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REPRESENTATIONS FOR TRANSFORMATIONS CONTINUOUS IN THE BV NORM

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Abstract. Riemann and Lebesgue-type integrations can be employed to represent operators on normed function spaces whose norms are not stronger than sup-norm by $T(f)=\int f\,d\mu$ where μ is determined by the action of T on the simple functions. The real-valued absolutely continuous functions on [0, 1] are not in the closure of the simple functions in the BV norm, and hence such an integral representation of an operator is not obtainable. In this paper the authors develop a v-integral whose structure depends on fundamental functions different than simple functions. This integral is as computable as the Riemann integral. By using these fundamental functions, the authors are able to obtain a direct, analytic representation of the linear functionals on AC which are continuous in the BV norm in terms of the v-integral. Further, the v-integral gives a characterization of the dual of AC in terms of the space of fundamentally bounded set functions which are convex with respect to length. This space is isometrically isomorphically identified with the space of Lipschitz functions anchored at zero with the norm given by the Lipschitz constant, which in turn is isometrically isomorphic to L^∞. Hence a natural identification exists between the classical representation and the one given in this paper. The results are extended to the vector setting.

1. **Introduction.** Riemann and Lebesgue integration are intimately related to the sup-norm topology on the function space and to function approximation in the sup-norm topology by step functions and simple functions respectively, where a simple function is a finite sum of scalars times characteristic functions of Lebesgue measurable sets. This close relationship is perpetuated in various abstractions of the integration process, including those given in [2], [3], [4], [11], [12], and [13]. In [3] the authors demonstrated that if a space of functions is contained in the sup-norm closure of a collection of simple functions, then any transformation $T(f)$ which is continuous in a norm weaker than sup-norm has an integral representation. In 1909 F. Riesz [10] showed that the dual of $C[0, 1]$ is BV_0^1 (the functions of bounded variation with $\|f\|_{BV}=V_0^1 f$) via his celebrated Stieltjes integral representation theorem. Obtaining a representation for the functionals on BV_0^1 has been a problem

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of interest since that time. T. H. Hildebrandt [6] has given a representation for the continuous functionals on the class BV_0^1 in terms of a Stieltjes-type integral involving functions defined on a space whose elements are collections of non-overlapping intervals. Using the notation introduced in [6] one can obtain a characterization for the continuous functionals on the absolutely continuous functions as follows: For I_1 and I_2 in $\mathfrak{J}$, define the distance between I_1 and I_2 to be $d(I_1, I_2) = \mu(I_1 \Delta I_2)$, the Lebesgue measure of the symmetric difference. Then β is continuous on the metric space $\mathfrak{J}$ if and only if α is absolutely continuous. Since any metric space is normal, it follows from Theorem 2 in [1, p. 262] that the continuous functionals are characterized by integrals with respect to regular bounded additive measures on the field generated by the closed sets in $\mathfrak{J}$. We remark that one can obtain similar representations for vector-valued functions using the results in [4]. Dunford-Schwartz indicate [1, p. 374] that such characterizations are not entirely satisfactory. Hildebrandt also has given [5] a representation with respect to a more conventional integral for the functionals on BV_0^1 which are continuous in the weak topology, and indicates that by his method a representation can be obtained for the functionals on BV_0^1 continuous in the sup-norm topology. Such a representation is also obtainable from the main theorem in [3]. Furthermore, the work in [3] seems to indicate that the historical type of integral representation theorem cannot be obtained for the functionals on a function space with norm stronger than sup-norm, or at least in a function space with a norm in which the simple functions are not dense. The space BV_0^1 with the variational norm is such a space since for any continuous BV_0^1 function f and any simple function s it follows that $\|f-s\|_{BV} \geqq \max\{V_0^1 f, V_0^1 s\}$.

The technique for obtaining integral type representation theorems has been to consider the action of a functional T on a dense subset of functions (historically the characteristic functions of a certain class of sets) from which a set function is obtained. Integration of a function f with respect to this set function then gives $T(f)$. It is well known that the step functions are dense in the set S of saltus functions [8, p. 219] or break functions [6] of bounded variation in the BV_0^1 norm and we shall show that the closure of the polygonal functions in the BV_0^1 norm is precisely the set AC of absolutely continuous functions. Identification of an appropriate decomposition of a polygonal function into elementary functions leads to the notion of a generalized Stieltjes-type integral of a set function K with respect to a point function f. For brevity we shall denote the integral by $v\int K\cdot df$, and shall refer to it as the v-integral. The characterization of $(S \oplus AC)^*$ is then made in terms of the v-integral and a class of set functions.

Dunford-Schwartz [1, p. 343] give a characterization of AC^* in terms of a Lebesgue integral of the product of the derivative f' of f and a function g from L^∞. The v-integral representation given in this paper enjoys a computability not shared by the above mentioned representation and requires no knowledge of Lebesgue integration. The space of Lipschitz functions on [0, 1] are shown to be isometric

and isomorphic to the collection of set functions which characterize AC^* and then the identification of the g in the Dunford-Schwartz representation with the K in the representation given herein is made. In this process we give a representation of AC^* in terms of an integral involving a Lipschitz function (which is a generalization of the Stieltjes integral and also is a generalization of the integral given by Lane [7]).

