TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 153, January 1971

REPRESENTATIONS OF FREE METABELIAN $\mathscr{D}_\pi$-GROUPS

BY
JOHN F. LEDLIE

Abstract. For π a set of primes, a $\mathscr{D}_\pi$-group is a group G with the property that, for every element g in G and every prime p in π, g has a unique pth root in G. Two faithful representations of free metabelian $\mathscr{D}_\pi$-groups are established: the first representation is inside a suitable power series algebra and shows that free metabelian $\mathscr{D}_\pi$-groups are residually torsion-free nilpotent; the second is in terms of two-by-two matrices and is analogous to W. Magnus' representation of free metabelian groups using two-by-two matrices. In a subsequent paper [12], these representations will be used to derive several properties of free metabelian $\mathscr{D}_\pi$-groups.

1. Introduction.

1.1. If g is an element of a group G and n is a positive integer, one may consider whether g has an *nth root* in G, that is, whether there is an element x in G such that $x^n = g$. In general, g need not have an nth root; on the other hand, it may have several. If every element of G has exactly one nth root for every positive integer n, G is called a *group with unique roots* or a *$\mathscr{D}$-group*.

More generally, with each nonempty set of primes π, there are associated three classes of groups: the class $\mathscr{E}_\pi$ consists of all groups in which pth roots (of all elements) exist for every prime $p \in \pi$; $\mathscr{U}_\pi$ consists of all groups in which pth roots (whenever they exist) are unique for every $p \in \pi$; the intersection of these two classes $\mathscr{E}_\pi \cap \mathscr{U}_\pi$ is denoted by $\mathscr{D}_\pi$; thus $\mathscr{D}_\pi$ comprises all groups in which pth roots (of all elements) exist and are unique for every $p \in \pi$. For $\mathscr{X} = \mathscr{E}$, $\mathscr{U}$ or $\mathscr{D}$, a group belonging to the class $\mathscr{X}_\pi$ is called an $\mathscr{X}_\pi$-group.

The class $\mathscr{D}_\pi$ forms a *variety of algebras*, that is, it consists of all algebras which admit a certain set of operations and satisfy a certain set of identical relations. To see this, we introduce a set Ω_π of operations, which consists of the group operations (multiplication and formation of inverses) together with a set $\{\rho_p \mid p \in \pi\}$ of unary operators in one-to-one correspondence with the set π; the set Ω_π is the set of operations for the variety $\mathscr{D}_\pi$. The identical relations or laws of $\mathscr{D}_\pi$ are the group laws together with the additional laws, two for each $p \in \pi$:

$$(x\rho_p)^p = x, \qquad x^p\rho_p = x.$$

It is easy to see that a group is a $\mathscr{D}_\pi$-group if and only if it admits this set of operators $\{\rho_p \mid p \in \pi\}$. Now a group G is *metabelian* if its second derived group

Received by the editors September 22, 1969.

AMS 1969 *subject classifications*. Primary 2040; Secondary 0840, 2027.

Key words and phrases. Metabelian group, $\mathscr{D}$-group, unique roots, power series algebra representation, matrix representation.

G'' is trivial, or equivalently, if the law

$$[[x_1, x_2], [x_3, x_4]] = 1$$

is identically satisfied in G (where $[x, y] = x^{-1}y^{-1}xy$). If we add this one further law to the set of laws which determine $\mathscr{D}_\pi$, the resulting variety of algebras, which we denote by $\mathfrak{M}_\pi$, is the variety of *metabelian* $\mathscr{D}_\pi$*-groups.*

If $\mathfrak{V}$ is a variety of algebras and X is an arbitrary set, the notion of *free* $\mathfrak{V}$*-algebra on the set* X may be defined by means of a universal mapping property. For example, a *free metabelian* $\mathscr{D}_\pi$*-group* or *free* $\mathfrak{M}_\pi$*-group on the set* X is a metabelian $\mathscr{D}_\pi$-group F_X for which there exists a mapping θ: $X \to F_X$, such that the following universal mapping property is satisfied:

"Given any metabelian $\mathscr{D}_\pi$-group G and any mapping ϕ: $X \to G$, there exists a unique homomorphism (of $\mathfrak{M}_\pi$-groups) $\bar{\phi}$: $F_X \to G$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "\phi"'] & F_X \arrow[d, "\bar{\phi}"] \\ & G \end{tikzcd}$$

commutes."

It is easy to see that such a free algebra is determined up to isomorphism by the cardinality of the set X, which is then called the *rank* of F_X. A general result of universal algebra due to G. Birkhoff [6] asserts the existence of free algebras of all ranks in any variety of algebras satisfying certain conditions; in particular, free $\mathscr{D}_\pi$-groups and free metabelian $\mathscr{D}_\pi$-groups of all ranks exist.

In this paper, it is almost always an arbitrary but fixed set of primes π which is under consideration. So, except in §3.5, the subscript π is omitted and we write $\mathscr{D}$, $\mathscr{E}$, $\mathscr{U}$ and $\mathfrak{M}$ in place of $\mathscr{D}_\pi$, $\mathscr{E}_\pi$, $\mathscr{U}_\pi$ and $\mathfrak{M}_\pi$ respectively.

1.2. Groups with roots were first considered by B. H. Neumann [19], who proved that every group can be embedded in an $\mathscr{E}$-group. In [18], A. I. Mal'cev proved that every torsion-free nilpotent group can be embedded in a $\mathscr{D}$-group which is nilpotent of the same class. The most important paper on groups with unique roots is due to G. Baumslag [1]. He considered free $\mathscr{D}$-groups and, in particular, showed how to construct a free $\mathscr{D}$-group by starting with an ordinary free group and gradually adjoining roots to it "in the freest possible manner" using free products with amalgamated subgroups; this construction enabled him to prove many interesting properties of free $\mathscr{D}$-groups. In [2], G. Baumslag constructed a free nilpotent-of-class-c $\mathscr{D}$-group as the direct limit of an ascending sequence of free nilpotent groups. This method is important to us here, because it gives a clue as to how one should try to construct a free metabelian $\mathscr{D}$-group (see §3.2). T. MacHenry [14] was the first to consider free metabelian $\mathscr{D}$-groups.

1.3. At least one important question about free $\mathscr{D}$-groups was left unanswered in G. Baumslag [1]. To explain this, let $Y=\{y_i \mid i\in I\}$ be a set of indeterminates and let $\boldsymbol{Z}[[Y]]$ be the "free" power series ring with integer coefficients in these indeterminates. Let C denote the set of all elements in $\boldsymbol{Z}[[Y]]$ which have constant component 1; then C is easily seen to be a group under multiplication (see [17, p. 310]). A well-known theorem of W. Magnus [15] asserts that gp $(1+y_i \mid i\in I)$ is a free group freely generated by the elements $1+y_i$, $i\in I$. Using the graded structure of the ring $\boldsymbol{Z}[[Y]]$, it is easy to show that C is residually torsion-free nilpotent (for a definition, see [21, p. 30]), so this representation of free groups implies that free groups are residually torsion-free nilpotent.

If now, instead of integer coefficients, we admit rational numbers, that is, if we consider the power series algebra $\boldsymbol{Q}[[Y]]$, then D, the set of elements in $\boldsymbol{Q}[[Y]]$ with constant component 1, is a $\mathscr{D}$-group under multiplication (see [17, p. 316]). So the question which naturally arises is whether $\mathscr{D}$-gp $(1+y_i \mid i\in I)$ (the smallest sub-$\mathscr{D}$-group of D containing the elements $1+y_i$, $i\in I$, see §2.2) is a free $\mathscr{D}$-group on the set $\{1+y_i \mid i\in I\}$. Since D is also residually torsion-free nilpotent, it is clear that if $\mathscr{D}$-gp $(1+y_i \mid i\in I)$ is a free $\mathscr{D}$-group, it follows that free $\mathscr{D}$-groups are residually torsion-free nilpotent. In another paper [4], G. Baumslag proved that the converse of this is also true; that is, he showed that if free $\mathscr{D}$-groups are residually torsion-free nilpotent, then $\mathscr{D}$-gp $(1+y_i \mid i\in I)$ is actually a free $\mathscr{D}$-group. Whether or not this is in fact the case is still not known.

The relevance of this to us is that we may consider the corresponding question for free metabelian $\mathscr{D}$-groups. This question, whether or not free metabelian $\mathscr{D}$-groups are residually torsion-free nilpotent, was one of the main motivating factors for the present work. It is answered in the affirmative in §4, not surprisingly, by establishing a power series algebra representation of free metabelian $\mathscr{D}$-groups closely analogous to Magnus' representation of free groups in $\boldsymbol{Z}[[Y]]$.

1.4. §2 of this paper gives some definitions and some basic facts about metabelian $\mathscr{D}$-groups. In §3, the so-called Main Theorem is proved. This is directly aimed at obtaining representations of free metabelian $\mathscr{D}$-groups and states, in essence, that if D is an "over-riding" metabelian $\mathscr{D}$-group and X is a subset of D, and if certain conditions are satisfied, then $\mathscr{D}$-gp (X) is a free metabelian $\mathscr{D}$-group on the set X.

The Main Theorem is utilized in §§4 and 5 to obtain two representations of free metabelian $\mathscr{D}$-groups. The first of these is inside a suitable power series algebra and shows that free metabelian $\mathscr{D}$-groups are residually torsion-free nilpotent as discussed above. The second representation is in terms of two-by-two matrices and is analogous to W. Magnus' representation of free metabelian groups using two-by-two matrices.

1.5. This paper is taken from my Ph.D. thesis at New York University. I would like to express my gratitude to Professor Gilbert Baumslag for his advice and encouragement during the period of this work. I would also like to thank Professor W. Magnus whose lectures and seminars stimulated my interest in group theory.

1.6. For the convenience of the reader, we list some of the notations used:

π an arbitrary nonempty set of primes,

$[\pi]$ the set of positive integers which are products of primes in π,

I a well-ordered index set,

$\boldsymbol{Z}$ the ring of integers,

$\boldsymbol{Q}$ the field of rational numbers,

Γ_π the additive subgroup of $\boldsymbol{Q}$ consisting of all rationals whose denominators are in the set $[\pi]$ (Γ_π is also a subring of $\boldsymbol{Q}$),

$\boldsymbol{Z}G$ the group-ring of the group G,

$|G|$ the order of the group G,

$|g|$ the order of the element g,

gp (S) where S is a subset of a group, the subgroup generated by S,

$\mathscr{D}$-gp (S) where S is a subset of a $\mathscr{D}_\pi$-group, the sub-$\mathscr{D}_\pi$-group generated by S,

$[h, g]$ the commutator $h^{-1}g^{-1}hg$ of h and g,

$[H, G]$ gp $([h, g] \mid h \in H, g \in G)$,

$\gamma_n G$ $[\gamma_{n-1}G, G]$, the nth group of the lower central series of G, where $\gamma_1 G = G$,

$\gamma_n^\pi G$ $\{g \in G \mid g^m \in \gamma_n G$ for some $m \in [\pi]\}$,

$\gamma_2^\pi G$ $\{g \in G \mid g^m \in \gamma_2 G$ for some $m \in [\pi]\}$, the *commutator π- ideal* of G,

$\mathfrak{A}$ the variety of all abelian $\mathscr{D}_\pi$-groups,

$\mathfrak{M}$ the variety of all metabelian $\mathscr{D}_\pi$-groups,

$\mathfrak{N}_c$ the variety of all $\mathscr{D}_\pi$-groups which are nilpotent of class at most c.

2. Some observations about metabelian $\mathscr{D}$-groups.

2.1. In this section, we introduce some definitions and prove some elementary properties of metabelian $\mathscr{D}$-groups, which will be used throughout the paper. Most of the definitions and results given in §§2.2–2.4 may be found in various forms in one of the references [1], [8], [9] or [11].

2.2. Let G be an arbitrary group. A subgroup H of G is called a *π-subgroup of G* if

$$g \in G,\ p \in \pi,\ g^p \in H \text{ implies } g \in H.$$

If G is a $\mathscr{D}$-group, it is easy to see that a π-subgroup of G is the same thing as a sub-$\mathscr{D}$-group of G.

Let G be an arbitrary group and let $N \lhd G$. We call N a *π-ideal of G* or simply an *ideal of G* (the set of primes π being understood as usual) if $G/N \in \mathscr{U}$. It is clear that an ideal of G is a π-subgroup of G; however, a normal π-subgroup of G need not be an ideal, even if $G \in \mathscr{D}$ (see [1, p. 301]).

Any intersection of π-subgroups of a group G is again a π-subgroup of G, and any intersection of ideals of G is again an ideal of G. If S is a subset of a group G, the unique minimal π-subgroup of G containing S is called the *π-closure of S in G* and is denoted by $\mathrm{cl}_\pi(S, G)$; similarly, the unique minimal ideal of G containing S is denoted by $\mathrm{id}_\pi(S, G)$.

If G is a $\mathscr{D}$-group, S a subset of G, and $H=\mathrm{cl}_\pi(S, G)$, then $H \in \mathscr{D}$ and H is the smallest sub-$\mathscr{D}$-group of G containing the set S; it is clear that H consists of those elements of G which are obtainable from the elements of S by a finite number of applications of the operations of multiplication, inversion and extraction of pth roots for primes $p \in \pi$. In this situation, we say that the $\mathscr{D}$-group H is *π-generated by the set S* and we write $H=\mathscr{D}\text{-gp}(S)$.

We next make the simple observation that if G and H are $\mathscr{D}$-groups and $\psi: G \to H$ is a group homomorphism, then ψ is a homomorphism of $\mathscr{D}$-groups, that is,

$$(g^{1/p})\psi = (g\psi)^{1/p} \quad \text{for } g \in G,\ p \in \pi,$$

(where $g^{1/p}$ denotes the unique pth root of g). Alternatively, we might say that the category of $\mathscr{D}$-groups and their homomorphisms is a full subcategory of the category of groups.

It is easy to see that the three Isomorphism Theorems and the Correspondence Theorem of group theory (see [22, pp. 24–28]) may be carried over to give analogous theorems for $\mathscr{D}$-groups, the role of "normal subgroup of a group" being played by "ideal of a $\mathscr{D}$-group". The proofs follow at once from the group theoretical results using the following lemma.

LEMMA 2.1. *Let G be a $\mathscr{D}$-group, let $H \leqq G$ be a sub-$\mathscr{D}$-group and let $N \lhd G$ be an ideal of G. Then HN is a sub-$\mathscr{D}$-group of G.*

Proof. Let $g \in G$ and $p \in \pi$ be such that $g^p \in HN$; we must prove that $g \in HN$. Since $H \in \mathscr{E}$ and, by an isomorphism theorem for groups, $H/H \cap N \cong HN/N$, we have that $HN/N \in \mathscr{E}$, so that HN/N is a sub-$\mathscr{D}$-group of the $\mathscr{D}$-group G/N. Therefore $(gN)^p=g^pN \in HN/N$ implies that $gN \in HN/N$. Hence $g \in HN$ as required.

We conclude these preliminary remarks by giving a realization of the free abelian $\mathscr{D}$-group on a set X. The subgroup of the additive group of rationals $\boldsymbol{Q}$, consisting of those rational numbers, whose denominators are products of primes in the set π, is denoted by Γ_π. It is clear that Γ_π may also be considered as a subring of $\boldsymbol{Q}$.

PROPOSITION 2.2. *The free abelian $\mathscr{D}$-group on a set X is isomorphic to the restricted direct product of $|X|$ copies of Γ_π.*

Proof. The proof consists of a straightforward verification that the direct product of $|X|$ copies of Γ_π satisfies the universal property which characterizes the free abelian $\mathscr{D}$-group on the set X. In this, it is useful to note that an abelian $\mathscr{D}$-group is the same thing as a Γ_π-module.

2.3. A positive integer n is called a *π-number* if n is a product of primes in the set π (see [8, p. 11]). The set of all π-numbers is denoted by $[\pi]$. We declare that $1 \in [\pi]$ for all sets π.

For any group G, we define

$$\tau_\pi(G) = \{g \in G \mid |g| \in [\pi]\}.$$

The elements of $\tau_\pi(G)$ are called *π-torsion* elements of G. If $\tau_\pi(G)=1$, G is said to be *π-free*.

In general, $\tau_\pi(G)$ is not a subgroup of G, as may be seen by considering the free product of two cyclic groups of order 2 and taking $\pi=\{2\}$. However, if G is locally nilpotent, we have

LEMMA 2.3. *Let G be a locally nilpotent group. Then*

(i) *$\tau_\pi(G)$ is a fully invariant subgroup of G,*

(ii) *$G/\tau_\pi(G) \in \mathscr{U}$,*

(iii) *$\tau_\pi(G)$ is the smallest normal subgroup of G which satisfies* (ii); *that is, if $N \lhd G$ is such that $G/N \in \mathscr{U}$, then $\tau_\pi(G) \leqq N$.*

The only difficulty in the proof of Lemma 2.3 lies in showing that $\tau_\pi(G)$ is a subgroup. This follows at once from the following result whose proof is well known and is omitted (see, for example, [8, p. 12]).

