TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 154, February 1971

A BOOLEAN ALGEBRA OF REGULAR CLOSED SUBSETS OF $\beta X - X$

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Abstract. Let X be a locally compact, σ-compact, noncompact Hausdorff space. Let βX denote the Stone-Čech compactification of X. Let $R(X)$ denote the Boolean algebra of all regular closed subsets of the topological space X. We show that the map $A \to (\mathrm{cl}_{\beta X} A) - X$ is a Boolean algebra homomorphism from $R(X)$ into $R(\beta X - X)$. Assuming the continuum hypothesis, we show that if X has no more than $2^{\aleph_0}$ zero-sets, then the image of a certain dense subalgebra of $R(X)$ under this homomorphism is isomorphic to the Boolean algebra of all open-and-closed subsets of $\beta N - N$ (N denotes the countable discrete space). As a corollary, we show that there is a continuous irreducible mapping from $\beta N - N$ onto $\beta X - X$. Some theorems on higher-cardinality analogues of Baire spaces are proved, and these theorems are combined with the previous result to show that if S is a locally compact, σ-compact noncompact metric space without isolated points, then the set of remote points of βS (i.e. those points of βS that are not in the βS-closure of any discrete subspace of S) can be embedded densely in $\beta N - N$.

Introduction. Assuming the continuum hypothesis, Parovičenko [8], and Rudin [10] have shown that if U is a Boolean algebra of cardinality $2^{\aleph_0}$ such that maximal chains in $U - \{0, 1\}$ are η_1-sets, then U is isomorphic to $B(\beta N - N)$, the Boolean algebra of all open-and-closed subsets of $\beta N - N$. Let $\sigma G(X)$ denote the smallest σ-complete subalgebra of $R(X)$ containing the family $\{\mathrm{cl}_X (\mathrm{int}_X Z) : Z$ is a zero-set of $X\}$. Using the result of Parovičenko and Rudin, we show in §2 that if the locally compact, σ-compact, noncompact space X has no more than $2^{\aleph_0}$ zero-sets, then the image of $\sigma G(X)$ under the above-defined homomorphism is isomorphic to $B(\beta N - N)$.

Let Y be a compact Hausdorff space. Let $S(B)$ denote the Stone space of the Boolean algebra B. In [5], Gleason has shown that there exists an irreducible mapping from $S(R(Y))$ onto Y. Let $\mathscr{A}$ be a subalgebra of $R(Y)$ that is also a base for the closed subsets of Y. Using Gleason's methods, we show that there exists an irreducible map f from $S(\mathscr{A})$ onto Y. We further show that there exists an embedding g of the (possibly empty) set $H(\mathscr{A}) = \bigcap_{A \in \mathscr{A}} [Y - \mathrm{bd}_Y A]$ into $S(\mathscr{A})$ such that $f \circ g$ is the natural inclusion of $H(\mathscr{A})$ in Y. Special cases in which $H(\mathscr{A})$ is dense in Y are considered.

Received by the editors November 5, 1969.

AMS 1969 *subject classifications*. Primary 5453; Secondary 0660.

Key words and phrases. Stone-Čech compactification, locally compact, σ-compact space, regular closed set, Boolean algebra homomorphism, projective cover, irreducible mapping, Stone space, remote points.

In [9], Plank has characterized the set of remote points of βX, where X is a separable metric space without isolated points. Plank also proves, assuming the continuum hypothesis, that if in addition X is locally compact, then the set of remote points of βX is dense in $\beta X - X$. We use these results of Plank and the results of §2 and §3 to prove the above mentioned theorem concerning the embedding of the remote points of βX in $\beta N - N$.

The terminology and notation used in this paper will, with only a few exceptions, be that of [6]. Material pertaining to Boolean algebras can be found in [11]. All spaces are completely regular Hausdorff spaces.

The results of this paper form part of a doctoral dissertation submitted to McGill University in July, 1969. The author wishes to thank Dr. Stelios Negrepontis for his valuable advice and encouragement. He also thanks the referee for his useful suggestions.

1. **Preliminaries.** The following theorem concerning the structure of locally compact, σ-compact spaces is well known (see [2, 11.7.2]).

1.1. THEOREM. *A Hausdorff space X is locally compact and σ-compact if and only if it can be expressed in the form $X = \bigcup_{n=0}^{\infty} V(n)$, where $V(n)$ is an open subset of X, $\mathrm{cl}_X V(n)$ is compact, and $\mathrm{cl}_X V(n) \subseteq V(n+1)$ for each nonnegative integer n.*

We may assume that each $V(n)$ is a regular open subset of X. If X is assumed to be noncompact, we may also assume that $V(n+1) - \mathrm{cl}_X V(n)$ is nonempty for each n. The symbol "$V(n)$" will be used with this meaning throughout this paper. We also recall that every locally compact, σ-compact space is normal [2, 8.2.2 and 11.7.3].

The family $R(X)$ of all regular closed subsets of X is a complete Boolean algebra under the following operations [11, §20 C]:

(i) $A \leqq B$ if and only if $A \subseteq B$,

(ii) $\bigvee_\alpha A_\alpha = \mathrm{cl}_X [\bigcup_\alpha A_\alpha]$,

(iii) $\bigwedge_\alpha A_\alpha = \mathrm{cl}_X [\mathrm{int}_X \bigcap_\alpha A_\alpha]$,

(iv) $A' = \mathrm{cl}_X (X - A)$ (A' denotes the complement of A).

The symbol $B(X)$ will denote the Boolean algebra of all open-and-closed subsets of X.

If U is a Boolean algebra, let $S(U)$ denote the set of all ultrafilters on U. For each $x \in U$, put $\lambda(x) = \{\alpha \in S(U) : x \in \alpha\}$. The following theorem of Stone is well known [11, 8.2]:

1.2. THEOREM. *If a topology τ is assigned to $S(U)$ by letting $\{\lambda(x) : x \in U\}$ be an open base for τ, then $(S(U), \tau)$ is a compact totally disconnected space and the map $x \to \lambda(x)$ is a Boolean algebra isomorphism from U onto $B(S(U))$.*

If S is a set, then $|S|$ will denote the cardinality of S. As is standard, we shall let c denote the cardinality $2^{\aleph_0}$ of the continuum. The symbol [CH] will be used to indicate that we are using the continuum hypothesis ($c=\aleph_1$) in the proof of a theorem.

A subset S of a Boolean algebra U is said to be a *dense* subset of U if, given $x \in U$, $x \neq 0$, there exists $s \in S$ such that $0 \neq s \leqq x$. The following result is well known [11, 35.2].