Consider the functional $T(f)=f(1)-f(0)$ on BV_0^1. If T is restricted to AC, then the representation given in [1] is $T(f)=L\int_0^1 f'g\,d\mu$ (we suppose $f(0)=0$ for simplicity). The $g\in L^\infty$ associated with T is thus in the equivalence class generated by $g\equiv 1$. If C represents the Cantor function, then $C\in BV_0^1$ but $C\notin AC$. Since $f'g$ is integrable (Lebesgue) for $f\in BV_0^1$ and $g\in L^\infty$, we can consider the action of the integral representation on C and find $T(C)=0$. In fact, for each T on AC, the integral representation in [1] gives $L\int_0^1 C'g=0$. However, the representation given herein for T which maps $f\in AC$ to $f(1)-f(0)$ extends immediately to all of BV_0^1 and gives for each $f\in BV_0^1$ the value $f(1)-f(0)$. We give in the last section of this paper a more complete discussion of this issue and a sufficient condition on the set function K (or on the associated Lipschitz function L) so that $v\int K\cdot df$ generates a continuous map on BV_0^1.

2. **A characterization of** AC. In this section we identify the desired dense set of functions and establish that an appropriate limit with respect to the net of partitions exists. Since $f-g=A$ for some constant A implies $\|f-g\|_{BV}=0$, we shall always choose the representative element f from each class in BV_0^1 so that $f(0)=0$.

2.1. THEOREM. *The space AC of absolutely continuous functions with the BV_0^1 norm is complete and is the closure of the set P of polygonal functions in the BV_0^1 norm.*

Proof. If f is absolutely continuous, then f' is an L^1 function. Hence there exists a sequence s_i of step functions converging to f' in the L^1 norm. Let $p_i(x)=\int_0^x s_i\,d\mu$. Then p_i is polygonal and

$$\|f-p_i\|_{BV} = L\int |(f-p_i)'|\,d\mu = \|f'-s_i\|_{L^1}.$$

Consequently $\langle p_i\rangle$ converges to f in the BV_0^1 norm.

Conversely if $\langle p_i\rangle$ is a Cauchy sequence of polygonal functions which is Cauchy in the BV_0^1 norm, then $\langle p_i'\rangle=\langle s_i\rangle$ is a sequence of L^1 functions Cauchy in the L^1 norm. Thus there exists an L^1 function f' such that $\langle s_i\rangle \underset{L^1}{\rightarrow} f'$, or equivalently $\langle p_i\rangle \underset{BV}{\rightarrow} f$ where f is the indefinite integral $\int_0^x f'\,d\mu$.

It is possible to show that Cauchy sequences $\langle p_i\rangle$ of polygonal functions converge to an AC function with a calculus type argument which does not require any knowledge of L^1. Since such a proof gives insight and shows that the sequence $\langle p_i\rangle$ is ultimately equi-absolutely continuous we next outline such a proof.

Since $\langle p_i \rangle$ is Cauchy in the BV_0^1 norm, it is Cauchy in the weaker sup-norm and the pointwise limit f is continuous. If σ is any partition of [0, 1], then

$$\sum_\sigma |f(x_{i+1})-f(x_i)| \leqq \sum_\sigma [|f(x_{i+1})-p_n(x_{i+1})|+|p_n(x_{i+1})-p_n(x_i)| + |p_n(x_i)-f(x_i)|]. \tag{1}$$

The sum of the first and third terms on the right of (1) can be made small by sufficiently large choice of n. The sum of the middle terms on the right of (1) is less than $\|p_n\|_{BV}$. But $\sup_n \|p_n\|_{BV}$ is finite and hence the left side of (1) is bounded independent of σ, and is an increasing function of σ. Thus $f \in BV$.

Choose M so that $n, m > M$ implies $\|p_n - p_m\|_{BV} < \varepsilon$ and let $\Delta_i f = f(x_{i+1}) - f(x_i)$. Then

$$\sum_\sigma |\Delta_i(f-p_m)| \leqq \sum_\sigma |\Delta_i(f-p_n)| + \sum_\sigma |\Delta_i(p_n-p_m)|, \tag{2}$$

and hence

$$\begin{aligned}\sum_\sigma |\Delta_i(f-p_m)| &\leqq \limsup_n \sum_\sigma |\Delta_i(f-p_n)| + \limsup_n \sum_\sigma |\Delta_i(p_n-p_m)| \\ &\leqq 0 + \limsup_n \|p_n-p_m\|_{BV} \leqq \varepsilon,\end{aligned}$$

and hence $\langle p_n \rangle \xrightarrow[BV]{} f$.

Finally we show that f is absolutely continuous. Given $\varepsilon > 0$, choose M so that $n, m \geqq M$ implies $\|p_n - p_m\|_{BV} < \varepsilon/3$. For p_M there is a δ such that if $\sum |y_i - x_i| < \delta$ then $\sum |p_M(y_i) - p_M(x_i)| < \varepsilon/3$. Hence for any finite collection $\{[x_i, y_i]\}$ of intervals such that $\sum |y_i - x_i| < \delta$ we have

$$\begin{aligned}\sum |f(y_i)-f(x_i)| &= \sum |f(y_i)-p_n(y_i)+p_n(y_i)-p_M(y_i)+p_M(y_i) \\ &\qquad -p_M(x_i)+p_M(x_i)-p_n(x_i)+p_n(x_i)-f(x_i)| \\ &\leqq \sum [|f(y_i)-p_n(y_i)|+|p_n(y_i)-p_M(y_i)+p_M(x_i)-p_n(x_i)| \\ &\qquad +|p_M(y_i)-p_M(x_i)|+|p_n(x_i)-f(x_i)|].\end{aligned} \tag{3}$$