LEMMA 2.4. *Let H be a nilpotent group generated by elements $h_1, h_2, \ldots, h_r$, where $|h_i| = m_i \in [\pi]$, $1 \leqq i \leqq r$. Then H is a finite group and $|H| \in [\pi]$.*

We now define a sequence of subgroups $\gamma_n^\pi G$, $n=1, 2, \ldots$, of an arbitrary group G. The subgroup $\gamma_n^\pi G$ contains $\gamma_n G$, the nth group of the lower central series of G, and it is defined by specifying $\gamma_n^\pi G/\gamma_n G = \tau_\pi(G/\gamma_n G)$. In other words,

$$\gamma_n^\pi G = \{g \in G \mid g^m \in \gamma_n G \text{ for some } m \in [\pi]\}.$$

The facts about $\gamma_n^\pi G$ given in the next lemma follow from the definition using Lemma 2.3.

LEMMA 2.5. *Let G be a group and let n be a positive integer. Then*

(i) *$\gamma_{n+1}^\pi G$ is a fully invariant subgroup of G,*

(ii) *$G/\gamma_{n+1}^\pi G$ is a $\mathscr{U}$-group and is nilpotent of class at most n,*

(iii) *$\gamma_{n+1}^\pi G$ is the smallest normal subgroup of G which satisfies* (ii); *that is, if $N \lhd G$ is such that G/N is a $\mathscr{U}$-group and is nilpotent of class at most n, then $\gamma_{n+1}^\pi G \leqq N$.*

2.4. We record a simple but useful lemma due to P. G. Kontorovič [10] (see [11, p. 244]).

LEMMA 2.6. *Let G be a $\mathscr{U}$-group, let $g, h \in G$, and let $m, n \in [\pi]$. If g^m and h^n commute, then g and h commute.*

2.5. In the study of metabelian $\mathscr{D}$-groups, the subgroup $\gamma_2^\pi G$ and the factor group $G/\gamma_2^\pi G$ play a particularly important role. For any group G, $\gamma_2^\pi G$ is called the *commutator ideal of G*, the set of primes π being understood as usual.

PROPOSITION 2.7. (i) *Let G be a metabelian $\mathscr{U}$-group. Then $\gamma_2^\pi G$ and $G/\gamma_2^\pi G$ are both abelian $\mathscr{U}$-groups.*

(ii) *Let $G \in \mathfrak{M}$. Then $\gamma_2^\pi G \in \mathfrak{A}$ and $G/\gamma_2^\pi G \in \mathfrak{A}$. That is, a metabelian $\mathscr{D}$-group is an extension of an abelian $\mathscr{D}$-group by an abelian $\mathscr{D}$-group.*

Proof. (i) The only fact which has to be proved is that $\gamma_2^\pi G$ is abelian. Let $g, h \in \gamma_2^\pi G$; then there exist $m, n \in [\pi]$ such that g^m, $h^n \in \gamma_2 G$. As G is metabelian, $\gamma_2 G$ is abelian, so g^m and h^n commute, which, by Lemma 2.6, implies that g and h commute. Hence $\gamma_2^\pi G$ is abelian as required.

(ii) This follows at once from part (i), using the facts that a π-subgroup of a $\mathscr{D}$-group is a $\mathscr{D}$-group and that a homomorphic image of an $\mathscr{E}$-group is an $\mathscr{E}$-group.

PROPOSITION 2.8. *Let F be a free metabelian $\mathscr{D}$-group freely π-generated by the elements x_i, $i \in I$. Then $F/\gamma_2^\pi F$ is a free abelian $\mathscr{D}$-group freely π-generated by the elements $x_i\gamma_2^\pi F$, $i \in I$, so that* (*by Proposition* 2.2)

$$F/\gamma_2^\pi F \cong \Gamma_\pi \times \Gamma_\pi \times \cdots \quad (|I| \textit{ copies}).$$

Proof. This requires only the usual argument to show that the abelian $\mathscr{D}$-group $F/\gamma_2^\pi F$ together with the set $\{x_i\gamma_2^\pi F \mid i \in I\}$ satisfy the appropriate universal mapping property.

2.6. Let G be an arbitrary metabelian $\mathscr{U}$-group, and let K be an abelian ideal of G such that G/K is also abelian. By Proposition 2.7(i), $\gamma_2^\pi G$ is an example of such an ideal; further, by Lemma 2.5(iii), we always have $K \geqq \gamma_2^\pi G$. Denote the quotient group G/K by B and the natural projection by η:

$$\eta\colon G \to G/K = B.$$

We shall be dealing with this kind of situation frequently throughout this paper; whenever it occurs, we shall adhere to the following conventions:

(1) Elements of the module K will be denoted by small Latin letters $a, b, c, \ldots$, and the group operation in K will be written multiplicatively (because K is a subgroup of the multiplicative group G).

(2) Elements of B will be denoted by Greek letters $\alpha, \beta, \gamma, \ldots$, and B will be written multiplicatively.

(3) Elements of $\boldsymbol{Z}B$ will be denoted by capital letters $S, T, U, \ldots$; thus a typical element of $\boldsymbol{Z}B$ is

$$S = n_1\beta_1 + n_2\beta_2 + \cdots + n_l\beta_l, \qquad n_j \in \boldsymbol{Z},\ \beta_j \in B, \quad 1 \leqq j \leqq l. \tag{2.1}$$

(4) The action of group-ring elements on elements of the module K will be written exponentially; thus, if $c \in K$, the element obtained by operating on c with $S \in \boldsymbol{Z}B$ is c^S.

The action of $\boldsymbol{Z}B$ on K is defined as follows: First choose a (left) transversal for K in G, that is, a subset $\{t_\alpha \mid \alpha \in B\} \subseteq G$, whose elements are indexed by the elements of the group B, such that $t_\alpha\eta=\alpha$ for every $\alpha\in B$ and $t_1=1$. Then if $S\in \boldsymbol{Z}B$ is given by (2.1) and $c\in K$, define

$$c^S = c^{n_1\beta_1+n_2\beta_2+\cdots+n_l\beta_l} = t_{\beta_1}^{-1}c^{n_1}t_{\beta_1}\cdot t_{\beta_2}^{-1}c^{n_2}t_{\beta_2}\cdots t_{\beta_l}^{-1}c^{n_l}t_{\beta_l}.$$

A standard verification shows that this action is independent of the choice of transversal, and that it does make K into a $\boldsymbol{Z}B$-module.

Note that there are many possible choices of transversal for K in G. But once a transversal $\{t_\alpha \mid \alpha\in B\}$ has been chosen, the associated factor set is uniquely determined; it is a mapping $(\ ,\ )\colon B\times B\to K$, defined by

$$t_\alpha t_\beta = t_{\alpha\beta}(\alpha,\beta), \qquad \alpha,\beta\in B.$$

Further, every element $g\in G$ is now uniquely expressible in the form $g=t_\alpha c$, with $\alpha\in B$, $c\in K$, and these multiply according to the formula

$$(t_\alpha c)\cdot(t_\beta d) = t_{\alpha\beta}(\alpha,\beta)c^\beta d, \qquad \alpha,\beta\in B, \quad c,d\in K.$$

At various points in this paper, it will be convenient to introduce such a transversal to investigate an extension. However, we emphasize that the structure of K as a $\boldsymbol{Z}B$-module is completely independent of this choice of transversal.

Of special importance to us will be situations in which K, considered as a $\boldsymbol{Z}B$-module is torsion-free. We recall the definition of this notion in our context: the $\boldsymbol{Z}B$-module K is said to be *torsion-free* if, whenever $c^S=1$, where $c\in K$ and $S\in\boldsymbol{Z}B$, then either $c=1$ or $S=0$.

We introduce two notations to be used whenever we are dealing with such extensions. Let G, K and B be as above; for m a positive integer and $\beta\in B$, we define an element $T_{\beta,m}\in\boldsymbol{Z}B$ by

$$T_{\beta,m} = 1+\beta+\beta^2+\cdots+\beta^{m-1}. \tag{2.2}$$

After choosing a transversal $\{t_\alpha \mid \alpha\in B\}$ for K in G, we define, again for m a positive integer and $\beta\in B$, an element $a_{\beta,m}\in K$ by

$$a_{\beta,m} = (\beta,\beta^{m-1})\cdot(\beta,\beta^{m-2})\cdots(\beta,\beta^2)\cdot(\beta,\beta). \tag{2.3}$$

2.7. The following proposition brings out the connection between the existence of roots of elements of a metabelian $\mathscr{U}$-group G, and the existence of solutions of certain $\boldsymbol{Z}B$-module equations.

Proposition 2.9. (i) *Let G be a metabelian $\mathscr{U}$-group, let K be an abelian ideal of G, denote G/K by B and suppose B is an abelian $\mathscr{D}$-group. Let m be a π-number, let $\beta\in B$ and let $c\in K$. Then the equation*

$$x^{T_{\beta,m}} = c, \tag{*}$$

for $x\in K$, has at most one solution.

(ii) *Choose a transversal $\{t_\alpha \mid \alpha\in B\}$ for K in G. Then the equation* (*) *has a solution if and only if the element $t_{\beta^m}a_{\beta,m}c$ has an mth root in G.*

Proof. We first carry out an important calculation; for $\alpha \in B$ and $y \in K$, we have

$$\begin{aligned}(t_\alpha y)^m &= t_\alpha y t_\alpha y \cdots t_\alpha y\\ &= t_\alpha^m(t_\alpha^{-(m-1)} y t_\alpha^{m-1})(t_\alpha^{-(m-2)} y t_\alpha^{m-2})\cdots(t_\alpha^{-1} y t_\alpha)y\\ &= t_\alpha^m y^{1+\alpha+\alpha^2+\cdots+\alpha^{m-1}}.\end{aligned}$$

But

$$\begin{aligned}t_\alpha^m = t_\alpha^{m-2}t_\alpha t_\alpha &= t_\alpha^{m-3}t_\alpha t_{\alpha^2}(\alpha,\alpha) = t_\alpha^{m-3}t_{\alpha^3}(\alpha,\alpha^2)(\alpha,\alpha)\\ = \quad \cdots \quad &= t_{\alpha^m}(\alpha,\alpha^{m-1})(\alpha,\alpha^{m-2})\cdots(\alpha,\alpha).\end{aligned}$$

Therefore

$$(2.4) \qquad (t_\alpha y)^m = t_{\alpha^m} a_{\alpha,m} y^{T_{\alpha,m}}.$$

(i) Suppose that equation (*) has two solutions $x, y \in K$, so that $x^{T_{\beta,m}} = c = y^{T_{\beta,m}}$; then, using (2.4), we have

$$(t_\beta x)^m = t_{\beta^m} a_{\beta,m} x^{T_{\beta,m}} = t_{\beta^m} a_{\beta,m} c.$$

Hence the element $t_\beta x$ is an mth root of the element $t_{\beta^m}a_{\beta,m}c$. But exactly the same argument shows that $t_\beta y$ is an mth root of this element $t_{\beta^m}a_{\beta,m}c$. As $m \in [\pi]$ and $G \in \mathscr{U}$, we must have $t_\beta x = t_\beta y$, so that $x = y$ as required.

(ii) Suppose first that the equation (*) has a solution $x \in K$. Then

$$(t_\beta x)^m = t_{\beta^m} a_{\beta,m} x^{T_{\beta,m}} = t_{\beta^m} a_{\beta,m} c,$$

so that the element $t_{\beta^m}a_{\beta,m}c$ has an mth root in G.

Conversely, suppose that $t_{\beta^m}a_{\beta,m}c$ has an mth root, $t_\alpha y$ say, where $\alpha \in B$, $y \in K$. Using (2.4), we then have

$$(2.5) \qquad t_{\alpha^m} a_{\alpha,m} y^{T_{\alpha,m}} = (t_\alpha y)^m = t_{\beta^m} a_{\beta,m} c.$$

Comparing the two outer terms of this equality and using that every element of G is uniquely of the form $t_\gamma d$, with $\gamma \in B$, $d \in K$, yields $t_{\alpha^m} = t_{\beta^m}$; therefore, $\alpha^m = \beta^m$ and so $\alpha = \beta$, because $m \in [\pi]$ and $B \in \mathscr{U}$. Replacing α by β in equation (2.5), we obtain

$$t_{\beta^m} a_{\beta,m} y^{T_{\beta,m}} = t_{\beta^m} a_{\beta,m} c.$$

Therefore $y^{T_{\beta,m}} = c$, so that equation (*) does indeed have a solution. This completes the proof of the proposition.

COROLLARY 2.10. *Let $G \in \mathfrak{M}$, let K be an abelian ideal of G, denote G/K by B and suppose that B is abelian. Let m be a π-number, let $\beta \in B$ and let $c \in K$. Then there is exactly one element $x \in K$ such that $x^{T_{\beta,m}} = c$.*

Proof. This is an immediate consequence of the preceding proposition.

2.8. Note that the ideas discussed above hold, in particular, when $K = \gamma_2^\pi G$. We consider now a situation in which we have two metabelian $\mathscr{U}$-groups G and H and a homomorphism $\phi: G \to H$.

Proposition 2.11. (i) *Let G and H be metabelian $\mathscr{U}$-groups and let $\phi\colon G \to H$ be a homomorphism. Denote $G/\gamma_2^\pi G = B$ and $H/\gamma_2^\pi H = B^*$. Then ϕ induces, in a natural way, the structure of a ZB-module in the abelian group $\gamma_2^\pi H$. If $\{t_\alpha \mid \alpha \in \beta\}$ is any transversal for $\gamma_2^\pi G$ in G, the action of ZB on $\gamma_2^\pi H$ is given by the formula*

$$
\begin{aligned}
(2.6)\qquad (a^*)^{n_1\beta_1+n_2\beta_2+\cdots+n_l\beta_l} &= (t_{\beta_1}\phi)^{-1}(a^*)^{n_1}(t_{\beta_1}\phi) \\
&\quad \cdot (t_{\beta_2}\phi)^{-1}(a^*)^{n_2}(t_{\beta_2}\phi)\cdots(t_{\beta_l}\phi)^{-1}(a^*)^{n_l}(t_{\beta_l}\phi),
\end{aligned}
$$

for $a^* \in \gamma_2^\pi H$, and $n_j \in Z$, $\beta_j \in B$, $1 \leqq j \leqq l$.

(ii) *Suppose, in addition, that H is a $\mathscr{D}$-group. Let $m \in [\pi]$, let $\beta \in B$ and let $c^* \in \gamma_2^\pi H$. Then there is exactly one element $x^* \in \gamma_2^\pi H$ such that*

$$(2.7)\qquad (x^*)^{T_{\beta,m}} = c^*.$$

Proof. (i) The homomorphism $\phi\colon G \to H$ induces, in turn, homomorphisms $\phi'\colon B \to B^*$ and $\hat{\phi}\colon ZB \to ZB^*$. As $\gamma_2^\pi H$ is already a ZB^*-module the homomorphism $\hat{\phi}$ makes $\gamma_2^\pi H$ into a ZB-module, the action being given by

$$(2.8)\qquad (a^*)^S = (a^*)^{S\hat{\phi}}, \qquad a^* \in \gamma_2^\pi H, \quad S \in ZB;$$

and formula (2.6) follows without difficulty.

(ii) By definition, the action of the element $T_{\beta,m} \in ZB$ on $\gamma_2^\pi H$ is exactly the same as the action of the element $T_{\beta,m}\hat{\phi}$ (see equation (2.8)). A simple calculation shows that $T_{\beta,m}\hat{\phi} = T_{\beta\phi',m}$. Since, by Corollary 2.10, there is exactly one element $x^* \in \gamma_2^\pi H$ such that

$$(x^*)^{T_{\beta\phi',m}} = c^*,$$

we have the required result.

3. The Main Theorem.

3.1. The Main Theorem of the paper is aimed at obtaining representations of free metabelian $\mathscr{D}$-groups.

Main Theorem. *Let π be an arbitrary set of primes. Let D be a metabelian $\mathscr{D}$-group and let $\{x_i \mid i \in I\}$ be a subset of D. Suppose that D has an abelian ideal K such that the following conditions are satisfied:*

(i) *$D/K(=B)$ is a free abelian $\mathscr{D}$-group freely π-generated by the elements x_iK, $i \in I$.*

(ii) *For every π-number m, $\mathrm{gp}\,(x_i^{1/m} \mid i \in I)$ is a free metabelian group freely generated by the elements $x_i^{1/m}$, $i \in I$.*

(iii) *K, considered as a ZB-module, is torsion-free.*

Then $F = \mathscr{D}$-$\mathrm{gp}\,(x_i \mid i \in I)$ is a free metabelian $\mathscr{D}$-group freely π-generated by the elements x_i, $i \in I$.

In §§4 and 5, we exhibit two different examples of $\mathscr{D}$-groups D with subsets $\{x_i \mid i \in I\}$ which satisfy the hypotheses of the Main Theorem. The present section is devoted to proving this theorem.

3.2. Before proceeding to the proof of the Main Theorem, we give some motivation for the ideas entering into it.

Let $\mathfrak{N}_c$ be the variety of $\mathscr{D}$-groups consisting of all $\mathscr{D}$-groups which are nilpotent of class at most c. In [2], G. Baumslag constructed the free algebras in the variety $\mathfrak{N}_c$ as direct limits of ascending sequences of free nilpotent-of-class-c groups. Guided by this construction, we are led to try to construct a free $\mathfrak{M}$-group as the direct limit of an ascending sequence of free metabelian groups. Let $G_0=\bigcup_{n=1}^{\infty} M_n$, where M_n is free metabelian on a set X_n and the ascending chain is formed exactly as in [2] (G_0 corresponds to the group G^* in [2]).