1.3. THEOREM. *If S_i is a dense subalgebra of the Boolean algebra U_i ($i=1, 2$) and if $f: S_1 \to S_2$ is a Boolean algebra isomorphism from S_1 onto S_2, then there exists a Boolean algebra isomorphism $g: U_1 \to U_2$ such that the restriction of g to S_1 is f.*

A Boolean algebra U is *σ-complete* if every countable subset of U has a supremum in U [11, §20]. Recall that every Boolean algebra can be embedded densely in a complete Boolean algebra [11, §35]. If S is a dense subset of the complete Boolean algebra U, then the *σ-completion* of S (symbolized σS) is the intersection of all the σ-complete subalgebras of U that contain S [11, 35.3]. Obviously σS is a σ-complete subalgebra of U, and if D is a countable subset of σS then the σS-supremum of D and the U-supremum of D are the same [11, 23.1]. Note that $|\sigma S| \leqq |S|^{\aleph_0}$.

2. **A Boolean algebra homomorphism.** Throughout this section X is a locally compact, σ-compact, noncompact Hausdorff space. We denote the set of nonnegative integers by N.

2.1. LEMMA. *Let $(A_n)_{n \in N}$ be a countable family of closed subsets of X. Define the nonnegative integer $k(n)$ as follows:*

$$k(n) = \min\{j \in N : A_n \cap V(j) \neq \varnothing\}$$

for each $n \in N$. Then

(i) *If $\lim_{n\to\infty} k(n)=\infty$ then $\bigcup_{n=0}^{\infty} A_n$ is closed in X.*

(ii) *If $\lim_{n\to\infty} k(n)=\infty$ and $A_n \in R(X)$ for each $n \in N$, then $\bigcup_{n=0}^{\infty} A_n = \bigvee_{n=0}^{\infty} A_n$.*

Proof. (i) Let $p \in \mathrm{cl}_X (\bigcup_{n=0}^{\infty} A_n)$ and let U be an open subset of X containing p. There exists $i \in N$ such that $p \in V(i)$; thus as $U \cap V(i)$ is open, it follows that $U \cap V(i) \cap (\bigcup_{n=0}^{\infty} A_n) \neq \varnothing$. As $\lim_{n\to\infty} k(n)=\infty$, there exists $m \in N$ such that $n \geqq m$ implies $A_n \cap V(i) = \varnothing$. Thus $U \cap V(i) \cap (\bigcup_{n=0}^{m-1} A_n) \neq \varnothing$. As U was an arbitrary open set containing p, it follows that p belongs to the closed set $\bigcup_{n=0}^{m-1} A_n$, and so $p \in \bigcup_{n=0}^{\infty} A_n$. Thus $\bigcup_{n=0}^{\infty} A_n$ is closed.

(ii) If $p \in \bigcup_{n=0}^{\infty} A_n$, there exists $k \in N$ such that $p \in A_k$. As $A_k \in R(X)$, it follows that $p \in \mathrm{cl}_X (\mathrm{int}_X A_k) \subseteq \mathrm{cl}_X (\mathrm{int}_X [\bigcup_{n=0}^{\infty} A_n])$. The result is now immediate.

2.2. DEFINITION. Let A be a closed subset of X. Then A^* will denote the set $(\mathrm{cl}_{\beta X} A) - X$ (thus $X^* = \beta X - X$).

If A and B are closed subsets of X, the following results are immediate:

(i) $(A \cup B)^* = A^* \cup B^*$;

(ii) $(A \cap B)^* = A^* \cap B^*$;

(iii) $A^* = \varnothing$ if and only if A is compact.

The second result follows from [6, 6.5 IV] and the normality of X.

We shall use the symbol "$\subset$" to denote proper set inclusion.

2.3. LEMMA. *Let A and B be closed subsets of X.*

(i) *$A^* \subseteq B^*$ if and only if there exists $n \in N$ such that $A - B \subseteq V(n)$.*

(ii) *If $A^* \subset B^*$ and if $B \in R(X)$, then* $(\mathrm{int}_X B) - (A \cup \mathrm{cl}_X V(n)) \neq \varnothing$ *for each $n \in N$.*

Proof. (i) Suppose that $A - B \subseteq V(n)$ for some $n \in N$. Then as $\mathrm{cl}_X V(n)$ is compact, $\mathrm{cl}_X (A - B)$ is also compact. As A is closed, it follows that $A = (A \cap B) \cup \mathrm{cl}_X (A - B)$. By 2.2,

$$A^* = (A \cap B)^* \cup [\mathrm{cl}_X (A - B)]^* = (A \cap B)^* = A^* \cap B^*.$$

Thus $A^* \subseteq B^*$.

Conversely, suppose that $A - B$ is not contained in $V(n)$ for any $n \in N$. Then there exists a sequence $(n_i)_{i \in N}$ of positive integers, with $\lim_{i \to \infty} n_i = \infty$, such that $(A - B) \cap [V(n_{i+1}) - \mathrm{cl}_X V(n_i)] \neq \varnothing$ for each $i \in N$. Let $p_i \in (A - B) \cap [V(n_{i+1}) - \mathrm{cl}_X V(n_i)]$ for each $i \in N$, and put $S = (p_i)_{i \in N}$. By 2.1 S is closed, and obviously S is not contained in any $V(n)$; hence S is not compact and so by 2.2(iii), $S^* \neq \varnothing$. Obviously $S \subseteq A - B$, so $S^* \subseteq A^*$ and by 2.2(ii), $S^* \cap B^* = \varnothing$. Consequently $A^* - B^* \neq \varnothing$ and (i) follows.

(ii) By (i) it follows that for each $n \in N$, $B - [A \cup V(n+1)] \neq \varnothing$ and so $B - (A \cup \mathrm{cl}_X V(n)) \neq \varnothing$. Thus $[\mathrm{cl}_X (\mathrm{int}_X B)] \cap [X - [A \cup \mathrm{cl}_X V(n)]] \neq \varnothing$ and so $(\mathrm{int}_X B) - [A \cup \mathrm{cl}_X V(n)] \neq \varnothing$.