Except for the peculiar grouping of the second term on the right of (3), this is a straightforward decomposition. The sum of all first and fourth terms can be made less than $\varepsilon/3$ for sufficiently large n. The sum of the second terms on the right of (3) is dominated by $\|p_n - p_M\|_{BV} < \varepsilon/3$ and the sum of the third terms on the right of (3) is less than $\varepsilon/3$ because of the absolute continuity of p_M. This shows that $f \in AC$ and it is also clear that if $\langle p_n \rangle$ is a Cauchy sequence of AC functions in the BV_0^1 norm, then for $\varepsilon > 0$ there exists δ and N such that $\sum |y_i - x_i| < \delta$ and $n > N$ implies $\sum |p_n(y_i) - p_n(x_i)| < \varepsilon$.

2.2. THEOREM. *Let pf_σ represent the polygonal function with corners at precisely the points $(x_i, f(x_i))$ for $x_i \in \sigma$. Then $f \in AC$ implies $\lim_\sigma pf_\sigma = f$ in the BV_0^1 norm.*

Proof. Let q_n be a sequence of polygonal functions converging to f in the BV_0^1 norm. Let σ represent the values of x at which q_n has corners. Then $\|pf_\sigma - q_n\|_{BV}$

$\leqq \|f-q_n\|$ since $\sum_\sigma |\Delta_i(pf_\sigma - q_n)|$ is precisely the variation of $pf_\sigma - q_n$ and is also an approximating sum to the value of $\|f-q_n\|_{BV}$ which is obtained as the limit with respect to subdivisions of a nondecreasing function of subdivisions. Thus if $\|f-q_n\|_{BV} < \varepsilon$ we conclude that for σ' beyond σ,

$$\|f-pf_{\sigma'}\|_{BV} \leqq \|f-q_n\|_{BV} + \|q_n - pf_{\sigma'}\|_{BV} \leqq \|f-q_n\|_{BV} + \|q_n - f\|_{BV} < 2\varepsilon.$$

2.3. THEOREM. *Let* $|\sigma|$ *represent* $\max\{|x_{i+1}-x_i|\}$ *for* $x_{i+1}, x_i \in \sigma$. *Then* $f \in AC$ *implies* $\lim_{|\sigma|\to 0} pf_\sigma = f$ *in* BV_0^1 *norm.*

Proof. Let σ_1 be a partition such that σ_2 finer than σ_1 implies $\|pf_{\sigma_2} - pf_{\sigma_1}\|_{BV} < \varepsilon/3$. Let δ_1 be a number such that $\sum |y_i - x_i| < \delta_1$ implies $\sum |f(y_i) - f(x_i)| < \varepsilon/3$. Let N be the number of points in σ_1 and choose σ' to be any partition so that $|\sigma'| = \delta < \delta_1/2N$. Let $\sigma = \sigma' * \sigma_1$, that is, σ is the union of all points x such that $x \in \sigma'$ or $x \in \sigma_1$. Since σ is finer than σ_1, we have $\|pf_\sigma - f\| < \varepsilon/3$. Let $A = \{[x_i, x_{i+1}] \mid x_i, x_{i+1} \in \sigma$ and either $x_i \in \sigma_1$ or $x_{i+1} \in \sigma_1\}$. Then if $[z_i, z_{i+1}] \in A$ we conclude that $|z_{i+1} - z_i| < \delta < \delta_1/2N$, and hence $\sum_A |z_{i+1} - z_i| < \delta_1$. Since $\|pf_\sigma - pf_{\sigma'}\|_{BV} = \sum_A \|pf_\sigma - pf_{\sigma'}\|_{BV} \leqq \sum_A \|pf_\sigma\|_{BV} + \sum_A \|pf_{\sigma'}\|_{BV} < \varepsilon/3 + \varepsilon/3$, we have

$$\|f-pf_{\sigma'}\|_{BV} \leqq \|f-pf_\sigma\|_{BV} + \|pf_\sigma - pf_{\sigma'}\|_{BV} < \varepsilon$$

for any σ' with $|\sigma'| < \delta$.

3. **The integral.** In this section we define and discuss the concepts relevant to the integral involved in the representation theorems.

3.1. DEFINITION. A half open interval $(a, b] \subset (0, 1]$ will be called a fundamental set.

3.2. DEFINITION. Let K be a set function defined on the fundamental sets. The generalized Stieltjes-type integral of the set function K with respect to f (or briefly, the v-integral), denoted by $v\int K \cdot df$ is defined to be

$$v\int K \cdot df = \lim_\sigma \sum_\sigma (f(x_{i+1}) - f(x_i)) K((x_i, x_{i+1}]).$$

3.3. THEOREM. *If f is a continuous function satisfying $f(0) = 0$, then the Riemann integral $R\int f\,dx = v\int K \cdot df$ where $K((a, b]) = 1 - b$.*

Proof. We have $v\int K \cdot df = \lim_\sigma \sum_\sigma (f(x_{i+1}) - f(x_i))(1 - x_{i+1}) = s\int (1-x)\,df$, where the last integral is the Stieltjes integral.