It is clear that G_0 is a metabelian $\mathscr{U}$-group. Also the construction is again the "freest possible" in the sense that if $H \in \mathfrak{M}$ and $\phi\colon X_1 \to H$ is any mapping, there is a unique homomorphism $\phi_0\colon G_0 \to H$ which extends ϕ. What fails here is that G_0 is not an $\mathscr{E}$-group. Černikov's theorem (see [11, p. 238]) was applicable to establish the corresponding fact in the nilpotent case, but here G_0 is not nilpotent so that this theorem does not apply.

Let B be a free abelian $\mathscr{D}$-group of rank $|X_1|$. It is not difficult to show that $G_0/\gamma_2^\pi G_0 \cong B$, so that, by Proposition 2.8, G_0 has the same commutator quotient group, modulo its commutator ideal, as the free $\mathfrak{M}$-group we set out to construct. Consider $\gamma_2^\pi G_0$ as a $\boldsymbol{ZB}$-module, as described in §2.6. Since $G_0 \notin \mathscr{E}$, there must be elements $g \in G_0$ and primes $p \in \pi$ such that g has no pth root in G_0. Equivalently, by Proposition 2.9, there must be elements $c \in \gamma_2^\pi G_0$ and $\beta \in B$ such that the equation

$$x^{T_{\beta,p}} = c \tag{3.1}$$

has no solution $x \in \gamma_2^\pi G_0$. So, whereas the quotient $G_0/\gamma_2^\pi G_0$ is correct, the commutator ideal $\gamma_2^\pi G_0$ is not sufficiently large. If we are to carry the construction further, we must adjoin to $\gamma_2^\pi G_0$ roots of equations of the form (3.1) and continue to do this until all such equations have roots which, by Proposition 2.9, will guarantee that we have arrived at an $\mathscr{E}$-group.

In addition we must ensure that these adjunctions are carried out in a suitably free manner, that is, in a manner such that the mapping $\phi\colon X_1 \to H$, which has already been extended to a homomorphism $\phi_0\colon G_0 \to H$, can continue to be extended to each successively enlarged group in the construction, until we have ϕ extended to the final $\mathscr{E}$-group. It turns out that in order to achieve this freeness, precisely what is needed is that the commutator ideals $\gamma_2^\pi G$ of the successive enlargements G should always be torsion-free, considered as $\boldsymbol{ZB}$-modules. If we could carry out this program, the final $\mathscr{E}$-group arrived at would be a free metabelian $\mathscr{D}$-group on the set X_1.

It would be quite possible to work out the details of the construction outlined above, and this is in essence what T. MacHenry attempted to do in [14].

3.3. As a first step in the proof of the Main Theorem, we note that, as an immediate consequence of the following lemma, it suffices to prove the theorem for the case in which the index set I is finite. The proof of the lemma requires only the usual verification of a universal property and is omitted.

LEMMA 3.2. *Let $\mathfrak{B}$ be any variety of $\mathscr{D}$-groups and let $F \in \mathfrak{B}$ be π-generated by the set $\{x_i \mid i \in I\}$. Then F is a free $\mathfrak{B}$-group on the set $\{x_i \mid i \in I\}$ if and only if for every finite subset $J \subseteq I$, $\mathscr{D}$-gp $(x_i \mid i \in J)$ is a free $\mathfrak{B}$-group on the set $\{x_i \mid i \in J\}$.*

3.4. We now give the proof of the Main Theorem for the case in which π is a finite set of primes. Throughout this subsection, the following notation will be in force:

$$\pi = \{p_1, p_2, \ldots, p_k\}, \qquad q = p_1 p_2 \cdots p_k,$$
$$r = \text{a fixed integer} > 1, \qquad I = \{1, 2, \ldots, r\},$$
$$B = \Gamma_\pi \times \Gamma_\pi \times \cdots \times \Gamma_\pi \quad (r \text{ copies}).$$

The proof will be carried out in a number of steps. We first describe the group which corresponds to the group G_0 in the discussion in §3.2. Define groups M_n by

$$M_n = \operatorname{gp}(x_i^{1/q^n} \mid 1 \leqq i \leqq r), \qquad n = 0, 1, 2, \ldots.$$

By hypothesis (ii), each M_n is a free metabelian group freely generated by x_i^{1/q^n}, $1 \leqq i \leqq r$, and clearly

$$M_0 \leqq M_1 \leqq \cdots \leqq M_n \leqq \cdots \leqq F.$$

We denote by G_0 the union of this sequence of subgroups of F:

$$G_0 = \bigcup_{n=0}^{\infty} M_n \leqq F. \tag{3.2}$$

PROPOSITION 3.3. *Let $H \in \mathfrak{M}$ and let $\phi\colon \{x_i \mid 1 \leqq i \leqq r\} \to H$ be any mapping. Then there is a unique homomorphism $\phi_0\colon G_0 \to H$ which extends ϕ.*

Proof. The uniqueness of such a homomorphism $\phi_0\colon G_0 \to H$ is obvious; we must prove its existence. Assume inductively that ϕ has been extended to a homomorphism $\phi_{n-1}\colon M_{n-1} \to H$. As $H \in \mathscr{D}$, each of the elements $x_i^{1/q^{n-1}}\phi_{n-1}$, $1 \leqq i \leqq r$, has a unique qth root in H, we define a mapping $\phi_n\colon \{x_i^{1/q^n} \mid 1 \leqq i \leqq r\} \to H$ by

$$\phi_n\colon x_i^{1/q^n} \mapsto (x_i^{1/q^{n-1}}\phi_{n-1})^{1/q}, \qquad 1 \leqq i \leqq r.$$

As M_n is free metabelian, the mapping ϕ_n extends to a homomorphism $\phi_n\colon M_n \to H$. Then $\phi_0 = \bigcup_{n=1}^{\infty} \phi_n\colon G_0 \to H$ is the unique homomorphism which extends the original mapping ϕ.

Choose once and for all a canonical free π-generating set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ for the free abelian $\mathscr{D}$-group B, and identify the isomorphic $\mathscr{D}$-groups D/K and B by $x_i K \leftrightarrow \alpha_i$, $1 \leqq i \leqq r$. If $\eta\colon D \to D/K = B$ is the natural projection, then $\eta\colon x_i \mapsto \alpha_i$, $1 \leqq i \leqq r$.

LEMMA 3.4. *For each $n = 0, 1, 2, \ldots, \gamma_2 M_n = M_n \cap K$.*

Proof. Consider the homomorphism $\eta_n = \eta|_{M_n}: M_n \to B$. It is clear that the image of η_n is a free abelian group of rank r and that the kernel of η_n is $M_n \cap K$. Therefore $\gamma_2 M_n \leqq M_n \cap K$ and $M_n/\gamma_2 M_n \cong M_n/M_n \cap K \cong$ free abelian of rank r. Hence $\gamma_2 M_n = M_n \cap K$.

PROPOSITION 3.5. (i) $D = G_0 K$,
(ii) $G_0 \cap K = \gamma_2^\pi G_0 = \gamma_2 G_0 = \bigcup_{n=0}^\infty \gamma_2 M_n$,
(iii) $G_0/\gamma_2^\pi G_0 = G_0/G_0 \cap K \cong B$.

Proof. (i) Let $d \in D$; certainly there exist integers $n, n_1, n_2, \ldots, n_r$ such that

$$d\eta = \alpha_1^{n_1/q^n}\alpha_2^{n_2/q^n}\cdots\alpha_r^{n_r/q^n}.$$

Therefore there exists $k \in K$ such that

$$d = x_1^{n_1/q^n}x_2^{n_2/q^n}\cdots x_r^{n_r/q^n}k,$$

which shows that $d \in G_0 K$.

(ii) First observe that the equality $\gamma_2 G_0 = \bigcup_{n=0}^\infty \gamma_2 M_n$ is immediate. Using result (i), we have

$$G_0/G_0 \cap K \cong G_0K/K = D/K \cong B. \tag{3.3}$$

In particular, $G_0/G_0 \cap K$ is an abelian $\mathscr{U}$-group, so since $\gamma_2^\pi G_0$ is the smallest normal subgroup of G_0 with this property (cf. Lemma 2.5(iii)), we have $\gamma_2^\pi G_0 \leqq G_0 \cap K$. Certainly $\gamma_2 G_0 \leqq \gamma_2^\pi G_0$, so the proof will be complete if we can prove that $G_0 \cap K \leqq \gamma_2 G_0$. But this follows at once from Lemma 3.4.

(iii) This is contained in (ii) together with line (3.3).

The result $D = G_0 K$ of Proposition 3.5(i) enables us to choose a left transversal $\{t_\alpha \mid \alpha \in B\}$ for D modulo K which is inside the group G_0. Thus $\{t_\alpha \mid \alpha \in B\} \subseteq G_0$, where $t_\alpha \eta = \alpha$ for all $\alpha \in B$ and $t_1 = 1$. This transversal, once chosen, is fixed throughout the proof. Note that the corresponding factor set $(\ ,\ ): B \times B \to K$ given by $t_\alpha t_\beta = t_{\alpha\beta}(\alpha, \beta)$, $\alpha, \beta \in B$, has its values inside $G_0 \cap K = \gamma_2^\pi G_0$, a fact which will be needed later on.

In the next proposition, we obtain some properties of the $\mathscr{D}$-group $F = \mathscr{D}\text{-gp}\,(x_i \mid 1 \leqq i \leqq r)$. All dependence on the over-riding $\mathscr{D}$-group D is thereby removed and the remainder of the proof takes place inside F.

PROPOSITION 3.6. (i) *Let G be any subgroup of D which contains G_0: $G_0 \leqq G \leqq D$. Then $G/G \cap K \cong B$; further, $G \cap K$, considered as a ZB-module, is torsion-free.*
(ii) $\gamma_2^\pi F = F \cap K$.

Proof. (i) Since the natural projection $\eta: D \to B$, when restricted to G_0, still maps G_0 onto B, we have $G/G \cap K \cong B$. A simple check shows that our transversal $\{t_\alpha \mid \alpha \in B\}$ may also be used as a transversal for G modulo $G \cap K$. Now the fact that $G \cap K$, considered as a ZB-module, is torsion-free follows from the hypothesis that K, considered as a ZB-module, is torsion-free.

(ii) By part (i), $F/F \cap K \cong B \in \mathfrak{A}$, so, by Lemma 2.5(iii), $\gamma_2^\pi F \leqq F \cap K$. Therefore the mapping

$$\nu: x_i\gamma_2^\pi F \mapsto x_i(F \cap K), \qquad 1 \leqq i \leqq r,$$

induces an epimorphism $\nu: F/\gamma_2^\pi F \to F/F \cap K$. Since $F/F \cap K$ is a free $\mathfrak{A}$-group freely π-generated by the elements $x_i(F \cap K)$, $1 \leqq i \leqq r$, ν has an inverse. Hence ν is an isomorphism which implies $\gamma_2^\pi F = F \cap K$ as required.

We are now ready to approach the most important step in the proof of the Main Theorem. We have to prove that F is a free metabelian $\mathscr{D}$-group on the set $\{x_i \mid 1 \leqq i \leqq r\}$. Given a metabelian $\mathscr{D}$-group H and a mapping $\phi: \{x_i \mid 1 \leqq i \leqq r\} \to H$, we have established, in Proposition 3.3, that ϕ extends to a homomorphism $\phi_0: G_0 \to H$; our aim now is to extend ϕ_0 to a homomorphism $\bar{\phi}: F \to H$. This will be achieved by gradually building up from G_0 to F, adjoining one root at a time, and proving that at each stage the homomorphism ϕ_1 from the group-before-the-adjunction to H can be extended to a homomorphism $\bar{\phi}_1$ from the group-after-the-adjunction to H.

We describe the adjunction-of-a-root for an arbitrary group G lying between G_0 and F: $G_0 \leqq G \leqq F$. Let $g \in G$ be any element of G such that the qth root of g, which exists as an element of F, does not belong to G. The enlarged group $\bar{G}$, with the root adjoined, is going to be simply gp $(G, g^{1/q})$, but we shall derive a more suitable expression for it. As $g \in G$, there are unique elements $\beta \in B$ and $c_1 \in G \cap K$ such that

$$g = t_{\beta^q}c_1. \tag{3.4}$$

Denote by c the element

$$c = a_{\beta,q}^{-1}c_1, \tag{3.5}$$

where $a_{\beta,q} = (\beta, \beta^{q-1})\cdot(\beta, \beta^{q-2})\cdots(\beta, \beta) \in G_0 \cap K$. Thus $c \in G \cap K$ and c is determined by g. By Corollary 2.10, as $F \in \mathfrak{M}$, there is a unique element $x \in F \cap K$ such that

$$x^{T_{\beta,q}} = c, \tag{3.6}$$

where $T_{\beta,q} = 1+\beta+\cdots+\beta^{q-1} \in ZB$. Denote by $\{x\}$ the normal subgroup of F generated by the single element x or, what is clearly the same thing, the sub-ZB-module of $F \cap K$ generated by x. Noting that G and $\{x\}$ are subgroups of F, with $\{x\} \lhd F$, we define our enlarged group $\bar{G}$ by

$$\bar{G} = G\cdot\{x\}. \tag{3.7}$$

Thus, $\bar{G}$ is a subgroup of F containing G: $G_0 \leqq G \leqq \bar{G} \leqq F$; and $\bar{G}$ contains the element $g^{1/q}$, indeed this motivated the selection of the element x; for $t_\beta \in G$ so that $t_\beta x \in \bar{G}$ and

$$\begin{aligned} (t_\beta x)^q &= t_\beta^q x^{1+\beta+\cdots+\beta^{q-1}} && \text{(see proof of Proposition 2.9)}, \\ &= t_{\beta^q}a_{\beta,q}c && \text{(see proof of Proposition 2.9 and line (3.6))}, \\ &= g && \text{(using (3.4) and (3.5))}, \end{aligned}$$

so, in fact, $t_\beta x = g^{1/q}$. The truth of our statement above that $\bar{G} = \text{gp}(G, g^{1/q})$ is now immediate.

It remains to be established that the enlargement from G to $\bar{G}$ has been sufficiently free.

PROPOSITION 3.7. *Let G, $\bar{G}$, g, c, x, etc., be as above. Let H be a metabelian $\mathscr{D}$-group and let $\phi: G \to H$ be a homomorphism such that $(G \cap K)\phi \leqq \gamma_2^\pi H$. Then there is a homomorphism $\bar{\phi}: \bar{G} \to H$ which extends ϕ. Further $(\bar{G} \cap K)\bar{\phi} \leqq \gamma_2^\pi H$.*

Proof. As a first step, we prove that $\gamma_2^\pi H$ may be regarded as a $\boldsymbol{Z}B$-module, and that then $\phi: G \cap K \to \gamma_2^\pi H$ is a $\boldsymbol{Z}B$-homomorphism. Let $\phi_0 = \phi|_{G_0}: G_0 \to H$; by Proposition 2.11(i), ϕ_0 makes $\gamma_2^\pi H$ into a $\boldsymbol{Z}B$-module, the action being given by the formula

$$(a^*)^{n_1\beta_1 + \cdots + n_l\beta_l} = (t_{\beta_1}\phi)^{-1}(a^*)^{n_1}(t_{\beta_1}\phi)\cdots(t_{\beta_l}\phi)^{-1}(a^*)^{n_l}(t_{\beta_l}\phi) \tag{3.8}$$

for $a^* \in \gamma_2^\pi H$, and $n_j \in \boldsymbol{Z}$, $\beta_j \in B$, $1 \leqq j \leqq l$. A simple calculation now shows that if $k, k' \in G \cap K$ and $S \in \boldsymbol{Z}B$, then $(kk')\phi = (k\phi)(k'\phi)$ and $(k^S\phi) = (k\phi)^S$, so that $\phi: G \cap K \to \gamma_2^\pi H$ is indeed a $\boldsymbol{Z}B$-homomorphism.

As a convention, we shall denote elements of H (or of $\boldsymbol{Z}B^*$, where $B^* = H/\gamma_2^\pi H$) which are images under ϕ (or under the induced ring homomorphism $\hat{\phi}: \boldsymbol{Z}B \to \boldsymbol{Z}B^*$) of elements of G (or $\boldsymbol{Z}B$) by superscripting with "$*$". For example, $c\phi = c^* \in \gamma_2^\pi H$.