2.4. PROPOSITION. *If A is a closed subset of X, then* $\mathrm{cl}_{X^*} (X^* - A^*) = [\mathrm{cl}_X (X - A)]^*$.

Proof. Since $A \cup \mathrm{cl}_X (X - A) = X$, by 2.2(i) it follows that $A^* \cup [\mathrm{cl}_X (X - A)]^* = X^*$. Thus $X^* - A^* \subseteq [\mathrm{cl}_X (X - A)]^*$, and as $[\mathrm{cl}_X (X - A)]^*$ is closed in X^*, it follows that

$$\mathrm{cl}_{X^*} (X^* - A^*) \subseteq [\mathrm{cl}_X (X - A)]^*. \tag{1}$$

Conversely, suppose that $x \notin \mathrm{cl}_{X^*} (X^* - A^*)$. Since the family $\{\mathrm{cl}_{\beta X} S : S \text{ closed in } X\}$ is a base for the closed subsets of βX [6, 6.5(b)], it follows that $\{X^* - S^* : S \text{ closed in } X\}$ is a base for the open sets of X^*. Thus since X^* is completely regular, there exists a closed subset B of X such that $x \in X^* - B^*$ and also $(X^* - B^*) \cap \mathrm{cl}_{X^*} (X^* - A^*) = \varnothing$; thus $(X^* - B^*) \cap (X^* - A^*) = \varnothing$ and thus by 2.2(i), $X^* = (A \cup B)^*$. By 2.3(i) there exists $i \in N$ such that $X - (A \cup B) \subseteq V(\mathrm{i})$. Thus

$$[\mathrm{cl}_X (X - A)] \cap (X - B) \subseteq \mathrm{cl}_X V(i); \tag{2}$$

for if not, $[\mathrm{cl}_X (X - A)] \cap (X - B) \cap [X - \mathrm{cl}_X V(i)] \neq \varnothing$ and as $(X - B) \cap [X - \mathrm{cl}_X V(i)]$ is open, it would follow that $(X - A) \cap (X - B) \cap [X - \mathrm{cl}_X V(i)] \neq \varnothing$, which contradicts $X - (A \cup B) \subseteq V(i)$. It follows from (2) that $[\mathrm{cl}_X (X - A)] - B \subseteq V(i+1)$, and so by 2.3(i) we have $[\mathrm{cl}_X (X - A)]^* \subseteq B^*$. As $x \in X^* - B^*$, it follows that $x \notin [\mathrm{cl}_X (X - A)]^*$. Thus $[\mathrm{cl}_X (X - A)]^* \subseteq \mathrm{cl}_{X^*} (X^* - A^*)$, and combining this with (1) yields the proposition.

2.5. PROPOSITION. *If A is a closed subset of X, then* $\mathrm{cl}_{X^*}(\mathrm{int}_{X^*} A^*)=[\mathrm{cl}_X(\mathrm{int}_X A)]^*$.

Proof. Since $\mathrm{int}_{X^*} A^*=X^*-\mathrm{cl}_{X^*}(X^*-A^*)$, by 2.4 it follows that $\mathrm{int}_{X^*} A^*= X^*-[\mathrm{cl}_X(X-A)]^*$ and so

$$\begin{aligned}\mathrm{cl}_{X^*}(\mathrm{int}_{X^*} A^*) &= \mathrm{cl}_{X^*}(X^*-[\mathrm{cl}_X(X-A)]^*)\\ &= [\mathrm{cl}_X(X-[\mathrm{cl}_X(X-A)])]^* \quad \text{(by 2.4)}\\ &= [\mathrm{cl}_X(\mathrm{int}_X A)]^*.\end{aligned}$$

The following result is an immediate consequence of 2.5.

2.6. COROLLARY. *If $A\in R(X)$ then $A^*\in R(X^*)$.*

2.7. REMARK. In Corollary 2.6, the hypothesis that X be a σ-compact space cannot be dropped. As an example, let W be the space of all countable ordinals and put $Y=W\times W$. Then Y is locally compact and noncompact, but not σ-compact. Let αW denote the one-point compactification of W, formed by adjoining the first uncountable ordinal ω_1 to W. It is shown in problems 8L and 8M of [6] that $\beta Y=\alpha W\times\alpha W$, and hence $Y^*=[\{\omega_1\}\times\alpha W]\cup[\alpha W\times\{\omega_1\}]$. Let A denote the "diagonal" of Y, that is $A=\{(\alpha,\alpha):\alpha\in W\}$. Since W is a Hausdorff space, A is closed. The point (α,α) is isolated in Y if and only if α is a nonlimit ordinal. It follows easily from this that $A\in R(Y)$. However, $A^*=\{(\omega_1,\omega_1)\}$ and evidently $\{(\omega_1,\omega_1)\}\notin R(Y^*)$. Hence we have the desired example.

2.8. THEOREM. *The map $A\to A^*$ is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$.*

Proof. By 2.6 the map $A\to A^*$ is well defined. Suppose that A and B belong to $R(X)$. Then by 2.2(i),

$$A^*\vee B^*=A^*\cup B^*=(A\cup B)^*=(A\vee B)^*.$$

It follows from 2.4 that

$$(A')^*=[\mathrm{cl}_X(X-A)]^*=\mathrm{cl}_{X^*}(X^*-A^*)=(A^*)'.$$

Thus the map preserves suprema and complements and hence is a Boolean algebra homomorphism.

Let $Z(X)$ denote the family of zero-sets of X (see [6, 1.10]).

2.9. NOTATION. (i) If $\mathscr{F}$ is a subfamily of $R(X)$, then $[\mathscr{F}]^*$ will denote the family $\{F^*:F\in\mathscr{F}\}$.

(ii) The family $\{\mathrm{cl}_X(\mathrm{int}_X Z):Z\in Z(X)\}$ will be denoted by $G(X)$. If X is a metric space then every closed subset of X is a zero-set, and so $G(X)=R(X)$.

The following result will be needed later.

2.10. PROPOSITION. *Let Y be any completely regular Hausdorff space. Then $[G(Y)]^*$ is a base for the closed subsets of Y^*.*

Proof. It suffices to show that $\{\mathrm{cl}_{\beta Y}(\mathrm{cl}_Y(\mathrm{int}_Y Z)) : Z \in Z(Y)\}$ is a base for the closed sets of βY. If A is closed in βY and $p \in \beta Y - A$, we can find $f \in C(\beta Y)$ such that $f(p)=0$ and $f[A]=\{1\}$. Put $Z = Y \cap f^{-1}[\frac{1}{2}, \infty)$. Then $Z \in Z(Y)$, $A \subseteq \mathrm{cl}_{\beta Y}(\mathrm{cl}_Y(\mathrm{int}_Y Z))$, and $p \notin \mathrm{cl}_{\beta Y}(\mathrm{cl}_Y(\mathrm{int}_Y Z))$. The result is now immediate.

The proof of the following result mimics that of [6, 13.5] and hence is not included.

2.11. LEMMA. *Let $\mathscr{A}$ be any subalgebra of $R(X)$ and let $\mathscr{E}$ be any countable subset of $[\mathscr{A}]^*$. Then there exists a subset $(E_n)_{n \in N}$ of $\mathscr{A}$ satisfying the following two conditions:*

(i) $\mathscr{E} = (E_n^*)_{n \in N}$;

(ii) $E_i \subseteq E_j$ *if and only if* $E_i^* \subseteq E_j^*$ $(i, j \in N)$.