3.4. DEFINITION. The set function K is said to be convex relative to length provided that if the fundamental set H is the union of a finite number of fundamental sets $\{H_i\}_{i=1}^n$, then $K(H) = \sum_{i=1}^n \lambda_i K(H_i)$ where λ_i is the ratio of the length of H_i to the length of H.

3.5. DEFINITION. The set function K is said to be fundamentally bounded if $|K(H)| \leqq B$ for some B and all fundamental sets H. If WK is the least bound, then we shall say WK is the fundamental bound for K.

3.6. EXAMPLE. Define $K((a, b])=0$ if $b \leqq \frac{1}{2}$ or if $a>\frac{1}{2}$ and define $K((\frac{1}{2}, b])=1/(b-\frac{1}{2})$. Finally for $a<\frac{1}{2}<b$ define $K((a, b])=[(b-\frac{1}{2})/(b-a)]K((\frac{1}{2}, b])=1/(b-a)$. Then K is a convex set function which is not fundamentally bounded and for each function whose derivative exists from the right at $\{\frac{1}{2}\}$ we have $v\int K\cdot df=f'(\frac{1}{2}^+)$. Thus $v\int K\cdot df$ generates a continuous linear functional on $C_1[0, 1]$, in the C_1 norm. We remark that Theorem 3.3 contains an example of a fundamentally bounded, but not convex, set function for which $v\int K\cdot df$ exists for a certain set of functions.

3.7. DEFINITION. Let $H=(a, b]$ be a fundamental set. We define the fundamental function

$$\begin{aligned}\Psi_H(t) &= 0 && \text{if } t \leqq a,\\ &= (t-a)/(b-a) && \text{if } a < t < b,\\ &= 1 && \text{if } t \geqq b.\end{aligned}$$

3.8. REMARK. The fundamental functions form a basis for the linear space of polygonal functions which are zero at zero.

4. The integral representation for AC^*.

4.1. LEMMA. *Let K denote a convex set function. Suppose p is a polygonal function and that $\sigma=\{x_i\}$ is the partition determined by the corners of p. If $\sigma'=\{x_{ij}\}$, $x_i<x_{ij}<x_{i+1}$ is a refinement of σ, then*

$$\sum_{\sigma}[p(x_{i+1})-p(x_i)]K(H_i) = \sum_{\sigma'}[p(x_{ij+1})-p(x_{ij})]K(H_{ij}).$$

Hence, it follows that $v\int K\cdot dp=\sum_\sigma \Delta_i pK(H_i)$.

Proof. For each i we show that

$$[p(x_{i+1})-p(x_i)]K(H_i) = \sum_j [p(x_{ij+1})-p(x_{ij})]K(H_{ij}),$$

from which the lemma follows. First observe that p has constant slope on the interval $[x_i, x_{i+1}]$. Hence, for each j we have

$$[p(x_{ij+1})-p(x_{ij})]/[p(x_{i+1})-p(x_i)] = [x_{ij+1}-x_{ij}]/[x_{i+1}-x_i].$$

Since K is convex, then

$$\begin{aligned}K(H_i) &= \sum_j ([x_{ij+1}-x_{ij}]/[x_{i+1}-x_i])K(H_{ij})\\ &= \sum_j ([p(x_{ij+1})-p(x_{ij})]/[p(x_{i+1})-p(x_i)])K(H_{ij}),\end{aligned}$$

from which the above desired equality follows.

4.2. THEOREM. *T is an element of AC^* (the dual of AC with BV norm) if and only if there exists a unique fundamentally bounded set function K which is convex with respect to length such that for $f\in AC$, $T(f)=v\int K\cdot df$. Furthermore if WK is the fundamental bound, then $\|T\|=WK$.*

Proof. For each fundamental set $H \subset (0, 1]$ let Ψ_H denote the corresponding elementary function and define the set function K by $K(H) = T(\Psi_H)$. If $f \in AC$, then from 2.2 we have that pf_σ converges to f in the BV norm. Hence $T(f) = \lim_\sigma T(pf_\sigma) = \lim_\sigma T(\sum_\sigma \Delta_i pf_\sigma \Psi_{H_i}) = \lim_\sigma \sum_\sigma \Delta_i fK(H_i) = v \int K \cdot df$. Further,

$$\|T\| = \sup_{\|f\|_{BV}=1} |T(f)| \geqq \sup_{\Psi_H} |T(\Psi_H)| = \sup |K(H)| = WK,$$

and

$$\begin{aligned}\|T\| &= \sup_{\|f\|_{BV}=1} |T(f)| = \sup_{\|f\|_{BV}=1} \left| v \int K \cdot df \right| = \sup_{\|f\|_{BV}=1} \left| \lim_\sigma \sum_\sigma \Delta_i fK(H_i) \right| \\ &\leqq \sup_{\|f\|_{BV}=1} \lim_\sigma \sum |\Delta_i f| \, |K(H_i)| \leqq \sup_{\|f\|_{BV}=1} \lim_\sigma \sum_\sigma |\Delta_i f| WK \\ &\leqq \sup_{\|f\|_{BV}=1} \|f\|_{BV} WK = WK.\end{aligned}$$

The uniqueness of K follows from 4.1.