We now define a $\boldsymbol{Z}B$-homomorphism $\phi': \{x\} \to \gamma_2^\pi H$. By Proposition 2.11(ii), since $H \in \mathfrak{M}$, there is a unique element $x^* \in \gamma_2^\pi H$ such that

$$(x^*)^{T_{\beta,q}} = c^*. \tag{3.9}$$

As K considered as a $\boldsymbol{Z}B$-module is torsion-free, every element of $\{x\}$ is *uniquely* of the form x^S, with $S \in \boldsymbol{Z}B$. A $\boldsymbol{Z}B$-homomorphism $\phi': \{x\} \to \gamma_2^\pi H$ is therefore defined by

$$\phi': x^S \mapsto (x^*)^S, \qquad S \in \boldsymbol{Z}B.$$

We now have two homomorphisms $\phi: G \to H$, $\phi': \{x\} \to \gamma_2^\pi H$. We claim that they agree on the intersection $G \cap \{x\} = (G \cap K) \cap \{x\}$. For if $k \in (G \cap K) \cap \{x\}$, say $k = x^S$, $S \in \boldsymbol{Z}B$, then

$$k^{T_{\beta,q}} = (x^S)^{T_{\beta,q}} = (x^{T_{\beta,q}})^S = c^S \qquad \text{(using (3.6))}.$$

Therefore

$$\begin{aligned}(k\phi')^{T_{\beta,q}} &= (x^S\phi')^{T_{\beta,q}} = ((x^*)^S)^{T_{\beta,q}} \\ &= ((x^*)^{T_{\beta,q}})^S = (c^*)^S;\end{aligned}$$

and

$$\begin{aligned}(k\phi)^{T_{\beta,q}} &= (k^{T_{\beta,q}})\phi = (c^S)\phi \\ &= (c\phi)^S = (c^*)^S,\end{aligned}$$

where we have used the fact that $\phi\colon G \cap K \to \gamma_2^\pi H$ is a ZB-homomorphism. Then, by Proposition 2.11(ii), $(k\phi')^{T_{\beta,q}} = (k\phi)^{T_{\beta,q}}$ implies $k\phi' = k\phi$ as required.

We are now in a position to define the homomorphism $\bar{\phi}\colon \bar{G} \to H$. Recalling that $\bar{G} = G \cdot \{x\}$, we define $\bar{\phi}\colon \bar{G} \to H$ by

$$\bar{\phi}\colon g_1 x^{S_1} \mapsto (g_1\phi)(x^{S_1}\phi'), \qquad g_1 \in G, \quad S_1 \in ZB.$$

To see that $\bar{\phi}$ is a well-defined mapping, let $g_1 x^{S_1} = g_2 x^{S_2}$, where $g_1, g_2 \in G$, $S_1, S_2 \in ZB$. Then $g_2^{-1} g_1 = x^{S_2 - S_1} \in G \cap \{x\}$; therefore, since ϕ and ϕ' agree on $G \cap \{x\}$, $(g_2^{-1} g_1)\phi = (x^{S_2 - S_1})\phi'$, hence $(g_1\phi)(x^{S_1}\phi') = (g_2\phi)(x^{S_2}\phi')$, as required.

To prove that $\bar{\phi}$ is a homomorphism, we must prove that if $\beta_1, \beta_2 \in B$, $k_1, k_2 \in G \cap K$ and $S_1, S_2 \in ZB$, then

$$((t_{\beta_1} k_1 x^{S_1})(t_{\beta_2} k_2 x^{S_2}))\bar{\phi} = ((t_{\beta_1} k_1 x^{S_1})\bar{\phi})((t_{\beta_2} k_2 x^{S_2})\bar{\phi}).$$

This is a straightforward calculation and is omitted.

The fact that $\bar{\phi}$ is an extension of ϕ is clear. Finally, we prove that $(\bar{G} \cap K)\bar{\phi} \leqq \gamma_2^\pi H$. Let $g_1 x^{S_1} \in \bar{G} \cap K$, where $g_1 \in G$, $S_1 \in ZB$; since $x^{S_1} \in K$, $g_1 \in K$ also; hence $g_1 \in G \cap K$ so that $g_1\phi \in \gamma_2^\pi H$, by hypothesis, and the result follows. This completes the proof of the proposition.

Once the adjunction-of-a-root process has been described and Proposition 3.7 has been established, the proof of the Main Theorem (with π a finite set of primes) is a fairly straightforward matter. We proceed by means of a well-known "tower of subgroups" method. We shall construct subgroups G_n, $n = 0, 1, 2, \ldots$, of F, where G_0 is the group defined by equation (3.2), such that $G_0 \leqq G_1 \leqq \cdots \leqq G_{n-1} \leqq G_n \leqq \cdots \leqq F$, and such that for all elements $g \in G_{n-1}$, $g^{1/q} \in G_n$. Then $\bigcup_{n=0}^\infty G_n$ is an $\mathscr{E}$-group, so that $\bigcup_{n=0}^\infty G_n = F$.

Assume inductively that G_{n-1} has been defined (G_0 starts the recursive definition); we show how to construct G_n. Since F is finitely π-generated, it is clear that F is countable. Enumerate the elements of G_{n-1} in a sequence:

$$G_{n-1} = \{g_1, g_2, \ldots, g_{j-1}, g_j, \ldots\}.$$

Define a sequence of groups $G_{n-1}^{(j)}$, $j = 0, 1, 2, \ldots$, such that

$$G_{n-1} = G_{n-1}^{(0)} \leqq G_{n-1}^{(1)} \leqq \cdots \leqq G_{n-1}^{(j-1)} \leqq G_{n-1}^{(j)} \leqq \cdots \leqq F \tag{3.10}$$

recursively as follows. Define $G_{n-1}^{(0)} = G_{n-1}$ and assume inductively that $G_{n-1}^{(j-1)}$ has been defined; to define $G_{n-1}^{(j)}$, consider the element $g_j \in G_{n-1} \leqq G_{n-1}^{(j-1)}$:

if $g_j^{1/q} \in G_{n-1}^{(j-1)}$, we simply set $G_{n-1}^{(j)} = G_{n-1}^{(j-1)}$;

if $g_j^{1/q} \notin G_{n-1}^{(j-1)}$, we carry out the adjunction-of-a-root process for the group $G_{n-1}^{(j-1)}$ and the element g_j, and denote the resulting group by $G_{n-1}^{(j)}$; thus, in this case, $G_{n-1}^{(j)} = \bar{G}_{n-1}^{(j-1)}$.

This yields a sequence of groups $G_{n-1}^{(j)}$, $j = 0, 1, 2, \ldots$, satisfying (3.10), and with the property $g_j^{1/q} \in G_{n-1}^{(j)}, j=1, 2, \ldots$. We define G_n to be the union of this sequence:

$$G_n = \bigcup_{j=0}^{\infty} G_{n-1}^{(j)}.$$

It is clear that our requirement $g^{1/q} \in G_n$ for all $g \in G_{n-1}$ is satisfied.

PROPOSITION 3.8. *Let H be a metabelian $\mathscr{D}$-group and let $\phi_{n-1}\colon G_{n-1} \to H$ be any homomorphism such that $(G_{n-1} \cap K)\phi_{n-1} \leqq \gamma_2^\pi H$. Then there is a homomorphism $\phi_n\colon G_n \to H$ which extends ϕ_{n-1}, and further $(G_n \cap K)\phi_n \leqq \gamma_2^\pi H$.*

Proof. It follows at once from Proposition 3.7 that we can define recursively a sequence of homomorphisms

$$\phi_{n-1}^{(j)}\colon G_{n-1}^{(j)} \to H, \qquad j = 0, 1, 2, \ldots,$$

such that, for $j=1, 2, \ldots$, $\phi_{n-1}^{(j)}$ is an extension of $\phi_{n-1}^{(j-1)}$, and $(G_{n-1}^{(j)} \cap K)\phi_{n-1}^{(j)} \leqq \gamma_2^\pi H$. Then $\phi_n = \bigcup_{j=0}^{\infty} \phi_{n-1}^{(j)} : G_n \to H$ does as required.

We can now, at last, complete the proof of the theorem. We use the expression for F,

$$F = \bigcup_{n=0}^{\infty} G_n, \tag{3.11}$$

established above. Let $H \in \mathfrak{M}$ and let $\phi\colon \{x_i \mid 1 \leqq i \leqq r\} \to H$ be any mapping. We must prove that there is a unique homomorphism $\bar{\phi}\colon F \to H$ which extends ϕ. The uniqueness of such a homomorphism is obvious because the set $\{x_i \mid 1 \leqq i \leqq r\}$ π-generates F. We proceed with the existence.

By Proposition 3.3, there is a homomorphism $\phi_0\colon G_0 \to H$ which extends the mapping ϕ. As $\gamma_2^\pi G_0$ is a fully invariant subgroup and since $\gamma_2^\pi G_0 = G_0 \cap K$, by Proposition 3.5(ii), we have $(G_0 \cap K)\phi_0 = (\gamma_2^\pi G_0)\phi_0 \leqq \gamma_2^\pi H$. It is now clear from Proposition 3.8 that we can define recursively a sequence of homomorphisms

$$\phi_n\colon G_n \to H, \qquad n = 0, 1, 2, \ldots,$$

such that, for $n=1, 2, \ldots$, ϕ_n is an extension of ϕ_{n-1}, and $(G_n \cap K)\phi_n \leqq \gamma_2^\pi H$. Finally, recalling (3.11), we see that the homomorphism

$$\bar{\phi} = \bigcup_{n=0}^{\infty} \phi_n\colon F \to H$$

extends the original mapping ϕ. This completes the proof of the Main Theorem (with π a finite set of primes).

3.5. Once the Main Theorem has been established for a finite set of primes, it is not difficult to prove it for an arbitrary set of primes. We now prove the Main Theorem exactly as stated except that, in accordance with the remarks in §3.3, we take the index set I to be finite: $I=\{1, 2, \ldots, r\}$.

In this one subsection only, we consider several different sets of primes, and we specify the set of primes being referred to by means of a subscript.

Let us denote by B_π a free abelian $\mathscr{D}_\pi$-group of rank r:

$$B_\pi \cong \Gamma_\pi \times \Gamma_\pi \times \cdots \times \Gamma_\pi \quad (r \text{ copies}).$$

Choose a free π-generating set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ for B_π, and identify the isomorphic $\mathscr{D}_\pi$-groups D/K and B_π by $x_iK \leftrightarrow \alpha_i$, $1 \leqq i \leqq r$. The projection $\eta\colon D \to D/K = B_\pi$ is then given by

$$\eta\colon x_i \mapsto \alpha_i, \qquad 1 \leqq i \leqq r.$$

We observe that if π' is any subset of the set of primes π, then $\mathscr{U}_{\pi'} \supseteq \mathscr{U}_\pi$, $\mathscr{E}_{\pi'} \supseteq \mathscr{E}_\pi$ and $\mathscr{D}_{\pi'} \supseteq \mathscr{D}_\pi$. The proofs of these inclusions are immediate from the definitions. In particular, B_π may be considered as a $\mathscr{D}_{\pi'}$-group and we may define

$$B_{\pi'} = \mathscr{D}_{\pi'}\text{-gp}\,(\alpha_i \mid 1 \leqq i \leqq r);$$

then $B_{\pi'} \leqq B_\pi$, and it is clear that $B_{\pi'} \cong \Gamma_{\pi'} \times \Gamma_{\pi'} \times \cdots \times \Gamma_{\pi'}$.

Now order the primes in the set π by increasing magnitude

$$\pi = \{p_1, p_2, \ldots, p_n, \ldots\},$$

and introduce finite subsets $\pi_n \subseteq \pi$ by

$$\pi_n = \{p_1, p_2, \ldots, p_n\}, \qquad n = 1, 2, \ldots.$$

As $F = \mathscr{D}_\pi\text{-gp}\,(x_i \mid 1 \leqq i \leqq r)$ may be considered as a $\mathscr{D}_{\pi_n}$-group, we may define

$$F_{\pi_n} = \mathscr{D}_{\pi_n}\text{-gp}\,(x_i \mid 1 \leqq i \leqq r), \qquad n = 1, 2, \ldots.$$

Our plan is to prove that each F_{π_n} is a free metabelian $\mathscr{D}_{\pi_n}$-group on the set $\{x_i \mid 1 \leqq i \leqq r\}$ and to deduce that F is a free metabelian $\mathscr{D}_\pi$-group on this same set by a simple "direct limit" argument.

Consider the projection $\eta\colon D \to B_\pi$. For $n = 1, 2, \ldots$, we may consider D and B_π as $\mathscr{D}_{\pi_n}$-groups and η as a homomorphism of D_{π_n}-groups; then the preimage, via η, of the sub-$\mathscr{D}_{\pi_n}$-group $B_{\pi_n} \leqq B_\pi$ must be a sub-$\mathscr{D}_{\pi_n}$-group of D. We denote

$$D_{\pi_n} = B_{\pi_n}\eta^{-1}, \qquad n = 1, 2, \ldots,$$

then, by an isomorphism theorem, $D_{\pi_n}/K \cong B_{\pi_n}$. We claim that the following facts are clear:

(1) D_{π_n} is a metabelian $\mathscr{D}_{\pi_n}$-group.

(2) The set $\{x_i \mid 1 \leqq i \leqq r\}$ is a subset of D_{π_n}, so that $F_{\pi_n} \leqq D_{\pi_n}$.

(3) K is an abelian π_n-ideal of D_{π_n}.

(4) $D_{\pi_n}/K \cong B_{\pi_n}$ is a free abelian $\mathscr{D}_{\pi_n}$-group freely π_n-generated by the elements x_iK, $1 \leqq i \leqq r$.

(5) For any π_n-number m, gp $(x_i^{1/m} \mid 1 \leqq i \leqq r)$ is a free metabelian group freely generated by the elements $x_i^{1/m}$, $1 \leqq i \leqq r$.

(6) K, considered as a ZB_{π_n}-module, is torsion-free.

All of these follow at once from the hypothesis of the Main Theorem and the above discussion. The only one requiring any comment is (6), which follows from hypothesis (iii) of the Main Theorem by writing down what it means to say that K considered as a ZB_{π_n}-module is torsion-free, and observing that it is an immediate consequence of the fact that K considered as a ZB-module is torsion-free. Therefore, since the Main Theorem has been established for a finite set of primes, we conclude that each F_{π_n}, $n=1, 2, \ldots$, is a free metabelian $\mathscr{D}_{\pi_n}$-group freely π_n-generated by the elements x_i, $1 \leqq i \leqq r$.

The conclusion of the Main Theorem now follows from the observation that F is the direct limit of the sequence

$$F_{\pi_1} \leqq F_{\pi_2} \leqq \cdots \leqq F_{\pi_n} \leqq \cdots \leqq F.$$

We claim that

$$F = \bigcup_{n=1}^{\infty} F_{\pi_n} \cong \text{inj lim } F_{\pi_n}. \tag{3.12}$$

For if $f \in F = \mathscr{D}_\pi\text{-gp}\,(x_i \mid 1 \leqq i \leqq r)$, f can be obtained from the elements x_i, $1 \leqq i \leqq r$, by a finite sequence of operations: multiplying, forming inverses and extracting pth roots for various primes $p \in \pi$. In particular, only a finite number of root extractions will be used. If N is an integer such that the primes corresponding to all these root extractions belong to π_N, it is clear that $f \in F_{\pi_N}$; thus equality (3.12) is established.

It is now immediate that F is a free metabelian $\mathscr{D}_\pi$-group on the set $\{x_i \mid 1 \leqq i \leqq r\}$. For let H be any metabelian $\mathscr{D}_\pi$-group and let $\phi : \{x_i \mid 1 \leqq i \leqq r\} \to H$ be any mapping. For each $n = 1, 2, \ldots$, H may be considered as a $\mathscr{D}_{\pi_n}$-group, so there is a unique homomorphism $\phi_n : F_{\pi_n} \to H$ which extends ϕ. Clearly $\phi_{n+1}|_{F_{\pi_n}} = \phi_n$, for $n = 1, 2, \ldots$, therefore

$$\phi = \bigcup_{n=1}^{\infty} \phi_n = \text{inj lim } \phi_n : F \to H$$

is the unique homomorphism from F into H which extends the mapping ϕ. This completes the proof of the Main Theorem.

4. Power series algebra representation.

4.1. In this section, our first representation of free metabelian $\mathscr{D}$-groups is derived from the Main Theorem. This representation is inside a power series algebra over the rationals and is comparable with W. Magnus' representation of a free group inside a "free" power series ring (see W. Magnus [15], or [17, p. 310]). Just as Magnus' representation implies that free groups are residually torsion-free nilpotent, our power series algebra representation implies that free metabelian $\mathscr{D}$-groups are residually torsion-free nilpotent.

4.2. The power series algebra in which our representation takes place was constructed by G. Baumslag [5] for an entirely different purpose, namely to obtain a representation of the wreath product of two arbitrary torsion-free abelian groups. The algebra A defined below is precisely the one used in [5], but our notation is different.

Let $\mathbf{Q}$ be the field of rational numbers and let I be a well-ordered index set. Let A be the power series algebra over $\mathbf{Q}$ in the indeterminates y_i, z_i, $i \in I$, subject to the defining relations

$$z_i y_j = y_i y_j z_k = y_i y_j - y_j y_i = z_i z_j - z_j z_i = 0, \qquad i, j, k \in I. \tag{4.1}$$

It is easy to see that every nonzero element in the multiplicative subsemigroup of A generated by the y_i and z_i is uniquely of one of the three forms

$$y_{i_1} y_{i_2} \cdots y_{i_n}, \tag{4.2}$$

$$z_{i_1} z_{i_2} \cdots z_{i_n}, \tag{4.3}$$

or

$$y_k z_{i_1} z_{i_2} \cdots z_{i_n}, \tag{4.4}$$

where $i_1, i_2, \ldots, i_n$, $k \in I$, and $i_1 \leqq i_2 \leqq \cdots \leqq i_n$. Elements of one of the forms (4.2), (4.3) or (4.4) are called *monomials*, and the *degree* of a monomial is defined in the obvious way. (The monomials in (4.2), (4.3) and (4.4) have degrees $n, n, n+1$ respectively.)