2.12. *η_1-sets.* Let S be a totally ordered set with subsets A and B. If $a < b$ for each $a \in A$ and $b \in B$ we will write $A < B$.

The totally ordered set S is called an η_1-set if, given subsets A and B of cardinality at most $\aleph_0$ and with $A < B$, there exists $c \in S$ such that $A < \{c\} < B$ [6, 13.6].

The following theorem is due to Rudin [10] and Parovičenko [8].

2.13. THEOREM [CH]. *Let U be a Boolean algebra of cardinality c. If maximal chains in the partially ordered set $U - \{0, 1\}$ are η_1-sets, then U is isomorphic to $B(\beta N - N)$.*

2.14. DEFINITION. Let Y be any completely regular Hausdorff space. A subalgebra $\mathscr{A}$ of $R(Y)$ is called a basic subalgebra of $R(Y)$ if $\mathscr{A}$ is a base for the closed subsets of Y.

The following result is immediate:

2.15. PROPOSITION. *If $\mathscr{A}$ is a basic subalgebra of $R(Y)$, then $\{\mathrm{int}_Y A : A \in \mathscr{A}\}$ is a base for the open subsets of Y.*

Obviously a basic subalgebra of $R(Y)$ is dense in $R(Y)$; the converse is untrue in general.

2.16. THEOREM. *Let $\mathscr{S}$ be a basic subalgebra of $R(X)$ with the property that if $(S_n)_{n \in N} \subseteq \mathscr{S}$ and if $\bigcup_{n=0}^{\infty} S_n \in R(X)$, then $\bigcup_{n=0}^{\infty} S_n \in \mathscr{S}$. Then maximal chains in $[\mathscr{S}]^* - \{\varnothing, X^*\}$ are η_1-sets.*

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be chains (with respect to set-theoretic inclusion) in $[\mathscr{S}]^* - \{\varnothing, X^*\}$, and assume that both $\mathscr{A}$ and $\mathscr{B}$ have cardinality no greater than $\aleph_0$. Then in order to prove the theorem, it suffices to show that if $\mathscr{A} < \mathscr{B}$, then there exists $C \in \mathscr{S}$ such that $\mathscr{A} < C^* < \mathscr{B}$. As $\mathscr{S}$ is a basic subalgebra of $R(X)$, it can be assumed without loss of generality that $X - V(n) \in \mathscr{S}$; for as $(\mathrm{int}_X S)_{S \in \mathscr{S}}$ is a base for the open subsets of X (2.15), there exists, for each $n \in N$, a family $(S_\alpha)_{\alpha \in \Sigma} \subseteq \mathscr{S}$ such that $V(n+1) = \bigcup_{\alpha \in \Sigma} \mathrm{int}_X S_\alpha$. Thus $\mathrm{cl}_X V(n) \subseteq \bigcup_{\alpha \in \Sigma} \mathrm{int}_X S_\alpha$, and so by the compactness of $\mathrm{cl}_X V(n)$ there exist $\alpha(1), \ldots, \alpha(k) \in \Sigma$ such that $\mathrm{cl}_X V(n) \subseteq \bigcup_{i=1}^k \mathrm{int}_X S_{\alpha(i)}$. Thus $\mathrm{cl}_X V(n) \subseteq \bigvee_{i=1}^k S_{\alpha(i)} \subseteq \mathrm{cl}_X V(n+1)$, and we may replace $X-V(n)$ by $(\bigvee_{i=1}^k S_{\alpha(i)})'$.

Let $\mathscr{A}=(A_n^*)_{n\in N}$ and $\mathscr{B}=(B_n^*)_{n\in N}$. There are several cases to consider.

Case 1. Assume that $\mathscr{A}$ either is empty or has a largest member, and that $\mathscr{B}$ has no smallest member. Let A^* be the largest member of $\mathscr{A}$ (for some $A \in \mathscr{S}$), and put $A^*=\varnothing$ if $\mathscr{A}$ is empty. By replacing B_n^* by $\bigwedge_{i=0}^n B_i^*$ if necessary, and noting that $\mathscr{B}$ has no smallest member, we can assume that $A^* \subset B_{n+1}^* \subset B_n^*$ for each $n \in N$. Thus by 2.11 we can assume that $A \subset B_{n+1} \subset B_n$ for each $n \in N$. By 2.3(ii) it is evident that $(\mathrm{int}_X B_n)-[A \cup \mathrm{cl}_X V(n)] \neq \varnothing$ for each $n \in N$. Thus for each $n \in N$, there exists $k_n \in N$ such that the open set

$$(\mathrm{int}_X B_n) \cap (X-A) \cap [X-\mathrm{cl}_X V(n)] \cap V(k_n)$$

is nonempty. As $\mathscr{S}$ is a basic subalgebra of $R(X)$, by Proposition 2.15 there exists $S_n \in \mathscr{S}$ such that

$$\varnothing \neq S_n \subseteq (\mathrm{int}_X B_n) \cap (X-A) \cap [X-\mathrm{cl}_X V(n)] \cap V(k_n). \tag{1}$$

Put $E=\bigcup_{n=0}^\infty S_n$. As $S_n \subseteq X-\mathrm{cl}_X V(n)$ for each $n \in N$, by 2.1(ii) $E \in R(X)$. Thus by hypothesis $E \in \mathscr{S}$. By (1), $E-V(n) \supseteq S_n - V(n) \neq \varnothing$ for each $n \in N$, so by 2.3(i) $E^* \neq \varnothing$. As $S_n \cap A=\varnothing$ for each $n \in N$ (see (1)), it follows that $E \cap A=\varnothing$, and so $E^* \cap A^*=\varnothing$. Put $C=A \cup E$; then it follows from the above remarks that $A^* \subset C^*$ and $C \in \mathscr{S}$. As $A \subset B_j$ for each $j \in N$, it follows that

$$C-B_j = E-B_j = \bigcup_{n=0}^{\infty} (S_n-B_j). \tag{2}$$

But by (1), $S_n-B_j \subseteq (\mathrm{int}_X B_n)-B_j$. If $n \geqq j$ then $B_n \subseteq B_j$ and so S_n-B_j is empty. Thus by (2),

$$C-B_j = \bigcup_{n=0}^{j-1} (S_n-B_j) \subseteq \bigcup_{n=0}^{j-1} V(k_n);$$

the second inclusion follows from (1). Thus $C-B_j \subseteq V(m)$ where $m=\max\{k_n : 0 \leqq n \leqq j-1\}$. Thus by 2.3(i), $C^* \subseteq B_j^* \subset B_{j-1}^*$ for each $j \in N$. Thus $\mathscr{A} < C^* < \mathscr{B}$.