Conversely, if K is a convex fundamentally bounded set function then from the definition of the integral we must show $\lim_\sigma \sum_\sigma \Delta_i fK(H_i)$ exists. From 2.1 we see $\sum_\sigma \Delta_i fK(H_i) = \sum_\sigma \Delta_i pf_\sigma K(H_i) = v \int K \cdot dpf_\sigma$. Since

$$\left| v \int K \cdot dpf_\sigma - v \int K \cdot dpf_{\sigma'} \right| = \left| \sum_{\sigma * \sigma'} \Delta_i (pf_\sigma - pf_{\sigma'}) K(H_i) \right| \leqq \|pf_\sigma - pf_{\sigma'}\|_{BV} WK$$

we conclude that $v \int K \cdot dpf_\sigma$ is Cauchy and $v \int K \cdot df$ exists. Furthermore, it is immediate that $|v \int K \cdot df| \leqq \|f\|_{BV} WK$ and hence that if $\langle f_\alpha \rangle$ converges to f in the BV norm, then $v \int K \cdot df_\alpha$ converges to $v \int K \cdot df$. Hence $T(f) = v \int K \cdot df$ is a continuous linear operator (the linearity of the integral follows from considering approximations to the integral).

4.3. Remark. We note that one does not need convexity of K to generate a linear functional from $v \int K \cdot df$. This is somewhat surprising until one realizes that in case of the Lebesgue integral one does not require additivity of a measure μ to generate a linear functional $L \int f \, d\mu$. The point is, once one has generated a linear functional from a bounded set function, then there exists an additive (in the case of Lebesgue) or a convex (in Theorem 4.2) set function which generates the same transformation. Thus in Theorem 3.3 in which the nonconvex set function $K_1((a, b]) = 1 - b$ generated the transformation $R \int f \, dx$, we see that the (unique) convex set function $K_2((a, b]) = 1 - (a+b)/2$ generates the same transformation via $v \int K_2 \cdot df$.

4.4. Remark. From 2.3 we conclude that Definition 3.2, Theorem 3.3, Lemma 4.1 and Theorem 4.2 can be restated in terms of $\lim |\sigma| \to 0$ rather than in terms of $\lim_\sigma$ which is a limit with respect to the net of partitions. Hence the v-integral is as computable as the Riemann integral.

5. **The integral representation for S^*.** If $s \in S$, that is, if s is a saltus function [8, p. 205] or break function [6], then it is easy to see that there exists a sequence

$\langle p_n \rangle$ of step functions converging to s in the BV norm. Hence the representation possible for S^* is in terms of more conventional integrals which resemble Stieltjes integrals and are usually called left Cauchy and right Cauchy integrals. We shall give only a brief sketch of the development for comparison with the v-integral of §§3 and 4, calling attention to [6] for further development. We again assume $s(0)=0$ for each $s \in S$.

5.1. LEMMA. *If $s \in S$ then $s=s_r+s_l$ where s_r is continuous from the right and s_l is continuous from the left at each point of* [0, 1].

5.2. LEMMA. *If s_r and s_l are as in Lemma* 5.1, *then there exists sequences $\langle p_n^r \rangle$ and $\langle p_n^l \rangle$ of step functions continuous from the right and left respectively which converge in the BV norm to s_r and s_l respectively.*

5.3. DEFINITION. If $\Phi_x^r(t)=0$ for $t<x$ and $\Phi_x^r(t)=1$ for $t \geqq x$ then Φ_x^r is said to be a right elementary function. A function $\Phi_x^l(t)$ is defined to be a left elementary function if $\Phi_x^l(t)=0$ for $t \leqq x$ and $\Phi_x^l(t)=1$ for $t>x$.

5.4. LEMMA. *The right and left elementary functions form a basis for the linear spaces of right and left continuous step functions respectively.*

5.5. DEFINITION. A set of points consisting of a single point will be defined to be an elementary set.

5.6. DEFINITION. The right Cauchy integral of a function f with respect to the elementary function (point function) K is defined to be

$$rv \int K \cdot df = \lim_{\sigma} \sum_{\sigma} [f(x_{i+1})-f(x_i)]K(x_i)$$

provided the limit with respect to the net of partitions exists. The left Cauchy integral $lv \int K \cdot df$ is defined to be $\lim_\sigma \sum_\sigma [f(x_{i+1})-f(x_i)]K(x_{i+1})$.

5.7. THEOREM. *T is an element of S^* (the dual of S with BV norm) if and only if there exist two unique bounded point functions K_l and K_r such that for $s \in S$, $T(s)=rv \int K_r\, ds_r + lv \int K_l\, ds_l$. Furthermore if $WK=\max\{\sup K_l(t), \sup K_r(t)\}$, then $\|T\|=WK$.*

Proof. Since $s=s_r+s_l$ and there exist sequences $\langle p_n^r \rangle$ and $\langle p_n^l \rangle$ converging to s_r and s_l respectively, we have

$$\begin{aligned} T(s) &= \lim_n T(p_n^r)+\lim_n T(p_n^l) \\ &= \lim_n \sum_{\sigma_n^r} [p_n^r(x_{i+1})-p_n^r(x_i)]T(\Phi_{x_{i+1}}^r)+\lim_n \sum_{\sigma_n^l} [p_n^l(x_{i+1})-p_n^l(x_i)]T(\Phi_{x_i}^l). \end{aligned}$$

If we define $K_l(t)=T(\Phi_t^l)$ and $K_r(t)=T(\Phi_t^l)$, then $T(s)=\lim_n T(p_n^r)+\lim_n T(p_n^l) = lv \int K_l\, ds_l + rv \int K_r\, ds_r$. The remainder of the proof is straightforward.