An element $a \in A$ is an infinite sum

$$a = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

where $a_0 \in \mathbf{Q}$, and a_n, the so-called *homogeneous component of degree n*, is either zero or a finite sum of $\mathbf{Q}$-multiples of monomials of degree n. For each $a \in A$, its *order* $v(a)$ is defined by

$$\begin{aligned} v(a) &= \min \{n \mid a_n \neq 0\} \quad && \text{if } a \neq 0, \\ &= \infty && \text{if } a = 0, \end{aligned}$$

and a sequence A_n, $n = 0, 1, 2, \ldots$, of ideals of A is defined by

$$A_n = \{a \in A \mid v(a) \geqq n\}.$$

It is clear that

$$\bigcap_{n=0}^{\infty} A_n = (0). \tag{4.5}$$

It is also immediate that every element $a \in A$ can be expressed uniquely in the form

$$a = a_0 + a_{yy} + a_{zz} + a_{yz}, \tag{4.6}$$

where $a_0 \in \mathbf{Q}$, and a_{yy} (respectively a_{zz}, a_{yz}) is a sum (in general infinite) of $\mathbf{Q}$-multiples of monomials of the form (4.2) (respectively (4.3), (4.4)).

Now let

$$C = \{a \in A \mid a_0 = 1\}.$$

It is easy to show that C is not only a subgroup of the multiplicative semigroup of A (see e.g. W. Magnus [15], or [17, p. 310]), but is also a $\mathscr{D}$-group, for any set of primes π (see [17, p. 316, Problem 4]). We denote

$$x_i = 1+y_i+z_i, \qquad i \in I,$$
$$F = \mathscr{D}\text{-gp}\,(x_i \mid i \in I).$$

The theorem which yields our power series algebra representation and which is proved in this section is

THEOREM 4.1. *F is a free metabelian $\mathscr{D}$-group freely π-generated by the elements x_i, $i \in I$.*

Observe that it suffices to prove Theorem 4.1 for the case in which π is the set of all primes. For if we can establish, for the set of all primes, that F is a free metabelian $\mathscr{D}$-group on the set $\{x_i \mid i \in I\}$, it follows from the work in §3.5 that, if π is any set of primes, the sub-$\mathscr{D}_\pi$-group of F π-generated by $\{x_i \mid i \in I\}$ is a free metabelian $\mathscr{D}_\pi$-group on this set. So throughout this section, it is to be understood that we are dealing with the set of all primes.

4.3. We begin by recording some results about groups inside the power series algebra which were proved by G. Baumslag in [5]. Introduce

$$Y = \text{gp}\,(1+y_i \mid i \in I), \qquad \overline{Y} = \mathscr{D}\text{-gp}\,(1+y_i \mid i \in I),$$
$$Z = \text{gp}\,(1+z_i \mid i \in I), \qquad \overline{Z} = \mathscr{D}\text{-gp}\,(1+z_i \mid i \in I).$$

It is clear that Y and Z are free abelian groups on the sets $\{1+y_i \mid i \in I\}$ and $\{1+z_i \mid i \in I\}$ respectively, and that $\overline{Y}$ and $\overline{Z}$ are free abelian $\mathscr{D}$-groups on these same sets. An isomorphism between $\overline{Y}$ and $\overline{Z}$ is determined by the mapping $1+y_i \to 1+z_i$, $i \in I$; if $1+c_{yy} \in \overline{Y}$ and $1+c'_{zz} \in \overline{Z}$ are two elements which correspond under this isomorphism, we denote this fact by $1+c_{yy} \leftrightarrow 1+c'_{zz}$.

The following results were established in G. Baumslag [5]:

THEOREM 4.2. *The subgroup of C generated by Y and Z is the (standard) wreath product of Y by Z.*

COROLLARY 4.3. (*Recall that $x_i = 1+y_i+z_i$, $i \in I$.*) *The group $M_0 = \text{gp}\,(x_i \mid i \in I)$ is a free metabelian group freely generated by the elements x_i, $i \in I$.*

THEOREM 4.4. *Let $M_0(n) = \{1+c \in M_0 \mid v(c) \geqq n\}$, then $M_0(n) = \gamma_n M_0$, $n = 1, 2, \ldots.$*

PROPOSITION 4.5. *C is residually torsion-free nilpotent.*

4.4. The next few results are to aid with computation inside the group C. Note that if $1+c$ is an arbitrary element of C, then (cf. line (4.6))

$$1+c = 1+c_{yy}+c_{zz}+c_{yz} = (1+c_{zz})(1+c_{yy})(1+c_{yz}). \tag{4.7}$$

This equality is immediate from the relations (4.1). As we are mainly concerned with the yy- and zz-components of elements, we adopt the convention of denoting by c^*_{yz} an unspecified element of this form; thus c^*_{yz} stands for any sum of $\boldsymbol{Q}$-multiples of monomials of the form (4.4).

LEMMA 4.6. *Let $1+c=1+c_{yy}+c_{zz}+c_{yz}$ be an element of C and let n be a positive integer. Then*

(i) $(1+c_{yz})(1+c_{yy})=(1+c_{yy})(1+c_{yz})$,

(ii) $(1+c_{yy})(1+c_{zz})=(1+c_{zz})(1+c_{yy})(1+c^*_{yz})$,

(iii) $(1+c^*_{yz})(1+c_{zz})=(1+c_{zz})(1+c^*_{yz})$,

(iv) $(c_{yy}+c_{zz}+c_{yz})^n = c^n_{yy}+c^n_{zz}+c^*_{yz}$.

Proof. These are all immediate from the relations (4.1), induction on n being used for (iv). Note that in (iii) we are using our convention, and the c^*_{yz} on the left-hand side is not the same as the c^*_{yz} on the right-hand side.

We observed, in the Introduction, that the class of $\mathscr{D}$-groups may be considered as a variety of algebras, the operations of the algebras being multiplication, inversion and extraction of pth roots for primes $p\in\pi$. Let S be any set; by a *$\mathscr{D}$-word* in the elements of S, we mean any expression obtained from the elements of S by a finite number of formal performances of these operations. For a general definition of the concept involved here, see B. H. Neumann [20, pp. 50–51].

PROPOSITION 4.7. *Let $w(s_1, s_2, \ldots, s_n)$ be any $\mathscr{D}$-word in the elements $s_1, s_2, \ldots, s_n$, and let*

$$1+c^{(j)} = 1+c^{(j)}_{yy}+c^{(j)}_{zz}+c^{(j)}_{yz}, \qquad 1\leqq j\leqq n,$$

be elements of C. Then

$$w(1+c^{(1)},\ldots,1+c^{(n)}) \\ = w(1+c^{(1)}_{zz},\ldots,1+c^{(n)}_{zz})\cdot w(1+c^{(1)}_{yy},\ldots,1+c^{(n)}_{yy})\cdot(1+c^*_{yz}). \tag{4.8}$$

Proof. Written in an obvious notation, equation (4.8) becomes

$$w(1+\boldsymbol{c}) = w(1+\boldsymbol{c}_{zz})\cdot w(1+\boldsymbol{c}_{yy})\cdot(1+c^*_{yz}).$$

We proceed by induction on the *length* $|w|$ of w as a $\mathscr{D}$-word (see [20, p. 51]). If $|w|=1$, $w=s_j$ for some j, $1\leqq j\leqq n$, and the result is given by equation (4.7). Let $|w|=l$ and assume the result has been proved for all $\mathscr{D}$-words of shorter length. Then w is of one of the forms u^{-1}, $u^{1/p}$ or $u\cdot v$, where u and v are $\mathscr{D}$-words of length less than l.

We deal with the first two possibilities simultaneously by supposing $w=u^q$, $q\in\boldsymbol{Q}$, where $|u|<l$. If $u(1+\boldsymbol{c})=1+c'_{yy}+c'_{zz}+c^*_{yz}$, the induction assumption gives

$$1+c'_{yy}+c'_{zz}+c^*_{yz} = u(1+\boldsymbol{c}_{zz})\cdot u(1+\boldsymbol{c}_{yy})\cdot(1+c^*_{yz}). \tag{4.9}$$

Therefore

$$\begin{aligned}
w(1+c) &= (1+c'_{yy}+c'_{zz}+c^*_{yz})^q \\
&= 1+q(c'_{yy}+c'_{zz}+c^*_{yz})+\frac{q(q-1)}{2!}(c'_{yy}+c'_{zz}+c^*_{yz})^2 \\
&\quad +\cdots+\frac{q(q-1)\cdots(q-m+1)}{m!}(c'_{yy}+c'_{zz}+c^*_{yz})^m+\cdots \\
&= (1+c'_{zz})^q\cdot(1+c'_{yy})^q\cdot(1+c^*_{yz}) \\
&\qquad\qquad \text{(after a short calculation using Lemma 4.6(iv))} \\
&= (u(1+c_{zz}))^q\cdot(u(1+c_{yy}))^q\cdot(1+c^*_{yz}) \quad \text{(by equation (4.9))} \\
&= w(1+c_{zz})\cdot w(1+c_{yy})\cdot(1+c^*_{yz})
\end{aligned}$$

which completes this case.

The other possibility is that $w=u\cdot v$, where $|u|<l$, $|v|<l$; we then have

$$\begin{aligned}
w(1+c) &= u(1+c)\cdot v(1+c) \\
&= u(1+c_{zz})\cdot u(1+c_{yy})\cdot(1+c^*_{yz})\cdot v(1+c_{zz})\cdot v(1+c_{yy})\cdot(1+c^*_{yz}) \\
&= u(1+c_{zz})\cdot v(1+c_{zz})\cdot u(1+c_{yy})\cdot v(1+c_{yy})\cdot(1+c^*_{yz}) \\
&\qquad\qquad \text{(using Lemma 4.6(i), (ii) and (iii))} \\
&= w(1+c_{zz})\cdot w(1+c_{yy})\cdot(1+c^*_{yz}).
\end{aligned}$$

This completes the proof of the proposition.

PROPOSITION 4.8 (COMMUTATOR FORMULA). *Let*

$$\begin{aligned}
1+c &= 1+c_{zz}+c_{yy}+c_{yz}\in C, \\
1+c' &= 1+c'_{zz}+c'_{yy}+c'_{yz}\in C.
\end{aligned}$$

Then

$$[1+c,\, 1+c'] = 1+c_{yy}c'_{zz}+c_{yz}c'_{zz}-c'_{yy}c_{zz}-c'_{yz}c_{zz}. \tag{4.10}$$

Proof. Denote $\Delta=c_{yy}c'_{zz}+c_{yz}c'_{zz}-c'_{yy}c_{zz}-c'_{yz}c_{zz}$ and observe that if $1+c''\in C$, then

$$(1+c'')\Delta = \Delta. \tag{4.11}$$

A simple calculation shows that $c(1+c')-(1+c')c=\Delta$; therefore

$$\begin{aligned}
[1+c,\, 1+c'] &= (1+c)^{-1}(1+c')^{-1}(1+c)(1+c') \\
&= (1+c)^{-1}\{1+(1+c')^{-1}c(1+c')\} \\
&= (1+c)^{-1}\{1+(1+c')^{-1}((1+c')c+\Delta)\} \\
&= 1+(1+c)^{-1}(1+c')^{-1}\Delta \\
&= 1+\Delta \qquad\qquad \text{(by equation (4.11)).}
\end{aligned}$$

This completes the proof of the proposition.

Corollary 4.9. (i) *Let* $K=\{1+c_{yz}\in C\}$. *Then* K *is an abelian normal subgroup of* C, *and* $\gamma_2 C\leqq K$.

(ii) C *is a metabelian group.*

Proof. (i) is immediate using the commutator formula (4.10). Then (ii) follows at once from (i).

4.5. We are now ready to introduce the $\mathscr{D}$-group D inside C which corresponds to the $\mathscr{D}$-group D of the Main Theorem. Referring to the definitions of $\bar{Y}$, $\bar{Z}$ and $1+c_{yy}\leftrightarrow 1+c'_{zz}$, see §4.3, we define

$$D = \{1+c_{yy}+c_{zz}+c_{yz}\in C \mid 1+c_{yy}\in \bar{Y},\ 1+c_{zz}\in\bar{Z},\ 1+c_{yy}\leftrightarrow 1+c_{zz}\}.$$

Thus (cf. equation (4.7)), a typical element of D is

$$(1+z_{i_1})^{q_1}(1+z_{i_2})^{q_2}\cdots(1+z_{i_l})^{q_l}\cdot(1+y_{i_1})^{q_1}(1+y_{i_2})^{q_2}\cdots(1+y_{i_l})^{q_l}\cdot(1+c_{yz}),$$
$$i_j\in I,\ q_j\in \mathbf{Q},\ 1\leqq j\leqq l.$$

It is clear, from Proposition 4.7, that D is closed under multiplication, inverses and extraction of nth roots for all positive integers n. As $D\leqq C\in\mathfrak{M}$, we have that D is a metabelian $\mathscr{D}$-group. Also, certainly, $x_i=1+y_i+z_i\in D$, for all $i\in I$.

We have already introduced $K=\{1+c_{yz}\in C\}$. By Corollary 4.9(i), K is an abelian normal subgroup of D. The fact that K is an ideal of D and the verification of hypothesis (i) of the Main Theorem are contained in the next proposition. As usual, B denotes a free abelian $\mathscr{D}$-group on the set $\{\alpha_i \mid i\in I\}$.

Proposition 4.10. *The mapping* $\eta\colon D\to B$ *defined by*

$$\eta\colon (1+z_{i_1})^{q_1}\cdots(1+z_{i_l})^{q_l}(1+y_{i_1})^{q_1}\cdots(1+y_{i_l})^{q_l}\cdot(1+c_{yz})\mapsto \alpha_{i_1}^{q_1}\alpha_{i_2}^{q_2}\cdots\alpha_{i_l}^{q_l} \tag{4.12}$$

is a surjective homomorphism with kernel K. *Furthermore,* D/K *is a free abelian* $\mathscr{D}$-*group freely* π-*generated by the elements* x_iK, $i\in I$.

Proof. It is immediate that η is a well-defined surjective mapping. The fact that η is a homomorphism follows from Proposition 4.7. It is also clear that $\ker\eta=K$, so that $D/K\cong B$. Finally, since $x_i\eta=((1+z_i)(1+y_i))\eta=\alpha_i$, the free abelian $\mathscr{D}$-group D/K is freely π-generated by the elements x_iK, $i\in I$.

Our aim is to prove that $\mathscr{D}$-gp $(x_i \mid i\in I)$ is a free metabelian $\mathscr{D}$-group freely π-generated by these elements, and the conclusion of the Main Theorem yields precisely that. The purpose of the next two subsections is to prove that hypotheses (ii) and (iii) of the Main Theorem are satisfied.

4.6. To verify hypothesis (ii) of the Main Theorem, we must prove that, if m is any positive integer, then

$$M'' = \mathrm{gp}\,(x_i^{1/m} \mid i\in I)$$

is a free metabelian group freely generated by the elements $x_i^{1/m}$, $i\in I$. To achieve this, it suffices to prove that for every finite subset $J\subseteq I$,

$$M' = \mathrm{gp}\,(x_i^{1/m} \mid i\in J)$$

is free metabelian freely generated by the elements $x_i^{1/m}$, $i \in J$. By Corollary 4.3, the subgroup

$$M_0' = \text{gp}\,(x_i \mid i \in J)$$

of M' is free metabelian freely generated by the elements x_i, $i \in J$. To continue, we need the following simple lemma:

LEMMA 4.11. *Let $\mathfrak{B}$ be a variety of groups and let $G \in \mathfrak{B}$ be an r-generator group which is nilpotent. Suppose that G has a subgroup H which is a free $\mathfrak{B}$-group of rank r. Then G is also a free $\mathfrak{B}$-group of rank r.*

Proof. Let $G = \text{gp}\,(g_1, g_2, \ldots, g_r)$ and let $\{h_1, h_2, \ldots, h_r\}$ be a set of free generators for H. The mapping $h_i \mapsto g_i$, $1 \leqq i \leqq r$, determines a homomorphism of H onto G with kernel N say. Suppose $N \neq 1$; we shall show that this leads to a contradiction.

Since $H/N \cong G$, H/N has a subgroup $H_2/N \cong H$ and a nontrivial normal subgroup N_2/N such that $(H_2/N)/(N_2/N) \cong G$. Continuing in this manner yields an ever-increasing sequence of subgroups of G:

$$1 \neq N < N_2 < N_3 < \cdots \leqq G.$$

But G, being a finitely generated nilpotent group, satisfies the maximum condition for subgroups. This contradiction implies $N = 1$, so that $H \cong G$ as was to be proved.

PROPOSITION 4.12. *Let n be a positive integer and let $\mathfrak{B}$ be the variety of groups consisting of all metabelian groups which are nilpotent of class at most n. Then $M'/\gamma_{n+1}M'$ is a free $\mathfrak{B}$-group of rank $|J|$.*

Proof. Let us denote $|J| = r$, and recall that $C(n+1) = \{1 + c \in C \mid v(c) \geqq n+1\}$. By Theorem 4.4,

$$\gamma_{n+1}M_0' = M_0' \cap C(n+1), \tag{4.13}$$

and, by the proof of Proposition 4.5,

$$\gamma_{n+1}M' \leqq M' \cap C(n+1). \tag{4.14}$$

Now

$$\frac{M_0'(M' \cap C(n+1))}{M' \cap C(n+1)} \cong \frac{M_0'}{M_0' \cap C(n+1)} \cong \frac{M_0'}{\gamma_{n+1}M_0'}, \tag{4.15}$$

which is a free $\mathfrak{B}$-group of rank r. Therefore, by Lemma 4.11, $M'/M' \cap C(n+1)$ is also a free $\mathfrak{B}$-group of rank r. Finally, from this fact, and from line (4.14) again, we have that there exist homomorphisms

$$M'/M' \cap C(n+1) \to M'/\gamma_{n+1}M' \to M'/M' \cap C(n+1),$$

whose composite is the identity on $M'/M' \cap C(n+1)$. Hence $M'/\gamma_{n+1}M'$ is also a free $\mathfrak{B}$-group of rank r. This completes the proof of the proposition.