Case 2. Assume that $\mathscr{B}$ either is empty or has a smallest member, and that $\mathscr{A}$ has no largest member. Let B^* be the smallest member of $\mathscr{B}$ (and put $B^*=X^*$ if $\mathscr{B}$ is empty). As in Case 1, since $\mathscr{A}$ has no largest member we can assume that $A_n^* \subset A_{n+1}^* \subset B^*$ for each $n \in N$, and thus by 2.11 that $A_n \subset A_{n+1} \subset B$ for each $n \in N$.

As $A_0^* \subset B^*$, it follows by 2.3(i) that $(B-A_0) \cap [X-\mathrm{cl}_X V(0)] \neq \varnothing$, and so we can choose $p_0 \in (B-A_0) \cap [X-\mathrm{cl}_X V(0)]$. Put $m_0=0$ and choose $m_1 \in N$ so that $p_0 \in V(m_1)$. Thus $m_1 > m_0$. Inductively, suppose that we have chosen $p_i \in X$, $0 \leqq i \leqq n-1$, and $m_i \in N$, $0 \leqq i \leqq n$, such that

$$\text{(3)} \quad \begin{array}{ll} \text{(i)} & m_{i+1} > m_i, \qquad 0 \leqq i \leqq n-1, \\ \text{(ii)} & p_i \in (B-A_i) \cap [X-\mathrm{cl}_X V(m_i)] \cap V(m_{i+1}), \qquad 0 \leqq i \leqq n-1. \end{array}$$

As $A_n^* \subset B^*$, by 2.3(i) there exists a point $p_n \in (B-A_n) \cap [X-\mathrm{cl}_X V(m_n)]$ and an integer $m_{n+1} \in N$ such that $p_n \in V(m_{n+1})$. Thus $(p_n)_{n\in N}$ and $(m_n)_{n\in N}$ satisfy (i) and (ii). Put

$$C = \bigcup_{n=0}^{\infty} [A_n \wedge X - V(m_{n+1})]. \tag{4}$$

By (i), $\lim_{n\to\infty} m_n = \infty$; hence by 2.1(ii) it follows that $C \in R(X)$. As $A_n \wedge [X - V(m_{n+1})] \in \mathscr{S}$ for each $n \in N$, it follows from the hypotheses that $C \in \mathscr{S}$. It is obvious from (4) that for each $n \in N$, $C^* \supseteq A_n^* \wedge [X - V(m_{n+1})]^*$. By 2.3(i), $[X - V(m_{n+1})]^* = X^*$ and so $C^* \supseteq A_n^* \supset A_{n-1}^*$ for each $n \in N$. Thus $\mathscr{A} < C^*$.

On the other hand, it is obvious from (4) that $C^* \subseteq (\bigvee_{n=0}^{\infty} A_n)^* \subseteq B^*$. Furthermore, for fixed $i \in N$, the set

$$(B - A_i) \cap [X - \mathrm{cl}_X V(m_i)] \cap V(m_{i+1}) \cap [A_n \wedge X - V(m_{n+1})]$$

is empty for each $n \in N$; for if $n \geqq i$, then $V(m_{i+1}) \cap X - V(m_{n+1}) = \varnothing$, and if $n < i$, then $(B - A_i) \cap A_n = \varnothing$. Thus by (3) and (4), $p_i \notin C$ for each $i \in N$. The set $S = (p_i)_{i\in N}$ is closed by 2.1(i) since $p_i \in X - \mathrm{cl}_X V(m_i)$; since S is disjoint from C, it follows from 2.2(ii) that $S^* \cap C^* = \varnothing$. But $S^* \neq \varnothing$ by 2.2(iii), and so $C^* \subset C^* \cup S^*$. By (3) $S^* \subseteq B^*$ and so $C^* \subset B^*$. Thus $\mathscr{A} < C^* < \mathscr{B}$.

Case 3. Assume that $\mathscr{A}$ either is empty or has a largest element, and that $\mathscr{B}$ either is empty or has a smallest element. The proof used in Case 2 applies here (with minor modifications).

Case 4. Assume that $\mathscr{A}$ has no largest member and $\mathscr{B}$ has no smallest member. As in Cases 1 and 2, it can be assumed that $A_n^* \subset A_{n+1}^* \subset B_{m+1}^* \subset B_m^*$ for each n, $m \in N$. By 2.11 it can also be assumed that $A_n \subset A_{n+1} \subset B_{m+1} \subset B_m$ for each n, $m \in N$. Put $C = \bigvee_{n=0}^{\infty} [A_n \wedge [X - V(n)]]$. As in earlier cases, it is evident that $C \in \mathscr{S}$. Obviously $C^* \supseteq A_n^* \wedge [X - V(n)]^* = A_n^* \supset A_{n-1}^*$ for each $n \in N$, and $C^* \subseteq (\bigvee_{n=0}^{\infty} A_n)^* \subseteq B_m^* \subset B_{m-1}^*$ for each $m \in N$. Thus $\mathscr{A} < C^* < \mathscr{B}$. This completes the proof of the theorem.

Recall our assumption that X is a locally compact, σ-compact, noncompact space.

2.17. Theorem [CH]. *Let* $|Z(X)| = c$. *Then* $[\sigma G(X)]^*$ *is a basic subalgebra of* $R(\beta X - X)$ *and is isomorphic to* $B(\beta N - N)$ (*see* 2.9 *and* §1 *for notation*). *In particular the Boolean algebras* $R(\beta X - X)$ *and* $R(\beta N - N)$ *are isomorphic.*

Proof. Since $|Z(X)| = c$, it follows that $|G(X)| = c$; thus $|\sigma G(X)| = c$ and so $[\sigma G(X)]^*$ has cardinality no greater than c. By [6, 3.6] and the complete regularity of X, $\sigma G(X)$ is a basic subalgebra of $R(X)$, and since $\sigma G(X)$ is σ-complete it satisfies all the conditions on $\mathscr{S}$ required in 2.16. Thus by Theorem 2.16 maximal chains in $[\sigma G(X)]^* - \{\phi, X^*\}$ are η_1-sets. Since every η_1-set has cardinality at least c [6, 13.6(b)], it follows that $|[\sigma G(X)]^*| = c$. Thus by Theorem 2.13 $[\sigma G(X)]^*$ is isomorphic to $B(\beta N - N)$. As $G(X) \subseteq \sigma G(X)$, it follows from Proposition 2.10 that

$[\sigma G(X)]^*$ is a basic subalgbra of $R(X^*)$. As $\beta N - N$ is totally disconnected, $B(\beta N - N)$ is dense in $R(\beta N - N)$; thus by Theorem 1.3 $R(\beta N - N)$ and $R(\beta X - X)$ are isomorphic.