5.8. REMARK. A more revealing representation for $T(s)$ can be obtained by representing $s(t)$ as follows:

$$\begin{aligned} s(t) &= \sum_x ((s(x+0)-s(x-0))\phi_x^1(t)+(s(x)-s(x-0))\delta(x, t)) \\ &= \sum_x a(x)\phi_x^1(t)+\sum_x b(x)\delta(x, t) \end{aligned}$$

where $a(x)$ and $b(x)$ are denumerably nonzero and $\delta(x, t)$ is the Kronecker δ function. T. H. Hildebrandt [5] has used such a representation to represent functionals continuous in the weak topology and observes (and we use his notation) that if $g \in BV$, then $\lim_\sigma g_\sigma = g$ in the sup-norm topology. We remark further that if $s \in BV$ is a saltus or break function, then $\lim_\sigma s_\sigma = s$ in the BV norm, and consequently we refer the interested reader to the development in [5] for a more revealing representation theorem than that given in this paper for S^*.

If we denote by B the set of all bounded point functions and if we denote by Ω the set of all fundamentally bounded convex set functions with norms given by the fundamental bound, then $(S \oplus AC)^* = S^* \oplus AC^* = (B \oplus B) \oplus \Omega$. If one desires to consider functions which are not zero at zero then one must form the direct sum of the preceding characterizations with R, where R denoted the real numbers.

We now return to a calculus for $v \int K \cdot df$ defined on $f \in AC$ and make several identifications.

6. **Some consequences of the calculus of the v-integral.** In this section we let $\mathscr{L}$ denote the normed space of Lipschitz functions on [0, 1] which are zero at zero and such that $\|f\|_{\mathscr{L}}$ is given by the Lipschitz constant.

6.1. DEFINITION. Suppose $K \in \Omega$. Then for a K-v-integrable function f define $v \int_a^b K \cdot df = v \int K \cdot df_{ab}$ where f_{ab} is defined by $f_{ab}(x) = f(a)$ for $x \leqq a$, $f_{ab}(x) = f(x)$ for $a \leqq x \leqq b$ and $f_{ab}(x) = f(b)$ for $x \geqq b$.

6.2. THEOREM. *Suppose $f \in AC$ and $K \in \Omega$. Then $v \int_a^b K \cdot df$ exists for $a, b \in [0, 1]$. Furthermore, for $a < b < c$, $v \int_a^c K \cdot df = v \int_a^b K \cdot df + v \int_b^c K \cdot df$.*

Proof. Since $f \in AC$ then $f_{ab} \in AC$. Hence the first result follows from 4.2. Suppose σ is finer than the partition $\{0, a, b, c, 1\}$. Then

$$\sum_{a\,\sigma}^{c} \Delta_i f K(x_i, x_{i+1}] = \sum_{a\,\sigma}^{b} \Delta_i f K(x_i, x_{i+1}] + \sum_{b\,\sigma}^{c} \Delta_i f K(x_i, x_{i+1}].$$

Therefore, the desired result follows by taking the limit over σ.

6.3. THEOREM. *Suppose $f \in AC$ and $K \in \Omega$. Define $F(x) = v \int_0^x K \cdot df$. Then $F(x)$ is AC.*

Proof. Suppose $\varepsilon>0$. Since $f\in AC$ then there is a $\delta>0$ such that if $\{(x_i, x_{i+1})\}$ is a collection of intervals such that $\sum \Delta x_i<\delta$, then $\sum |\Delta_i f|<\varepsilon/WK$. Then

$$\sum |\Delta_i F| = \sum \left| \int_0^{x_{i+1}} K\cdot df - \int_0^{x_i} K\cdot df \right| = \sum \left| \int_{x_i}^{x_{i+1}} K\cdot df \right|$$
$$\leqq \sum WK\|f_{x_i x_{i+1}}\|_{BV} = WK\sum \|f_{x_i x_{i+1}}\|_{BV} \leqq WK\varepsilon/WK = \varepsilon.$$

Hence $F\in AC$.

In the next three theorems we establish the isomorphism between Ω and $\mathscr{L}$.

6.4. LEMMA. *If $K\in\Omega$, then there is a unique Lipschitz function F_K anchored at zero given by $F_K(x)=xK((0, x])$. Furthermore, the Lipschitz constant is WK.*

Proof. Since K is convex with respect to length, $F_K(b)-F_K(a)=(b-a)K((a, b])$. Since K is fundamentally bounded it follows that F_K is Lipschitz, and since WK is the fundamental bound for K it follows that WK is the Lipschitz constant for F_K.

6.5. LEMMA. *If F is a Lipschitz function such that $F(0)=0$ with Lipschitz constant L, then the set function K_F given by $K_F((a, b])=(F(b)-F(a))/(b-a)$ where $a, b\in[0, 1]$ is convex with respect to length and is fundamentally bounded with fundamental bound $WK_F=L$.*

Proof. Since F is Lipschitz it follows that K_F is fundamentally bounded and that $WK_F=L$. We now show that K is convex with respect to length. Suppose $(a, b]\subset[0, 1]$ and that $\{x_i\}$ partition $(a, b]$. Then

$$\sum [\Delta x_i/(b-a)]K_F((x_i, x_{i+1}]) = \sum [\Delta x_i/(b-a)][\Delta_i F/\Delta x_i]$$
$$= (b-a)^{-1}\sum \Delta_i F$$
$$= (F(b)-F(a))/(b-a) = K_F((a, b]).$$

6.6. REMARK. In view of 6.5, an alternate form for the representation theorem 4.2 is $T(f)=\lim_\sigma \sum_\sigma \Delta_i f\Delta_i F/\Delta_i x=\int df\, dF/dx$, the last integral being the Hellinger integral(1).