PROPOSITION 4.13. *M' is a free metabelian group freely generated by the elements $x_i^{1/m}$, $i \in J$.*

Proof. The mapping $\psi: x_i \mapsto x_i^{1/m}$ determines an epimorphism $\psi: M_0' \to M'$; we claim that ψ is an isomorphism. For let $1 \neq g \in \ker \psi$; since M_0' is residually nilpotent (by Proposition 4.5, for example), there is an integer n such that $g \notin \gamma_{n+1} M_0'$. Now ψ induces an epimorphism

$$\hat{\psi}: M_0'/\gamma_{n+1}M_0' \to M'/\gamma_{n+1}M',$$

which, since both these groups are free $\mathfrak{B}$-groups of rank r and since such groups are Hopfian, must be an isomorphism. But $g\gamma_{n+1}M_0'$ is a nontrivial element in the kernel of ψ. This contradiction shows that $\ker \psi = 1$, so that $M_0' \cong M'$ as required.

4.7. The preceding proposition completes the proof that hypothesis (ii) of the Main Theorem is satisfied. We turn now to hypothesis (iii), that K, considered as a $\boldsymbol{Z}B$-module is torsion-free. The next two lemmas will be used in the proof of this fact.

Recall that $\bar{Z}$ is a free abelian $\mathscr{D}$-group freely π-generated by the elements $1+z_i$, $i \in I$. For convenience of notation, let us denote by $\{1+z_i \mid 1 \leqq i \leqq r\}$ an arbitrary finite subset of $\{1+z_i \mid i \in I\}$.

LEMMA 4.14. *Let $s \geqq 2$ be an integer and let $1+v_j$, $1 \leqq j \leqq s$, be distinct elements of $\bar{Z}$ given by*

$$1+v_j = (1+z_1)^{q_{1,j}}(1+z_2)^{q_{2,j}} \cdots (1+z_r)^{q_{r,j}}, \tag{4.16}$$

where $q_{i,j} \in \boldsymbol{Q}$, $1 \leqq i \leqq r$, $1 \leqq j \leqq s$. Let $n_1, n_2, \ldots, n_s$ be nonzero integers. Then

$$n_1 v_1 + n_2 v_2 + \cdots + n_s v_s \neq 0. \tag{4.17}$$

Proof. *Case* 1. If all the $q_{i,j}$ are (strictly) positive integers, each $1+v_j$ is determined by its leading term $z_1^{q_{1,j}} z_2^{q_{2,j}} \cdots z_r^{q_{r,j}}$; since the $1+v_j$ are all distinct, these leading terms are all distinct, so the result is immediate.

Case 2. Assume that $q_{i,j} \in \boldsymbol{Z}$ for all i, j. Without loss of generality, we may assume that all the v_j's are nonzero; for at most one of them could be zero; if there is one equal to zero, we simply omit it (noting that $s \geqq 2$), and the result (4.17) to be proved remains the same.

Now suppose that the result is false, that is, suppose that

$$n_1(1+v_1) + n_2(1+v_2) + \cdots + n_s(1+v_s) - (n_1 + n_2 + \cdots + n_s) = 0. \tag{4.18}$$

From the form (4.16) of each $1+v_j$, where here every $q_{i,j} \in \boldsymbol{Z}$, it is clear that there is a large positive integer N such that if $1+v = (1+z_1)^N (1+z_2)^N \cdots (1+z_r)^N$, then, for each j, $1 \leqq j \leqq s$, the product $(1+v)(1+v_j) = 1+v_j'$, when expressed in the form (4.16) will involve only positive integer $q_{i,j}$'s. Further, the $1+v_j'$ are all different from each other because the $1+v_j$ were; and the $1+v_j'$ are all different from $1+v$ because no v_j was zero. Multiplying across (4.18) by $1+v$ gives

$$n_1(1+v_1') + n_2(1+v_2') + \cdots + n_s(1+v_s') - (n_1 + n_2 + \cdots + n_s)(1+v) = 0.$$

If $n_1+n_2+\cdots+n_s=0$, this immediately contradicts Case 1. If $n_1+n_2+\cdots+n_s\neq 0$, it also contradicts Case 1, because the $1+v_j'$'s are not only different from one another but also from $1+v$.

Case 3 (*General Case*). Here the $q_{i,j}$ are arbitrary rational numbers. Choose a large positive integer N such that $Nq_{i,j}\in \boldsymbol{Z}$, $1\leqq i\leqq r$, $1\leqq j\leqq s$, and define an endomorphism ν of the power series algebra A by specifying:

$$\nu: y_i \mapsto 0, \qquad i\in I,$$
$$\nu: z_i \mapsto (1+z_i)^N-1, \qquad i\in I.$$

This does define an endomorphism of A because the images of the y_i and z_i, $i\in I$, satisfy the relations (4.1).

Now assume that the result (4.17) is false so that equation (4.18) holds. Applying the endomorphism ν to equation (4.18) gives

$$(4.19)\qquad n_1(1+v_1'')+n_2(1+v_2'')+\cdots+n_s(1+v_s'')-(n_1+n_2+\cdots+n_s) = 0,$$

where

$$1+v_j'' = (1+z_1)^{Nq_{1,j}}(1+z_2)^{Nq_{2,j}}\cdots(1+z_r)^{Nq_{r,j}}, \qquad 1\leqq j\leqq s.$$

Since these exponents $Nq_{i,j}$ are all integers and since the $1+v_j''$ are still all distinct (because $1+v_j''=(1+v_j)^N$), equation (4.19) is impossible by Case 2. This completes the proof of the lemma.

LEMMA 4.15. *Let $1+c=1+c_{yy}+c_{zz}+c_{yz}\in C$, let $1+k_{yz}\in K$ and let n be a positive integer. Then*

(i) $(1+c)^{-1}(1+k_{yz})(1+c)=(1+c_{zz})^{-1}(1+k_{yz})(1+c_{zz})$,

(ii) $(1+c)^{-1}(1+k_{yz})^n(1+c)=1+nk_{yz}(1+c_{zz})$.

Proof. Both of these are simple calculations, the second using induction on n.

PROPOSITION 4.16. *K, considered as a $\boldsymbol{Z}B$-module, is torsion-free.*

Proof. Let $n_1\beta_1+n_2\beta_2+\cdots+n_s\beta_s\in \boldsymbol{Z}B$ be any nonzero element, where $n_j\neq 0$, $1\leqq j\leqq s$, and the β_j's are all distinct elements of B. Let $1\neq 1+k_{yz}\in K$. We must prove that

$$(4.20)\qquad (1+k_{yz})^{n_1\beta_1+n_2\beta_2+\cdots+n_s\beta_s} \neq 1.$$

For each j, $1\leqq j\leqq s$, choose an element $(1+v_j)(1+u_j)(1+c_{yz}^*)$ in D which projects, via η, onto β_j (cf. line (4.12)), where $1+u_j\in \overline{Y}$, $1+v_j\in\overline{Z}$. As $1+u_j\leftrightarrow 1+v_j$, the elements $1+v_j$, $1\leqq j\leqq s$, are all different from one another, because the β_j's are all distinct. Therefore, the conditions of Lemma 4.14 are satisfied, so that

$$(4.21)\qquad v = n_1v_1+n_2v_2+\cdots+n_sv_s \neq 0.$$

From the definition of the action of $\boldsymbol{Z}B$ on K, and using Lemma 4.15(i), we see that equivalent to equation (4.20) is

$$(4.22)\qquad (1+v_1)^{-1}(1+k_{yz})^{n_1}(1+v_1)\cdots(1+v_s)^{-1}(1+k_{yz})^{n_s}(1+v_s) \neq 1.$$

Assume that this result is false; that is, assume

$$(4.23)\qquad (1+v_1)^{-1}(1+k_{yz})^{n_1}(1+v_1)\cdots(1+v_s)^{-1}(1+k_{yz})^{n_s}(1+v_s) = 1.$$

Using Lemma 4.15(ii) and denoting $n=n_1+n_2+\cdots+n_s$, this becomes

$$1+n_1k_{yz}(1+v_1)+n_2k_{yz}(1+v_2)+\cdots+n_sk_{yz}(1+v_s) = 1,$$

that is

$$(4.24)\qquad k_{yz}(n+v) = 0.$$

Now from lines (4.1) and (4.4), we see that k_{yz} is of the form

$$k_{yz} = \sum_{i\in I} y_i w_i,$$

where each w_i involves only z's (no y's). On substituting this expression for k_{yz} in equation (4.24), it follows at once that either $k_{yz}=0$ or $v=0$. But these are both impossible: $k_{yz}=0$ contradicts the original assumption that $1+k_{yz}\neq 1$, while $v=0$ contradicts equation (4.21).

Thus assumption (4.23) leads to a contradiction, so (4.23) is false and (4.22) is established. This completes the proof of the proposition.

4.8. The verification that the hypotheses of the Main Theorem are satisfied by the situation within the power series algebra is now complete. By the Main Theorem, we conclude that $F=\mathcal{D}\text{-gp}\,(x_i \mid i\in I)$ is a free metabelian $\mathcal{D}$-group freely π-generated by the elements x_i, $i\in I$, and this completes the proof of Theorem 4.1.

The most important consequence of this representation is the following theorem; it follows at once from Proposition 4.5.

THEOREM 4.17. *Free metabelian $\mathcal{D}$-groups are residually torsion-free nilpotent.*

5. **Matrix representation.**

5.1. Our second representation of free metabelian $\mathcal{D}$-groups is in terms of two-by-two matrices. It is a natural analog of a representation of free metabelian groups by two-by-two matrices due to W. Magnus [16].

5.2. Let

$$\begin{aligned} I &= \text{a well-ordered index set,}\\ A &= \text{free abelian group on the set } \{\alpha_i \mid i\in I\} \text{ (written multiplicatively),}\\ ZA &= \text{group-ring of } A,\\ W &= \text{free } ZA\text{-module on the set } \{e_i \mid i\in I\}. \end{aligned}$$

It is easy to check that the set of all matrices of the form $\left(\begin{smallmatrix}\alpha & 0\\ w & 1\end{smallmatrix}\right)$, $\alpha\in A$, $w\in W$, is a metabelian group under matrix multiplication. W. Magnus [16] proved that the subgroup

$$(5.1)\qquad M_0 = \text{gp}\left(\begin{pmatrix}\alpha_i & 0\\ e_i & 1\end{pmatrix} \;\middle|\; i\in I\right)$$

is a free metabelian group freely generated by these matrices

$$\begin{pmatrix} \alpha_i & 0 \\ e_i & 1 \end{pmatrix}, \qquad i \in I.$$

(Magnus' result is actually more general than this; he represents any group $X/[R, R]$, where X is a free group and R is an arbitrary normal subgroup; we have described the special case in which $R = \gamma_2 X$.)

We would like to obtain a similar representation of free metabelian $\mathscr{D}$-groups. In particular, we must be able to extract mth roots for all π-numbers m; if we try extracting an mth root of one of these matrices, we find, working purely formally, that the matrix which when raised to the mth power gives $\left(\begin{smallmatrix} \alpha & 0 \\ w & 1 \end{smallmatrix}\right)$, where $\alpha \neq 1$, is

$$\begin{pmatrix} \alpha^{1/m} & 0 \\ \left(\dfrac{\alpha^{1/m}-1}{\alpha-1}\right) w & 1 \end{pmatrix}.$$

Thus in order to be able to extract mth roots, we need to be able to extract them in the upper left-hand corner, and we must also be able to "multiply" by elements of the form $((\alpha^{1/m}-1)/(\alpha-1))$ in the module in the lower left-hand corner. We are therefore led to make the following definitions:

$$\begin{aligned} B &= \text{free abelian } \mathscr{D}\text{-group on the set } \{\alpha_i \mid i \in I\} \text{ (written multiplicatively)}, \\ ZB &= \text{group-ring of } B. \end{aligned}$$

Now B is a torsion-free abelian group, so by a theorem of F. W. Levi [13], B is an ordered group; from this it follows at once that ZB is an integral domain, and we may therefore define

$$\begin{aligned} F &= \text{quotient field of } ZB, \\ V &= \text{vector space over } F \text{ on the basis } \{e_i \mid i \in I\}. \end{aligned}$$

Preparing for an application of the Main Theorem, let

$$D = \left\{ \begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix} \mid \beta \in B, v \in V \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \mid v \in V \right\},$$

$$x_i = \begin{pmatrix} \alpha_i & 0 \\ e_i & 1 \end{pmatrix}, \qquad i \in I.$$

PROPOSITION 5.1. *D is a metabelian $\mathscr{D}$-group under matrix multiplication:*

$$\begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta' & 0 \\ v' & 1 \end{pmatrix} = \begin{pmatrix} \beta\beta' & 0 \\ \beta' v + v' & 1 \end{pmatrix}, \qquad \beta, \beta' \in B, \quad v, v' \in V.$$

K is an abelian ideal of F, and D/K is a free abelian $\mathscr{D}$-group freely π-generated by the elements $x_i K$, $i \in I$.

Proof. The proof requires only simple verifications and is omitted. The last part of the proposition may be proved by considering the epimorphism $\eta\colon D \to B$ given by

$$\eta\colon \begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix} \mapsto \beta.$$

As usual, we denote $F = \mathscr{D}\text{-gp}\,(x_i \mid i \in I)$. Our matrix representation will be established once we prove that hypotheses (ii) and (iii) of the Main Theorem are satisfied; for then we can conclude that F is a free metabelian $\mathscr{D}$-group freely generated by the matrices x_i, $i \in I$.

5.3. The verification of hypothesis (iii) of the Main Theorem is very easy, so we shall dispose of it first. First of all, we record some formulas which are useful to have at hand when computing inside the $\mathscr{D}$-group D.

LEMMA 5.2. *Let $\alpha, \beta \in B$, $u, v \in V$, $q \in \Gamma_\pi$ and $S \in ZB$. Then*

(i)
$$\begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix}^q = \begin{pmatrix} \beta^q & 0 \\ \left(\dfrac{\beta^q - 1}{\beta - 1}\right) v & 1 \end{pmatrix} \quad \text{if } \beta \neq 1,$$
$$= \begin{pmatrix} 1 & 0 \\ qv & 1 \end{pmatrix} \qquad \text{if } \beta = 1;$$

(ii)
$$\left[\begin{pmatrix} \alpha & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ (\beta - 1)u - (\alpha - 1)v & 1 \end{pmatrix};$$

(iii)
$$\begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta u & 1 \end{pmatrix};$$

(iv)
$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^S = \begin{pmatrix} 1 & 0 \\ Su & 1 \end{pmatrix}.$$

Proof. The proofs of formulas (i), (ii) and (iii) are straightforward calculations and are omitted. Note that in (i), q is an arbitrary number in Γ_π and so may be positive or negative, integral or fractional.

To prove (iv), let $S = n_1\beta_1 + n_2\beta_2 + \cdots + n_t\beta_t$, where $0 \neq n_j \in Z$, $1 \leqq j \leqq t$, and $\beta_1, \beta_2, \ldots, \beta_t$ are distinct elements of B. For $1 \leqq j \leqq t$, let $\left(\begin{smallmatrix} \beta_j & 0 \\ * & 1 \end{smallmatrix}\right)$ be any matrix in D which projects via η onto β_j. Then

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^S &= \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^{n_1\beta_1 + n_2\beta_2 + \cdots + n_t\beta_t} \\ &= \begin{pmatrix} \beta_1 & 0 \\ * & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^{n_1} \begin{pmatrix} \beta_1 & 0 \\ * & 1 \end{pmatrix} \cdots \begin{pmatrix} \beta_t & 0 \\ * & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^{n_t} \begin{pmatrix} \beta_t & 0 \\ * & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ Su & 1 \end{pmatrix} \qquad \text{(using (i) and (iii))}. \end{aligned}$$

This completes the proof of the lemma.

PROPOSITION 5.3. *K, considered as a ZB-module, is torsion-free.*

Proof. Let $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\neq\left(\begin{smallmatrix}1&0\\u&1\end{smallmatrix}\right)\in K$ and let $0\neq S\in ZB$; then using Lemma 5.2(iv), we have

$$\begin{pmatrix}1&0\\u&1\end{pmatrix}^S = \begin{pmatrix}1&0\\Su&1\end{pmatrix} \neq \begin{pmatrix}1&0\\0&1\end{pmatrix},$$

for clearly $Su\neq 0$ because $S\neq 0$ and $u\neq 0$. This completes the proof of the proposition.