Let Y be a compact space. Gleason [5] has shown that $S(R(Y))$ is the projective cover of Y (in the categorical sense) in the category of compact Hausdorff spaces and continuous maps. The following theorem is also proved by Gleason in Theorem 3.2 of [5] for the special case in which $\mathscr{A} = R(Y)$. As the proof of the present result is essentially the same as his proof, we shall not include it here. Recall that a continuous map f from the compact space T onto the compact space Y is called *irreducible* if the image under f of a proper closed subset of T is a proper closed subset of Y.

2.18. Theorem. *Let Y be a compact space and let $\mathscr{A}$ be a basic subalgebra of $R(Y)$. Then there exists an irreducible surjection $f\colon S(\mathscr{A}) \to Y$ defined by $f(x) = \bigcap \{A \in \mathscr{A} : x \in \lambda(A)\}$ (see 1.2 for notation).*

We can now prove one of the principal results of this paper. Recall that X is locally compact, σ-compact, and noncompact.

2.19. Theorem [CH]. *Assume that $|Z(X)| = c$. Then there exists an irreducible surjection f from $\beta N - N$ onto $\beta X - X$.*

Proof. If we put $Y = \beta X - X$ and $\mathscr{A} = [\sigma G(X)]^*$, then by 2.17 the conditions of 2.18 are satisfied, and $S(\mathscr{A})$ is homeomorphic to $\beta N - N$. Thus by 2.18 there is an irreducible surjection from $\beta N - N$ onto $\beta X - X$.

2.20. Remarks. (i) Note again that if X is a locally compact, σ-compact space in which every closed set is a zero-set and if $|Z(X)| = c$, then $\sigma G(X) = G(X) = R(X)$ and so $[R(X)]^*$ is isomorphic to $B(\beta N - N)$.

(ii) The fact that $R(\beta X - X)$ and $R(\beta N - N)$ are isomorphic (equivalently, $S(R(\beta X - X))$ and $S(R(\beta N - N))$ are homeomorphic) can be deduced from the following two known results:

(a) [CH] If X is locally compact, σ-compact, noncompact, and $|Z(X)| = c$ then $\beta X - X$ and $\beta N - N$ have homeomorphic dense subspaces of P-points [1, 3.6].

(b) If T is a dense subspace of Y, then $R(T)$ and $R(Y)$ are isomorphic.

A discussion of P-points can be found in [6, 4JKL].

(iii) Let Y be a compact space and $\mathscr{A}$ a dense subalgebra of $R(Y)$. It is not necessarily true that there exists an irreducible surjection from $S(\mathscr{A})$ onto Y. For example, if Y is the projective cover of $\beta N - N$, then $R(Y)$ contains a dense copy of $B(\beta N - N)$. If there were an irreducible surjection from $\beta N - N$ onto Y, then by Theorem 3.2 of [5] $\beta N - N$ would be homeomorphic to Y. This is untrue; see Problem 6.W of [6]. Hence the word "basic" cannot be replaced by the word "dense" in Theorem 2.18.

We conclude this section by giving a necessary condition that a compact space Y have a projective cover homeomorphic to that of $\beta N - N$.

2.21. Theorem [CH]. *Let Y be a compact space whose projective cover is homeomorphic to that of $\beta N - N$. Then dense G_δ-sets of Y have dense interiors.*

Proof. If Y and $\beta N - N$ have homeomorphic projective covers, then $R(Y)$ and $R(\beta N - N)$ are isomorphic. Thus $R(Y)$ contains a dense copy $\mathscr{F}$ of $B(\beta N - N)$. Let $G = \bigcap_{n=0}^{\infty} U_n$ be a dense G_δ-set in Y, where $(U_n)_{n \in N}$ is a countable family of dense open subsets of Y. Let W be any nonempty open subset of Y. Then $W \cap U_0$ is nonempty, and as $\mathscr{F}$ is dense in $R(Y)$ there exists $F_0 \in \mathscr{F}$ such that $\varnothing \neq F_0 \subseteq W \cap U_0$. As U_1 is dense in Y, it follows that $(\mathrm{int}_Y F_0) \cap U_1$ is nonempty. Thus there exists $F_1 \in \mathscr{F}$ such that $\varnothing \neq F_1 \subseteq (\mathrm{int}_Y F_0) \cap U_1$.

Inductively, suppose we have found $(F_k)_{0 \leqq k \leqq n}$ in $\mathscr{F}$ such that $\varnothing \neq F_i \subseteq (\mathrm{int}_Y F_{i-1}) \cap U_i$ $(1 \leqq i \leqq n)$. Then as U_{n+1} is dense in Y, it follows that $(\mathrm{int}_Y F_n) \cap U_{n+1}$ is nonempty and so there exists $F_{n+1} \in \mathscr{F}$ such that $\varnothing \neq F_{n+1} \subseteq (\mathrm{int}_Y F_n) \cap U_{n+1}$.

Thus we have a sequence $(F_n)_{n \in N} \subset \mathscr{F}$ such that $\varnothing \neq F_n \subseteq (\mathrm{int}_Y F_{n-1}) \cap U_n$ for each $n \in N$. Thus $(F_n)_{n \in N}$ is a countable chain of nonempty members of $\mathscr{F}$, and since $\mathscr{F}$ is isomorphic to $B(\beta N - N)$, whose maximal chains are η_1-sets (2.13), it follows from the definition of an η_1-set that there exists $H \in \mathscr{F}$ such that

$$\varnothing \neq H \subseteq \bigcap_{n=0}^{\infty} F_n \subseteq \bigcap_{n=0}^{\infty} (U_n \cap W).$$

As $\mathrm{int}_Y H$ is nonempty, it follows that $(\mathrm{int}_Y G) \cap W$ is nonempty. The theorem follows.

2.22. Remark. It is not necessarily true that if Y is compact and has a projective cover homeomorphic to that of $\beta N - N$, then *every* nonempty G_δ-set of Y has a nonempty interior. As an example, let Y be the projective cover of $\beta N - N$. Then Y is extremally disconnected [11, 22.4], and so every regular closed subset of Y is open-and-closed. If every nonempty G_δ-set of Y had a nonempty interior, then every zero-set of Y would be regular closed and hence open-and-closed. It follows from [6, 4J3] that Y would be a compact P-space; hence by [6, 4K2] Y would be finite, which is impossible. Thus the projective cover of $\beta N - N$ contains nonempty G_δ-sets with empty interiors.

3. **Baire m-spaces.** In this section we shall prove some general results about a certain class of compact space which we shall call Baire m-spaces. These results will be applied in §4 to spaces of the form $\beta X - X$.

3.1. Definition. Let X be a compact space and let m be an arbitrary cardinal number. Then X will be called a *Baire m-space* if, given a family $\mathscr{F}$ of dense open subsets of X such that $|\mathscr{F}| \leqq m$, the set $\bigcap \mathscr{F}$ is dense in X. The Baire category theorem states that every compact space is a Baire $\aleph_0$-space.