We summarize in the following theorem.

6.7. THEOREM. *The space Ω is isometrically isomorphic to $\mathscr{L}$ with the isomorphism given by $K\leftrightarrow F_K$. Hence, it follows that AC^* is isometrically isomorphic to $\mathscr{L}$.*

We now relate 4.2 to the classical representation theorem given in [1] and show that, for at least some elements $T\in AC^*$, Theorem 4.2 gives a more delineating representation of the norm preserving extension of T on BV than that of [1]. First we observe the following lemma without proof.

6.8. LEMMA. *The closure of C_1, the continuously differentiable functions on* [0, 1] *anchored at zero, in the BV norm is AC.*

(1) The authors were unaware of the Hellinger integral when the original manuscript was prepared and wish to acknowledge the referee for this contribution.

6.9. THEOREM. *Suppose that $g_n \in L^\infty$, $n=0, 1, 2, \ldots$, and that g_n converges to g_0 in the L^1 norm. Further, suppose $G_n(x)=L\int_0^x g_n$ (Lebesgue integral) for each n. Then for $f \in \mathscr{L}$, $v\int K_{G_n}\cdot df$ converges to $v\int K_{G_0}\cdot df$.*

Proof. Observe that

$$\begin{aligned}\left|v\int K_{G_n}\cdot df - v\int K_{G_0}\cdot df\right| &= \left|\lim_\sigma \sum \Delta_i f[K_{G_n}((x_i, x_{i+1}]) - K_{G_0}((x_i, x_{i+1}])]\right| \\ &= \lim_\sigma \left|\sum \Delta_i f\left[\left(\int_{x_i}^{x_{i+1}} g_n - g_0\right)\Big/\Delta x_i\right]\right| \\ &\leqq \limsup_\sigma \sum |\Delta_i f/\Delta x_i| \int_{x_i}^{x_{i+1}} |g_n - g_0| \\ &\leqq L\int_0^1 |g_n - g_0|\end{aligned}$$

where L is the Lipschitz constant for f. Since g_n converges to g in the L^1 norm, it follows that $v\int K_{G_n}\cdot df$ converges to $v\int K_{G_0}\cdot df$.

6.10. THEOREM. *Suppose $f \in AC$ and $g \in L^\infty$ and suppose $G \in \mathscr{L}$ is given by $G(x)=L\int_0^x g$. Then $v\int K_G\cdot df = L\int f'g$.*

This theorem essentially says that the most natural of all possible relations exists between our representation and that in [1], i.e., the isomorphism between the characterization of AC^* as L^∞ and the characterization of AC^* as Ω isomorphic to $\mathscr{L}$ is given by $g \leftrightarrow T \leftrightarrow K_G \leftrightarrow G = L\int g$.

Proof of 6.10. We first establish 6.10 for the case $f \in C_1$ and $g \in C$. Then $v\int K\cdot df$ is given by

$$\begin{aligned}\lim_\sigma \sum \Delta_i f K_G((x_i, x_{i+1}]) &= \lim_\sigma \sum (\Delta_i f)(\Delta_i G/\Delta x_i) \\ &= \lim_\sigma \sum (\Delta_i f/\Delta x_i)(\Delta G_i/\Delta x_i)\,\Delta x_i \\ &= \lim_\sigma \sum f'(\xi_i) g(\eta_i)\,\Delta x_i\end{aligned}$$

by the Mean Value Theorem, where $x_i < \xi_i$, $\eta_i < x_{i+1}$. But by Bliss' Theorem [9, p. 224] the above limit is the Riemann integral $R\int f'(x)g(x)\,dx$, which establishes the result in this special case.

Next the result is established for $f \in C_1$ and $g \in L^\infty$. There is a bounded sequence $\{g_n\}$ in C which converges to g in the L^1 norm. Then Theorem 6.9 implies $v\int K_{G_n}\cdot df$ converges to $v\int K_G\cdot df$ where $G_n(x)=L\int_0^x g_n$ for each n.

For each n, we know $v\int K_{G_n}\cdot df = L\int f'g_n$ and by the Bounded Convergence Theorem $L\int f'g_n$ converges to $L\int f'g$. Hence, $v\int K_G\cdot df = L\int f'g$.

Finally we establish the full result. We suppose $f \in AC$ and $g \in L^\infty$. Lemma 6.8 implies that there is a sequence $\{f_n\}$ in C_1 which converges to f in the BV norm. Hence, it follows that $\lim_n v\int K_G\cdot df_n = v\int K_G\cdot df$. For each n we have shown that

$v\int K_G\cdot f_n=\int f_n'g$. Since f_n converges to f in the BV norm, it follows that f_n' converges to f' in the L_1 norm. Since for f in L^1 and g in L^∞ we have $|\int fg|<\|g\|_\infty\int|f|$, it follows that $\lim_n L\int f_n'g=L\int f'g$, from which we have $v\int K_G\cdot df=L\int f'g$.