5.4. We proceed to the proof that hypothesis (ii) of the Main Theorem is satisfied:

PROPOSITION 5.4. *Let m be a π-number and let $M=\mathrm{gp}\,(x_i^{1/m} \mid i\in I)$; then M is a free metabelian group freely generated by the matrices $x_i^{1/m}$, $i\in I$.*

Proof. Let $M_0=\mathrm{gp}\,(x_i \mid i\in I)$; it is immediate from Magnus' representation (cf. line (5.1)) that M_0 is a free metabelian group freely generated by the elements x_i, $i\in I$. An epimorphism $\psi\colon M_0\to M$ is determined by the mapping

$$\psi\colon x_i\mapsto x_i^{1/m}, \qquad i\in I;$$

the proposition will be established if we can prove that ψ is an isomorphism.

Let $x=\left(\begin{smallmatrix}\beta&0\\v&1\end{smallmatrix}\right)\in\ker\psi$; we must show that $\beta=1$ and $v=0$. Now x can be expressed as a group-word in the elements x_i, $i\in I$, say

$$x = x_{i_1}^{l_1}x_{i_2}^{l_2}\cdots x_{i_s}^{l_s}, \qquad l_j\in Z,\ i_j\in I, \quad 1\leqq j\leqq s;$$

then

$$x\psi = x_{i_1}^{l_1/m}x_{i_2}^{l_2/m}\cdots x_{i_s}^{l_s/m}.$$

Computing the matrices x and $x\psi$ from these expressions yields

$$x = \begin{pmatrix}\alpha_{i_1}^{l_1}\alpha_{i_2}^{l_2}\cdots\alpha_{i_s}^{l_s} & 0\\ v & 1\end{pmatrix},$$

$$x\psi = \begin{pmatrix}\alpha_{i_1}^{l_1/m}\alpha_{i_2}^{l_2/m}\cdots\alpha_{i_s}^{l_s/m} & 0\\ v' & 1\end{pmatrix},$$

where

$$\begin{aligned} v = {} & \frac{\alpha_{i_1}^{l_1}-1}{\alpha_{i_1}-1}\,\alpha_{i_2}^{l_2}\alpha_{i_3}^{l_3}\cdots\alpha_{i_s}^{l_s}e_{i_1}+\frac{\alpha_{i_2}^{l_2}-1}{\alpha_{i_2}-1}\,\alpha_{i_3}^{l_3}\cdots\alpha_{i_s}^{l_s}e_{i_2}\\ & +\cdots+\frac{\alpha_{i_{s-1}}^{l_{s-1}}-1}{\alpha_{i_{s-1}}-1}\,\alpha_{i_s}^{l_s}e_{i_{s-1}}+\frac{\alpha_{i_s}^{l_s}-1}{\alpha_{i_s}-1}\,e_{i_s}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} v' = {} & \frac{\alpha_{i_1}^{l_1/m}-1}{\alpha_{i_1}-1}\,\alpha_{i_2}^{l_2/m}\alpha_{i_3}^{l_3/m}\cdots\alpha_{i_s}^{l_s/m}e_{i_1}+\frac{\alpha_{i_2}^{l_2/m}-1}{\alpha_{i_2}-1}\,\alpha_{i_3}^{l_3/m}\cdots\alpha_{i_s}^{l_s/m}e_{i_2}\\ & +\cdots+\frac{\alpha_{i_{s-1}}^{l_{s-1}/m}-1}{\alpha_{i_{s-1}}-1}\,\alpha_{i_s}^{l_s/m}e_{i_{s-1}}+\frac{\alpha_{i_s}^{l_s/m}-1}{\alpha_{i_s}-1}\,e_{i_s}. \end{aligned} \tag{5.3}$$

Since $x \in \ker \psi$, $\alpha_{i_1}^{l_1/m}\alpha_{i_2}^{l_2/m}\cdots\alpha_{i_s}^{l_s/m}=1$ and $v'=0$, so we have at once that $\beta = \alpha_{i_1}^{l_1}\alpha_{i_2}^{l_2}\cdots\alpha_{i_s}^{l_s}=1$.

Let e_i be any basis vector which appears in the above expressions for v and v', and let $f_i, f_i' \in \boldsymbol{F}$ be its coefficients in v, v' respectively. We see from equations (5.2) and (5.3) that, with obvious notation, if

$$f_i = \frac{\alpha_i^{n_{1,1}}-1}{\alpha_i-1}\,\alpha_{j_{1,2}}^{n_{1,2}}\alpha_{j_{1,3}}^{n_{1,3}}\cdots\alpha_{j_{1,t_1}}^{n_{1,t_1}}+\frac{\alpha_i^{n_{2,1}}-1}{\alpha_i-1}\,\alpha_{j_{2,2}}^{n_{2,2}}\cdots\alpha_{j_{2,t_2}}^{n_{2,t_2}}+\cdots,$$

then

$$f_i' = \frac{\alpha_i^{n_{1,1}/m}-1}{\alpha_i-1}\,\alpha_{j_{1,2}}^{n_{1,2}/m}\alpha_{j_{1,3}}^{n_{1,3}/m}\cdots\alpha_{j_{1,t_1}}^{n_{1,t_1}/m}+\frac{\alpha_i^{n_{2,1}/m}-1}{\alpha_i-1}\,\alpha_{j_{2,2}}^{n_{2,2}/m}\cdots\alpha_{j_{2,t_2}}^{n_{2,t_2}/m}+\cdots;$$

therefore

$$(5.4)\quad (\alpha_i-1)f_i = (\alpha_i^{n_{1,1}}-1)\alpha_{j_{1,2}}^{n_{1,2}}\alpha_{j_{1,3}}^{n_{1,3}}\cdots\alpha_{j_{1,t_1}}^{n_{1,t_1}}+(\alpha_i^{n_{2,1}}-1)\alpha_{j_{2,2}}^{n_{2,2}}\cdots\alpha_{j_{2,t_2}}^{n_{2,t_2}}+\cdots,$$

$$(5.5)\quad \begin{aligned}(\alpha_i-1)f_i' = {}&(\alpha_i^{n_{1,1}/m}-1)\alpha_{j_{1,2}}^{n_{1,2}/m}\alpha_{j_{1,3}}^{n_{1,3}/m}\cdots\alpha_{j_{1,t_1}}^{n_{1,t_1}/m}\\ &+(\alpha_i^{n_{2,1}/m}-1)\alpha_{j_{2,2}}^{n_{2,2}/m}\cdots\alpha_{j_{2,t_2}}^{n_{2,t_2}/m}+\cdots.\end{aligned}$$

Certainly since $m \in [\pi]$, the endomorphism θ of B induced by the mapping

$$\theta\colon \alpha_i \mapsto \alpha_i^{1/m}, \qquad i \in I,$$

is an automorphism. Therefore, the endomorphism θ of ZB, given by

$$(n_1\beta_1+n_2\beta_2+\cdots+n_u\beta_u)\theta = n_1(\beta_1\theta)+n_2(\beta_2\theta)+\cdots+n_u(\beta_u\theta)$$

is also an automorphism. By inspection of equations (5.4) and (5.5), we see that $((\alpha_i-1)f_i)\theta=(\alpha_i-1)f_i'$. But $f_i'=0$, because $v'=0$, so we have $((\alpha-1)f_i)\theta=0$. As θ is an automorphism of ZB, this implies $(\alpha_i-1)f_i=0$, so that $f_i=0$. Since this holds for all $i\in I$, we finally have $v=0$ as required. This completes the proof of the proposition.

5.5. All the hypotheses of the Main Theorem have now been verified, and our matrix representation is established. The power series algebra representation had the advantage of immediately implying that free metabelian $\mathscr{D}$-groups are residually torsion-free nilpotent. The matrix representation is perhaps even more useful for deriving properties of free metabelian $\mathscr{D}$-groups as we shall see in [12]. The advantage here is that we can actually write down, in closed form, an expression for the general matrix belonging to the $\mathscr{D}$-group F. Our next task is to determine this closed form.

5.6. For one of the applications in [12], it is convenient to be a little more general; we introduce an arbitrary set of vectors $v_i \in V$, $i \in I$, and denote

$$x_i' = \begin{pmatrix}\alpha_i & 0\\ v_i & 1\end{pmatrix}, \quad i\in I, \qquad F' = \mathscr{D}\text{-gp}\,(x_i' \mid i \in I).$$

Throughout the following work, the prime (′) indicates that we are dealing with this more general situation; thus to omit the primes is equivalent to setting $v_i=e_i$, $i\in I$. We shall determine the most general form of an element of F': we shall prove, in Theorem 5.6, that F' is equal to the set of matrices $\tilde{F}'$ defined at line (5.10); the form of elements of F will be given by the special case in which $v_i=e_i$, $i\in I$.

Our method is directly motivated by the adjunction-of-a-root step in the proof of the Main Theorem (cf. §3.4). Another contributing factor is the result of Proposition 3.5(ii) that $G\cap K=\bigcup_{n=0}^{\infty}\gamma_2 M_n$, together with the fact that if $M=\operatorname{gp}(y_i \mid i\in I)$ is a metabelian group, then $\gamma_2 M$ is generated as a $Z(M/\gamma_2 M)$-module, by the set of simple commutators $\{[y_i, y_j] \mid i,j\in I, i<j\}$.

We need some rather complicated definitions. First, let E be the following subset of the ring ZB:

$$E=\{p\mid p\in\pi\}\cup\left\{\left(\frac{\beta^p-1}{\beta-1}\right)\,\Big|\, 1\neq\beta\in B, p\in\pi\right\}.$$

Define L to be the multiplicative subsemigroup (containing 1) of ZB generated by the set E; a typical element of L is

$$l=p_1p_2\cdots p_s\left(\frac{\beta_1^{p_1'}-1}{\beta_1-1}\right)\left(\frac{\beta_2^{p_2'}-1}{\beta_2-1}\right)\cdots\left(\frac{\beta_t^{p_t'}-1}{\beta_t-1}\right),$$

where $p_j\in\pi$, $p_k'\in\pi$, $\beta_k\in B$, $1\leqq j\leqq s$, $1\leqq k\leqq t$. Now E may be considered as a subset of the field F, so it is reasonable to denote the subset of F consisting of all inverses of elements of E by

$$E^{-1}=\left\{\frac{1}{p}\,\Big|\, p\in\pi\right\}\cup\left\{\left(\frac{\beta-1}{\beta^p-1}\right)\,\Big|\, 1\neq\beta\in B, p\in\pi\right\}.$$

The field F may be considered as an (associative) ZB-algebra. Define A to be the sub-ZB-algebra (containing 1) of F generated by the set E^{-1}; a typical element of A is easily seen to be expressible in the form

$$\text{(5.6)}\quad a=\sum_{j=1}^{s}S_j\left\{\frac{1}{p_1^{m(j,1)}p_2^{m(j,2)}\cdots p_t^{m(j,t)}}\left(\frac{\beta_1-1}{\beta_1^{p_1'}-1}\right)^{n(j,1)}\left(\frac{\beta_2-1}{\beta_2^{p_2'}-1}\right)^{n(j,2)}\cdots\left(\frac{\beta_u-1}{\beta_u^{p_u'}-1}\right)^{n(j,u)}\right\},$$

where $S_j\in ZB$, $p_k\in\pi$, $p_l'\in\pi$, $0\leqq m(j,k)\in Z$, $0\leqq n(j,l)\in Z$, $1\leqq j\leqq s$, $1\leqq k\leqq t$, $1\leqq l\leqq u$. In particular, A is a subring of F.

Next we introduce a subset W' of the vector space V:

$$W'=\{w_{i,j}^{(n)}\mid i,j\in I, i<j, n\in[\pi]\},$$

where $w_{i,j}^{(n)}$ is defined by

$$[(x_i')^{1/n}, (x_j')^{1/n}]=\begin{pmatrix}1 & 0\\ w_{i,j}^{(n)} & 1\end{pmatrix}.$$

Using the formulas of Lemma 5.2, we obtain explicitly

$$w^{(n)}_{i,j} = \frac{(\alpha_i^{1/n}-1)(\alpha_j^{1/n}-1)}{(\alpha_i-1)}v_i - \frac{(\alpha_i^{1/n}-1)(\alpha_j^{1/n}-1)}{(\alpha_j-1)}v_j, \qquad i,j\in I,\quad i<j,\quad n\in[\pi].$$

Now V is a vector space over F and A is a subring of F; so V may be considered as an A-module. Define U' to be the sub-A-module of V generated by the set W', a typical element of U' is

$$u = a_1 w^{(n_1)}_{i_1,j_1} + a_2 w^{(n_2)}_{i_2,j_2} + \cdots + a_t w^{(n_t)}_{i_t,j_t}, \qquad a_l \in A,\quad 1 \leqq l \leqq t. \tag{5.7}$$

V may also be considered as a ZB-module. Define U_0' to be the sub-ZB-module of V generated by the set W'; a typical element of U_0' is

$$u_0 = S_1 w^{(n_1)}_{i_1,j_1} + S_2 w^{(n_2)}_{i_2,j_2} + \cdots + S_t w^{(n_t)}_{i_t,j_t}, \qquad S_l \in ZB,\quad 1 \leqq l \leqq t. \tag{5.8}$$

The reader may begin to see some sense in these definitions now. Before continuing with the definitions we indicate, in Lemma 5.5(iii), what the sub-ZB-module U_0' of V represents. Referring to the discussion in §3.2 and the proof of the Main Theorem in §3.4, and recalling the purpose of the primes, let

$$M_n' = \mathrm{gp}\,((x_i')^{1/n} \mid i\in I), \qquad n\in[\pi],$$

$$G_0' = \bigcup_{n\in[\pi]} M_n' = \mathrm{gp}\,((x_i')^{1/n} \mid i\in I,\, n\in[\pi]).$$

LEMMA 5.5. (i) *For* $n\in[\pi]$,

$$\gamma_2 M_n' = M_n' \cap K = \left\{\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix} \in M_n'\right\}.$$

(ii) *If* $\left(\begin{smallmatrix}1 & 0\\ v & 1\end{smallmatrix}\right) \in G_0'$, *then* $v\in U_0'$.

(iii) $\gamma_2 G_0' = \{\left(\begin{smallmatrix}1 & 0\\ u_0 & 1\end{smallmatrix}\right) \mid u_0\in U_0'\}$.

Proof. (i) The projection $\eta\colon \left(\begin{smallmatrix}\beta & 0\\ v & 1\end{smallmatrix}\right) \mapsto \beta$ maps M_n' onto a free abelian group freely generated by $\alpha_i^{1/n}$, $i\in I$, which shows that $M_n'/(M_n'\cap K)$ is free abelian freely generated by $(x_i')^{1/n}(M_n'\cap K)$, $i\in I$. Therefore the homomorphism $M_n'/\gamma_2 M_n' \to M_n'/(M_n'\cap K)$, induced by the identity on M_n', has an inverse, so that $\gamma_2 M_n' = M_n'\cap K$ as required.

(ii) If $\left(\begin{smallmatrix}1 & 0\\ v & 1\end{smallmatrix}\right)\in G_0'$, clearly $\left(\begin{smallmatrix}1 & 0\\ v & 1\end{smallmatrix}\right)\in M_n'$ for some $n\in[\pi]$, therefore, by part (i), $\left(\begin{smallmatrix}1 & 0\\ v & 1\end{smallmatrix}\right)\in\gamma_2 M_n'$. But $\gamma_2 M_n'$ is generated, as a $Z(M_n'/\gamma_2 M_n')$-module, by the matrices

$$[(x_i')^{1/n}, (x_j')^{1/n}] = \begin{pmatrix}1 & 0\\ w^{(n)}_{i,j} & 1\end{pmatrix}, \qquad i,j\in I,\quad i<j.$$

Therefore, using Lemma 5.2(iv), v is contained in the ZB-module generated by the $w^{(n)}_{i,j}$, that is $v\in U_0'$.

(iii) The inclusion $\leqq$ follows at once from part (ii). The reverse inclusion $\geqq$ follows because $\gamma_2 G_0'$ is a ZB-module which contains all the ZB-module generators

$$\begin{pmatrix}1 & 0\\ w^{(n)}_{i,j} & 1\end{pmatrix}$$

of the ZB-module

$$\left\{\begin{pmatrix} 1 & 0 \\ u_0 & 1 \end{pmatrix} \mid u_0 \in U_0'\right\}.$$

Further, as we shall shortly prove, the A-module U' consists of precisely the set of vectors needed to describe the commutator ideal $\gamma_2^\pi F'$; that is

$$\gamma_2^\pi F' = \left\{\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mid u \in U'\right\}.$$

Returning now to our definitions, with each element $\beta \in B$ we associate a vector $v_\beta \in V$ as follows: β can be expressed uniquely in the form $\beta = \alpha_{i_1}^{q_1}\alpha_{i_2}^{q_2}\cdots\alpha_{i_r}^{q_r}$, where $q_l \in \Gamma_\pi$, $i_l \in I$, $1 \leqq l \leqq r$, and $i_1 < i_2 < \cdots < i_r$; the vector $v_\beta \in V$ is then defined by the equation

$$\begin{pmatrix} \alpha_{i_1} & 0 \\ v_{i_1} & 1 \end{pmatrix}^{q_1}\begin{pmatrix} \alpha_{i_2} & 0 \\ v_{i_2} & 1 \end{pmatrix}^{q_2}\cdots\begin{pmatrix} \alpha_{i_r} & 0 \\ v_{i_r} & 1 \end{pmatrix}^{q_r} = \begin{pmatrix} \beta & 0 \\ v_\beta & 1 \end{pmatrix}.$$

Explicitly, using the formulas of Lemma 5.2,

$$v_\beta = \frac{\alpha_{i_1}^{q_1}-1}{\alpha_{i_1}-1}\alpha_{i_2}^{q_2}\alpha_{i_3}^{q_3}\cdots\alpha_{i_r}^{q_r}v_{i_1} + \frac{\alpha_{i_2}^{q_2}-1}{\alpha_{i_2}-1}\alpha_{i_3}^{q_3}\cdots\alpha_{i_r}^{q_r}v_{i_2} + \cdots + \frac{\alpha_{i_{r-1}}^{q_{r-1}}-1}{\alpha_{i_{r-1}}-1}\alpha_{i_r}^{q_r}v_{i_{r-1}} + \frac{\alpha_{i_r}^{q_r}-1}{\alpha_{i_r}-1}v_{i_r}.$$

Note that, for every $\beta \in B$,

$$\begin{pmatrix} \beta & 0 \\ v_\beta & 1 \end{pmatrix} \in G_0'. \tag{5.9}$$

Finally, we define a subset $\tilde{F}'$ of the matrix group D by

$$\tilde{F}' = \left\{\begin{pmatrix} \beta & 0 \\ v_\beta + u & 1 \end{pmatrix} \mid \beta \in B, \mu \in U'\right\}. \tag{5.10}$$

Recall that

$$F' = \mathscr{D}\text{-gp}\left(\begin{pmatrix} \alpha_i & 0 \\ v_i & 1 \end{pmatrix} \mid i \in I\right).$$

The statement which gives our explicit representation of the elements of F' is

THEOREM 5.6. $F' = \tilde{F}'$.