The idea used in the proof of the following theorem was first employed by Rudin [10, 4.2] to illustrate the existence of P-points in $\beta N - N$. Rudin's approach was used by Plank in [9, 3.2] in a more general setting, as described below.

3.2. THEOREM. *Let X be locally compact, noncompact, and realcompact (see* [6, 5.15]). *Then $\beta X - X$ is a Baire $\aleph_1$-space.*

Proof. In Theorem 3.1 of [3], Fine and Gillman show that if X is locally compact, realcompact, and noncompact, then every nonempty zero-set of X^* has a nonempty interior. Let Y be a locally compact space every nonempty zero-set of which has a nonempty interior. Let $\mathscr{F}$ be a family of dense open subsets of Y. In Theorem 3.2 of [9], Plank shows that if $|\mathscr{F}| \leqq \aleph_1$, then $\bigcap \mathscr{F}$ is dense in Y. The theorem now follows.

3.3. DEFINITION. Let Y be a space and let $\mathscr{F}$ be a subfamily of $R(Y)$. Then $H(\mathscr{F})$ is defined to be the set $\bigcap_{F \in \mathscr{F}} (Y - \mathrm{bd}_Y F)$ ($\mathrm{bd}_Y F$ denotes the topological boundary of F with respect to Y).

Note that $H(\mathscr{F})$ may be empty.

3.4. REMARK. There seems to be some formal similarity between the notions of a basic subalgebra $\mathscr{A}$ of $R(Y)$ and the associated subset $H(\mathscr{A})$ of Y and the concepts, defined by Plank [9, 2.2 and 3.1], of a β-subalgebra A of $C(X)$ and the associated set of A-points of $\beta X - X$. However, the exact relationship between these concepts is unclear.

3.5. THEOREM. *Let Y be a compact space and $\mathscr{A}$ a basic subalgebra of $R(Y)$.*

(i) *There exists a topological embedding g: $H(\mathscr{A}) \to S(\mathscr{A})$ such that $f \circ g$ is the natural inclusion of $H(\mathscr{A})$ in Y (where f is the mapping defined in* 2.18).

(ii) *If Y is a Baire m-space and if $|\mathscr{A}| \leqq m$ then $H(\mathscr{A})$ is dense in Y, $g[H(\mathscr{A})]$ is dense in $S(\mathscr{A})$, and $f[S(\mathscr{A}) - g[H(\mathscr{A})]] = Y - H(\mathscr{A})$.*

Proof. (i) Let $y \in H(\mathscr{A})$. If $A \in \mathscr{A}$, then from the definition of $H(\mathscr{A})$ it is apparent that $y \in A$ if and only if $y \in A - \mathrm{bd}_Y A = \mathrm{int}_Y A$. Define $\mathscr{U}(y) = \{\lambda(A) : A \in \mathscr{A}$ and $y \in A\}$ (see 1.2 for notation). This is an ultrafilter on $B(S(\mathscr{A}))$; for if $\lambda(A_1)$ and $\lambda(A_2)$ belong to $\mathscr{U}(y)$ then $y \in \mathrm{int}_Y A_1 \cap \mathrm{int}_Y A_2 = \mathrm{int}_Y (A_1 \wedge A_2)$. Thus $\lambda(A_1 \wedge A_2) = \lambda(A_1) \cap \lambda(A_2)$ is a member of $\mathscr{U}(y)$. Obviously $\varnothing \notin \mathscr{U}(y)$ and if $\lambda(A_1) \in \mathscr{U}(y)$ and $\lambda(A_1) \subseteq \lambda(A_2)$ then $\lambda(A_2) \in \mathscr{U}(y)$. Thus $\mathscr{U}(y)$ is a filter on $B(S(\mathscr{A}))$. Finally, if $\lambda(A) \notin \mathscr{U}(y)$ for some $A \in \mathscr{A}$, then $y \notin A$ and since $\mathscr{A}$ is a basic subalgebra of $R(Y)$, there exists $B \in \mathscr{A}$ such that $y \in B$ and $A \cap B = \varnothing$. Thus $\lambda(B) \in \mathscr{U}(y)$ and $\lambda(A) \cap \lambda(B) = \varnothing$; hence $\mathscr{U}(y)$ is an ultrafilter on $B(S(\mathscr{A}))$. Thus $\bigcap \mathscr{U}(y)$ is a single point of $S(\mathscr{A})$ and so we can define $g(y)$ by $g(y) = \bigcap \mathscr{U}(y)$.

Suppose that x and y are distinct members of $H(\mathscr{A})$. Since $\mathscr{A}$ is a basic subalgebra of $R(Y)$, there exists $A \in \mathscr{A}$ such that $y \in A$ and $x \in A'$. Thus $g(y) \in \lambda(A)$, $g(x) \in \lambda(A')$, and $\lambda(A) \cap \lambda(A') = \varnothing$. Thus g is one-to-one.

We next claim that if $y \in H(\mathscr{A})$ and if $B \in \mathscr{A}$, then $g(y) \in \lambda(B)$ if and only if $y \in B$. It is obvious from the definition of g that $y \in B$ implies $g(y) \in \lambda(B)$. Conversely, if $y \notin B$ then $y \in B'$ and so $g(y) \in \lambda(B') = \lambda(B)'$; thus $g(y) \notin \lambda(B)$ and our claim is valid.

We now show that g is continuous. It follows from the above remarks that

$$g^{-1}[\lambda(A)] = \{y \in H(\mathscr{A}) : g(y) \in \lambda(A)\} = H(\mathscr{A}) \cap A$$

for each $A \in \mathscr{A}$. Since $\{\lambda(A) : A \in \mathscr{A}\}$ is a base for the closed subsets of $S(\mathscr{A})$ and since $H(\mathscr{A}) \cap A$ is closed in $H(\mathscr{A})$, it follows that g is continuous.

Finally, for each $A \in \mathscr{A}$ we have

$$g[H(\mathscr{A}) \cap A] = \{g(y) : y \in H(\mathscr{A}) \cap A\} = \lambda(A) \cap g[H(\mathscr{A})].$$

Since $\mathscr{A}$ is a basic subalgebra of $R(Y)$, the family $\{H(\mathscr{A}) \cap A : A \in \mathscr{A}\}$, which is identical with $\{H(\mathscr{A}) \cap \operatorname{int}_Y A : A \in \mathscr{A}\}$, is a base for the open sets of $H(\mathscr{A})$, and so g is an open mapping onto its range. It follows that $H(\mathscr{A})$ and $g[H(\mathscr{A})]$ are homeomorphic, and so g is a topological embedding.

