corresponding to a given w. As pointed out in [15, §2], the latter happens for $m=2$, 7, and 30, but probably not in any other case. (For $m=2$, both sets consist of $z=1$.) Thus the possible types $\tau < \log(3+2\sqrt{2})$ are determined from

$$e^{\tau}+e^{-\tau} = 4+2\cos 2\pi/N \qquad (N = 2, 3, 4, 5, \ldots),$$

which yields

$$\tau = \log[2+\cos 2\pi/N+\sqrt{((1+\cos 2\pi/N)(3+\cos 2\pi/N))}] \qquad (N = 2, 3, 4, 5, \ldots).$$

In general, $F(t)$ will have this type at both ends of the real axis. However, at least for $N=7$ and $N=30$, this need not be the case. It is not known whether there are any other exceptions.

The critical type $\tau=\log(3+2\sqrt{2})$ is attained at both ends of the real axis by

$$F(t) = (3+2\sqrt{2})^t+(3-2\sqrt{2})^t,$$

but can also be attained by functions which are not exponential polynomials. Indeed, for every $\tau \geqq \log(3+2\sqrt{2})$, we can find a completely integer-valued function $F(t)$ which is not an exponential polynomial, and for which $h(\pm\pi/2)=0$ and $h(0)=h(\pi)=\tau$. For there are a nondenumerable infinity of nonrational functions $q(w)$ which are regular outside of the interval $[e^{\tau}+e^{-\tau}-4, e^{\tau}+e^{-\tau}]$, have integer coefficients at ∞, and vanish there. The ends of the interval must be singular points. The corresponding functions $j(z)=q(z+1/z)$ cannot all be rational, and must have e^{τ} and $e^{-\tau}$ as singular points, which yields the required result. We can also see that the possible types for exponential polynomials are everywhere dense in $(\log(3+2\sqrt{2}), \infty)$.

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