5.7. In this subsection, we give the proof of Theorem 5.6. Clearly for $i \in I$, $v_{\alpha_i} = v_i$; therefore the π-generators of F' belong to $\tilde{F}'$. So to establish that $F' \leqq \tilde{F}'$, it suffices to prove that F' is a $\mathscr{D}$-group. Toward this end, we prove the following lemma.

LEMMA 5.7. *Let* $\alpha, \beta \in B$ *and let* $p \in \pi$, *then*

(i) $\beta v_\alpha + v_\beta - v_{\alpha\beta} \in U_0'$,

(ii) $v_{\alpha^{-1}} + \alpha^{-1}v_\alpha \in U_0'$,

(iii) $((\alpha^{1/p}-1)/(\alpha-1))v_\alpha - v_{\alpha^{1/p}} \in U'$ *if* $\alpha \neq 1$.

Proof. (i) By the remark at line (5.9),

$$\begin{pmatrix}\alpha\beta & 0\\ v_{\alpha\beta} & 1\end{pmatrix}^{-1}\begin{pmatrix}\alpha & 0\\ v_\alpha & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ v_\beta & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ \beta v_\alpha+v_\beta-v_{\alpha\beta} & 1\end{pmatrix} \in G_0';$$

hence, by Lemma 5.5(ii), $\beta v_\alpha+v_\beta-v_{\alpha\beta}\in U_0'$.

(ii) Similarly,

$$\begin{pmatrix}\alpha & 0\\ v_\alpha & 1\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ v_{\alpha^{-1}} & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ v_{\alpha^{-1}}+\alpha^{-1}v_\alpha & 1\end{pmatrix} \in G_0';$$

hence, by Lemma 5.5(ii), $v_{\alpha^{-1}}+\alpha^{-1}v_\alpha\in U_0'$.

(iii) Similarly,

$$\begin{pmatrix}\alpha^{1/p} & 0\\ v_{\alpha^{1/p}} & 1\end{pmatrix}^{-p}\begin{pmatrix}\alpha & 0\\ v_\alpha & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ v_\alpha-\left(\dfrac{\alpha-1}{\alpha^{1/p}-1}\right)v_{\alpha^{1/p}} & 1\end{pmatrix} \in G_0';$$

hence, by Lemma 5.5(ii),

$$v_\alpha-\left(\frac{\alpha-1}{\alpha^{1/p}-1}\right)v_{\alpha^{1/p}}\in U_0'\subseteq U'.$$

But U' is an A-module, and $(\alpha^{1/p}-1)/(\alpha-1)\in A$, since $\alpha\neq 1$; therefore

$$\frac{\alpha^{1/p}-1}{\alpha-1}\left(v_\alpha-\frac{\alpha-1}{\alpha^{1/p}-1}\,v_{\alpha^{1/p}}\right)\in U',$$

as was to be proved.

PROPOSITION 5.8. *$\tilde{F}'$ is a $\mathscr{D}$-group.*

Proof. Let $\alpha,\beta\in B$, $u', u''\in U'$. First, $\tilde{F}'$ is closed under multiplication:

$$\begin{pmatrix}\alpha & 0\\ v_\alpha+u' & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ v_\beta+u'' & 1\end{pmatrix} = \begin{pmatrix}\alpha\beta & 0\\ v_{\alpha\beta}+u_1 & 1\end{pmatrix},$$

where $u_1=(\beta v_\alpha+v_\beta-v_{\alpha\beta})+\beta u'+u''\in U'$, using Lemma 5.7(i) and the fact that U' is a ZB-module. Secondly, $\tilde{F}'$ is closed with respect to inverses

$$\begin{pmatrix}\alpha & 0\\ v_\alpha+u' & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha^{-1} & 0\\ v_{\alpha^{-1}}+u_2 & 1\end{pmatrix},$$

where $u_2=-(v_{\alpha^{-1}}+\alpha^{-1}v_\alpha)-\alpha^{-1}u'\in U'$, using Lemma 5.7(ii) and the fact that U' is a ZB-module. Finally, $\tilde{F}'$ is closed under extraction of pth roots for primes $p\in\pi$. For $\alpha\neq 1$, we have

$$\begin{pmatrix}\alpha & 0\\ v_\alpha+u' & 1\end{pmatrix}^{1/p} = \begin{pmatrix}\alpha^{1/p} & 0\\ v_{\alpha^{1/p}}+u_3 & 1\end{pmatrix},$$

where

$$u_3 = \left(\frac{\alpha^{1/p}-1}{\alpha-1}\right)v_\alpha-v_{\alpha^{1/p}}+\left(\frac{\alpha^{1/p}-1}{\alpha-1}\right)u'\in U',$$

using Lemma 5.7(iii) and the fact that U' is an A-module. For $\alpha=1$, we have

$$\begin{pmatrix} 1 & 0 \\ u' & 1 \end{pmatrix}^{1/p} = \begin{pmatrix} 1 & 0 \\ u'/p & 1 \end{pmatrix},$$

and $u'/p \in U'$, because $1/p \in A$ and U' is an A-module. This completes the proof of the proposition.

As observed before, the last proposition implies that $F' \leqq \tilde{F}'$; we now establish the reverse inclusion.

PROPOSITION 5.9. $F' \geqq \tilde{F}'$.

Proof. First, from the equality

$$\begin{pmatrix} \beta & 0 \\ v_\beta & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ v_\beta + u & 1 \end{pmatrix},$$

where $\left(\begin{smallmatrix} \beta & 0 \\ v_\beta & 1 \end{smallmatrix}\right) \in G_0' \leqq F'$, we see that, to prove the proposition, it suffices to prove that $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in F'$, for all elements $u \in U'$. Recall that L is the multiplicative sub-semigroup of $\boldsymbol{ZB}$ generated by the set

$$E = \{p \mid p \in \pi\} \cup \left\{\left(\frac{\beta^p - 1}{\beta - 1}\right) \mid 1 \neq \beta \in B, p \in \pi\right\}.$$

We want to have some measure of the "length" of elements of L. If $l \in L$ and t is a positive integer, we define $|l| \leqq t$ to mean that l can be expressed as a product of t elements of the set E (each being counted as many times as it appears). For example, if

$$l = p_1 p_2 \cdots p_s \left(\frac{\beta_1^{p_1'} - 1}{\beta_1 - 1}\right)\left(\frac{\beta_2^{p_2'} - 1}{\beta_2 - 1}\right) \cdots \left(\frac{\beta_t^{p_t'} - 1}{\beta_t - 1}\right),$$

then $|l| \leqq s+t$. We also specify $|1| \leqq 0$ (where $1 \in L$).

It is clear from the definitions of L and A (the sub-$\boldsymbol{ZB}$-algebra of $\boldsymbol{F}$ generated by the set E^{-1}, see line (5.6)) that for every $a \in A$, there exists an element $l_a \in L$ such that $l_a a \in \boldsymbol{ZB}$. Using this fact, it is also clear that for every element $u \in U'$ (see lines (5.7) and (5.8)), there exists an element $l_u \in L$ such that $l_u u \in U_0'$ (where we note that, as sets, $L \subseteq \boldsymbol{ZB} \subseteq A \subseteq \boldsymbol{F}$).

To prove that for every $u \in U'$, $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in F'$, we proceed by induction on n, where n is the length of the "shortest" element $l \in L$ such that $lu \in U_0'$. More precisely, our induction assumption is the following: if $u' \in U'$ is such that there exists $l' \in L$ with $|l'| \leqq n-1$ such that $l'u' \in U_0'$, then $\left(\begin{smallmatrix} 1 & 0 \\ u' & 1 \end{smallmatrix}\right) \in F'$. In the initial case of the induction, we have an element $u \in U_0'$ already (because $l=1$ is the only element of L such that $|l| \leqq 0$). In this case, Lemma 5.5(iii) gives that $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in F'$; so this starts the induction.

Now let $u \in U'$ be such that there is an element $l_u \in L$ with $|l_u| \leqq n$ such that $l_u u \in U_0'$. Clearly (assuming $l_u \neq 1$), there exist elements $e \in E$ and $l' \in L$ such that

$l_u = el'$, $|l'| \leqq n-1$. Noting that $eu \in U'$, we then have $l'(eu) = (el')u \in U_0'$. Since $|l'| \leqq n-1$, the induction assumption yields

$$\begin{pmatrix} 1 & 0 \\ eu & 1 \end{pmatrix} \in F'. \tag{5.11}$$

There are now two cases to consider, depending on whether $e = p$, $p \in \pi$, or $e = (\beta^p - 1)/(\beta - 1)$, $\beta \in B$, $p \in \pi$. If $e = p$, $p \in \pi$, line (5.11) gives $\left(\begin{smallmatrix} 1 & 0 \\ pu & 1 \end{smallmatrix}\right) \in F'$; then, as $F' \in \mathscr{D}$ and $p \in \pi$, we have

$$\begin{pmatrix} 1 & 0 \\ pu & 1 \end{pmatrix}^{1/p} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in F',$$

which completes this case. If, on the other hand, $e = (\beta^p - 1)/(\beta - 1)$, $\beta \in B$, $p \in \pi$, line (5.11) gives

$$\begin{pmatrix} 1 & 0 \\ \left(\dfrac{\beta^p - 1}{\beta - 1}\right)u & 1 \end{pmatrix} \in F';$$

then, using the facts that $\left(\begin{smallmatrix} \beta & 0 \\ v_\beta & 1 \end{smallmatrix}\right) \in F'$ and that $F' \in \mathscr{D}$ and $p \in \pi$, we have

$$\begin{pmatrix} \beta & 0 \\ v_\beta & 1 \end{pmatrix}^{-1} \begin{pmatrix} \beta & 0 \\ v_\beta & 1 \end{pmatrix}^{p} \begin{pmatrix} 1 & 0 \\ \left(\dfrac{\beta^p - 1}{\beta - 1}\right)u & 1 \end{pmatrix}^{1/p} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in F'.$$

This completes the proof of the induction step and of the proposition.

5.8. The proof of Theorem 5.6 is now complete, so the explicit form (see line (5.10)) of those matrices which occur in the $\mathscr{D}$-group F' is known. It remains for us to deduce some useful consequences.

COROLLARY 5.10. $\gamma_2^\pi F' = \{\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \mid u \in U'\}$.

Proof. By Theorem 5.6, we have

$$F' \cap K = \tilde{F}' \cap K = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mid u \in U' \right\}.$$

The projection $\eta\colon \left(\begin{smallmatrix} \beta & 0 \\ v & 1 \end{smallmatrix}\right) \mapsto \beta$ maps F' onto B, which shows that $F'/F' \cap K$ is a free abelian $\mathscr{D}$-group freely π-generated by the elements $x_i'(F' \cap K)$, $i \in I$. Therefore the homomorphism $F'/\gamma_2^\pi F' \to F'/F' \cap K$, induced by the identity on F', has an inverse, so that $\gamma_2^\pi F' = F' \cap K$ as required.

COROLLARY 5.11. *Let $c' \in \gamma_2^\pi F'$. Then there exist primes $p_1, p_2, \ldots, p_s, p_1', p_2', \ldots, p_t' \in \pi$, and elements $\beta_1, \beta_2, \ldots, \beta_t \in B$, such that*

$$(c')^{p_1 p_2 \cdots p_s ((\beta_1^{p_1'} - 1)/(\beta_1 - 1))((\beta_2^{p_2'} - 1)/(\beta_2 - 1)) \cdots ((\beta_t^{p_t'} - 1)/(\beta_t - 1))} \in \gamma_2 G_0'.$$

Proof. This is immediate from Corollary 5.10 and the proof of Proposition 5.9, once we recall (cf. Lemma 5.2(iv)) that

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^{S} = \begin{pmatrix} 1 & 0 \\ Su & 1 \end{pmatrix}, \qquad u \in V,\ S \in ZB.$$

COROLLARY 5.12. *Let F be a free metabelian $\mathscr{D}$-group freely π-generated by the elements x_i, $i \in I$; denote $F/\gamma_2^\pi F$ by B and let*

$$G_0 = \mathrm{gp}\,(x_i^{1/m} \mid i \in I, m \in [\pi]).$$

If $c \in \gamma_2^\pi F$, then there exist primes $p_1, p_2, \ldots, p_s$, $p_1', p_2', \ldots, p_t' \in \pi$ and elements $\beta_1, \beta_2, \ldots, \beta_t \in B$, such that

$$c^{p_1 p_2 \cdots p_s ((\beta_1^{p_1'} - 1)/(\beta_1 - 1))((\beta_2^{p_2'} - 1)/(\beta_2 - 1)) \cdots ((\beta_t^{p_t'} - 1)/(\beta_t - 1))} \in \gamma_2 G_0.$$

Proof. If we remove the primes from the preceding work, that is, if we set $v_i = e_i$, $i \in I$, the resulting matrix $\mathscr{D}$-group F is a free metabelian $\mathscr{D}$-group freely π-generated by the matrices x_i, $i \in I$. Therefore the result is a special case of Corollary 5.11.

REFERENCES

1. G. Baumslag, *Some aspects of groups with unique roots*, Acta. Math. **104** (1960), 217–303 MR **23** #A191.

2. ———, *Some remarks on nilpotent groups with roots*, Proc. Amer. Math. Soc. **12** (1961), 262–267. MR **23** #A934.

3. ———, *Some subgroup theorems for free $\mathfrak{B}$-groups*, Trans. Amer. Math. Soc. **108** (1963), 516–525. MR **27** #4862.

4. ———, *Residual nilpotence and relations in free groups*, J. Algebra **2** (1965), 271–282. MR **31** #3487.

5. ———, *A representation of the wreath product of two torsion-free abelian groups in a power series ring*, Proc. Amer. Math. Soc. **17** (1966), 1159–1165. MR **34** #7623.

6. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

7. S. N. Černikov, *Periodic ZA-extensions of complete groups*, Mat. Sb. **27** **(69)** (1950), 117–128. (Russian) MR **12**, 156.

8. P. Hall, *Nilpotent groups*, Lecture Notes, Canadian Math. Congress, University of Alberta, 1957.

9. G. Higman, *Groups and rings having automorphisms without non-trivial fixed elements*, J. London Math. Soc. **32** (1957), 321–334. MR **19**, 633.

10. P. G. Kontorovič, *Groups with a basis of partition*. III, Mat. Sb. **22** **(64)** (1948), 79–100. (Russian) MR **9**, 493.

11. A. G. Kuroš, *The theory of groups*, 2nd ed., GITTL, Moscow, 1953; English transl., Vol. II, Chelsea, New York, 1960. MR **15**, 501; MR **22** #727.

12. J. F. Ledlie, *Properties of free metabelian $\mathscr{D}_\pi$-groups* (to appear).

13. F. W. Levi, *Ordered groups*, Proc. Indian Acad. Sci. Sect. A. **16** (1942), 256–263. MR **4**, 192.

14. T. MacHenry, *Free metabelian $\mathscr{D}_\pi$-groups: A construction*, Ph.D. thesis, Adelphi University, Garden City, N. J., 1962.

15. W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. **111** (1935), 259–280.

16. ———, *On a theorem of Marshall Hall*, Ann. of Math. (2) **40** (1939), 764–768.

17. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966. MR **34** #7617.

18. A. I. Mal'cev, *Nilpotent torsion-free groups*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 201–212. (Russian) MR **10**, 507.

19. B. H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. **18** (1943), 4–11. MR **5**, 58.

20. ———, *Special topics in algebra*: *Universal algebra*, Lecture notes prepared by P. M. Neumann, Courant Inst. Math. Sci., New York University, 1962.

21. H. Neumann, *Varieties of groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, New York, 1967. MR **35** #6734.

22. J. J. Rotman, *The theory of groups. An introduction*, Allyn and Bacon, Boston, Mass., 1965. MR **34** #4338.

Rice University,
Houston, Texas 77001