Suppose that $y \in H(\mathscr{A})$. Since $g(y) \in \lambda(A)$ if and only if $y \in A$ for each $A \in \mathscr{A}$, it follows that $f(g(y)) = \bigcap \{A \in \mathscr{A} : y \in A\} = y$.

(ii) Since $|\mathscr{A}| \leqq m$, the family $\mathscr{F} = \{Y - \operatorname{bd}_Y A : A \in \mathscr{A}\}$ is a family of not more than m dense open subsets of Y. Since Y is a Baire m-space, the set $H(\mathscr{A}) = \bigcap \mathscr{F}$ is dense in Y. Thus if A is any member of $\mathscr{A}$, it follows that $(\operatorname{int}_Y A) \cap H(\mathscr{A}) \neq \varnothing$. Choose $y \in (\operatorname{int}_Y A) \cap H(\mathscr{A})$; then $g(y) \in \lambda(A) \cap g[H(\mathscr{A})]$, as seen above. As $\{\lambda(A) : A \in \mathscr{A}\}$ is a base for the open subsets of $S(\mathscr{A})$, it follows that $g[H(\mathscr{A})]$ is dense in $S(\mathscr{A})$. Finally, since by (i) the restriction of f to the dense subset $g[H(\mathscr{A})]$ of $S(\mathscr{A})$ is a homeomorphism onto $H(\mathscr{A})$, it follows from [6, 6.11] that

$$f[S(\mathscr{A}) - g[H(\mathscr{A})]] = Y - H(\mathscr{A}).$$

4. **Applications to the remote points of $\beta X - X$.** The following notion is due to Fine and Gillman [4], who proved, assuming the continuum hypothesis, the existence of a set of remote points in $\beta \boldsymbol{R}$ that is dense in $\beta \boldsymbol{R} - \boldsymbol{R}$ ($\boldsymbol{R}$ denotes the reals).

4.1. Definition. A point $p \in \beta X$ is called a *remote point* of βX if p is not in the βX-closure of any discrete subset of X.

Obviously all the remote points of βX lie in $\beta X - X$. We shall denote the set of remote points of βX by $T(X^*)$.

Plank [9] has obtained a number of results concerning the set of remote points of βX when X is a separable metric space without isolated points. The following result comprises a portion of theorems 5.4, 5.5, and 6.2 of [9].

4.2. Theorem. *If X is a locally compact, σ-compact, noncompact metric space without isolated points, then*

$$\begin{aligned} T(X^*) &= \bigcap_{Z \in \boldsymbol{Z}(X)} [(\beta X - X) - (\operatorname{bd}_X Z)^*] \\ &= \bigcap_{Z \in \boldsymbol{Z}(X)} [(\beta X - X) - \operatorname{bd}_{X^*} Z^*]. \end{aligned}$$

Assuming the continuum hypothesis, both $T(X^*)$ and $(\beta X - X) - T(X^*)$ are dense subspaces of $\beta X - X$ of cardinality $2^{\mathfrak{c}}$.

We now combine 3.5 and 4.2 to obtain the following result:

4.3. THEOREM [CH]. *Let X be a locally compact, σ-compact, noncompact metric space without isolated points. Then $T(X^*)$ can be embedded densely in $\beta N-N$.*

Proof. Since X obviously is separable, clearly $|\boldsymbol{Z}(X)|=c$. By 2.17 and 2.20(i) it follows that $[R(X)]^*$ is a basic subalgebra of $R(X^*)$ and is isomorphic to $B(\beta N-N)$. Thus $S([R(X)]^*)$ is homeomorphic to $\beta N-N$, and by 3.5(i) there is an embedding g of $H([R(X)]^*)$ into $\beta N-N$. Since $|[R(X)]^*|=c$ and X^* is a Baire $\aleph_1$-space (3.2), it follows from 3.2(ii) that $g[H([R(X)]^*)]$ is dense in $\beta N-N$. Using 4.2 and recalling that in a metric space $R(X)\subseteq \boldsymbol{Z}(X)$, we have

$$T(X^*) \subseteq \bigcap_{A\in R(X)} [X^*-\mathrm{bd}_{X^*} A^*] = H([R(X)]^*).$$

As $T(X^*)$ is dense in X^* and hence in $H([R(X)]^*)$, it follows that $g[T(X^*)]$ is dense in $g[H([R(X)]^*)]$ and thus in $\beta N-N$. As g is a topological embedding, $g[T(X^*)]$ is homeomorphic to $T(X^*)$ and dense in $\beta N-N$.

If X is a locally compact, σ-compact, noncompact metric space without isolated points, it is obvious from the definitions and the fact that $R(X)\subseteq Z(X)$ that $T(X^*)\subseteq H([R(X)]^*)$. Whether these sets are equal in general is an open question. If $X=\boldsymbol{R}$, we can employ the following lemma, proved independently by ourselves and Mandelker [7, 2.3], to show that $T(\boldsymbol{R}^*)=H([R(\boldsymbol{R})]^*)$.

4.9. LEMMA. *Let K be a closed nowhere dense subset of $\boldsymbol{R}$. Then there exists a regular closed subset A of $\boldsymbol{R}$ such that $K\subseteq \mathrm{bd}_{\boldsymbol{R}} A$.*

Proof. See [7, 2.3].

4.10. THEOREM. $T(\boldsymbol{R}^*)=H([R(\boldsymbol{R})]^*)$.

Proof. Recall (4.2) that $T(\boldsymbol{R}^*)=\bigcap_{Z\in \boldsymbol{Z}(\boldsymbol{R})} [\boldsymbol{R}^*-(\mathrm{bd}_{\boldsymbol{R}} Z)^*]$ and that $H([R(\boldsymbol{R})]^*)= \bigcap_{A\in R(\boldsymbol{R})} [\boldsymbol{R}^*-(\mathrm{bd}_{\boldsymbol{R}} A)^*]$. As $R(\boldsymbol{R})\subseteq \boldsymbol{Z}(\boldsymbol{R})$, obviously $T(\boldsymbol{R}^*)\subseteq H([R(\boldsymbol{R})]^*)$. Conversely, if $Z\in \boldsymbol{Z}(\boldsymbol{R})$ then by 4.9 there exists $A\in R(\boldsymbol{R})$ such that $\mathrm{bd}_{\boldsymbol{R}} Z\subseteq \mathrm{bd}_{\boldsymbol{R}} A$; thus $\boldsymbol{R}^*-(\mathrm{bd}_{\boldsymbol{R}} A)^*\subseteq \boldsymbol{R}^*-(\mathrm{bd}_{\boldsymbol{R}} Z)^*$. Thus $H([R(\boldsymbol{R})]^*)\subseteq T(\boldsymbol{R}^*)$ and the theorem follows.

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