6.11. COROLLARY. *Suppose $f\in AC$ and $K\in\Omega$, and suppose $F(x)=v\int_0^x K\cdot df$. Then, the relation between F and f is given by $F'=f'G_K'$ a.e.*

We close this section with the following development which gives a sufficient condition on a convex set function K such that the v-integral with respect to K of every BV function will exist (Corollary 6.14).

6.12. THEOREM(2). *Suppose $\{K_n\}$ is a sequence of fundamentally bounded set functions which converge to the set function K in the fundamental bound sense. Suppose further that $f\in BV$ which is v-integrable with respect to each K_n. Then K is fundamentally bounded, f is v-integrable with respect to K, and* $\lim_n v\int K_n\cdot df=v\int K\cdot df$.

Proof. Observe that

$$\left\|v\int K_n\cdot df-v\int K_m\cdot df\right\|=\left\|v\int(K_n-K_m)\cdot df\right\|\leqq W(K_n-K_m)\|f\|_{BV}.$$

Hence, $\lim_n v\int K_n\cdot df$ exists. Call the limit V. For $\varepsilon>0$, there is an N such that $n\geqq N$ implies $W(K_n-K)<(\varepsilon/3)\|f\|_{BV}$, and such that $\|V-v\int K_n\cdot df\|<\varepsilon/3$. Choose σ' such that σ finer than σ' implies $\|v\int K_n\cdot df-\sum_\sigma(\Delta_i f)K_n((x_i,x_{i+1}])\|<\varepsilon/3$. Then for σ finer than σ',

$$\begin{aligned}\left\|V-\sum_\sigma(\Delta_i f)K((x_i,x_{i+1}])\right\|&\leqq\left\|V-v\int K_n\cdot df\right\|+\left\|v\int K_n\cdot df-\sum_\sigma(\Delta_i f)K_n((x_i,x_{i+1}])\right\|\\&\quad+\left\|\sum_\sigma(\Delta_i f)K_n((x_i,x_{i+1}])-\sum_\sigma(\Delta_i f)K((x_i,x_{i+1}])\right\|\\&<\varepsilon/3+\varepsilon/3+W(K_n-K)\|f\|_{BV}<\varepsilon.\end{aligned}$$

Hence, the theorem is established.

6.13. THEOREM. *If G is polygonal, the $v\int K_G\cdot df$ exists for every $f\in BV$.*

Proof. Suppose $f\in BV$. Let $\sigma_G=\{a_i\}$ denote the partition of $[0,1]$ determined by G. Then for $\sigma>\sigma_G$, $\sum_\sigma\Delta_i fK_G((x_i,x_{i+1}])=\sum_\sigma(\Delta_i f)(\Delta_i G/\Delta x_i)=\sum_\sigma(\Delta_i f)\alpha_j$ where α_j is the slope of G in the interval (a_j,a_{j+1}), where $[x_i,x_{i+1}]\subset[a_j,a_{j+1}]$. Therefore,

$$\sum_\sigma(\Delta_i f)K_G((x_i,x_{i+1}])=\sum_j(f(a_{j+1})-f(a_j))\alpha_j=v\int K_G\cdot df.$$

Observe that if $K\in\Omega$ and $f\in BV$, then $|\lim_\sigma\sum\Delta_i fK((x_i,x_{i+1}])|\leqq WK\|f\|_{BV}$. Therefore, if $K\in\Omega$ integrates all BV functions in the v-sense, then it defines a linear

(2) The original Theorem 6.12 and proof require that the K_n's be convex. The authors gratefully acknowledge the referee for this improved version.

functional on BV. This observation along with Theorems 6.12 and 6.13 yield the following.

6.14. COROLLARY. *If g is in the sup-norm closure of the step functions, and if $G(x)=L\int_0^x g$, then $T(f)=v\int K_G\cdot df$ exists for every $f\in BV$ and hence $T\in BV^*$.*

7. **The v-integral in the vector-valued setting.** In this section we suppose X is a Banach space and that Y is an LNS, and $B[X, Y]$ is the LNS of bounded operators from X into Y. The variation of an X-valued function f is defined to be $V_0^1 f=\sup_\sigma \sum \|\Delta_i f\|$ and $BV(X)$ denotes the LNS of X-valued functions of bounded variation which are θ_X (the additive identity of X) at 0, with $\|f\|_{BV(X)}=V_0^1 f$. Define the space of X-valued absolutely continuous functions, $AC(X)$, in the natural, analogous fashion. We mean by an X-valued polygonal function a function of the form $p=\sum \Psi_{H_i}\cdot x_i$, where $x_i\in X$ for each i. Let Γ denote the closure of P, the X-valued polygonal functions, in the BV norm. It is not known if $\Gamma=AC$. However, we show $\Gamma\subset AC$ in the following theorem.

7.1. THEOREM. *If $\{p_n\}$ is a sequence of polygonal functions which is Cauchy in the BV norm, then there is a function $f\in AC$ such that $\{p_n\}$ converges to f in the BV norm.*

The proof follows as in the second proof of 2.1.

The following theorem generalizes two theorems from earlier sections. The proof follows as earlier and is therefore omitted.

7.2. THEOREM. (i) *If $f\in\Gamma$, then $\lim_\sigma pf_\sigma=f$ where the convergence is in the BV norm.* (ii) *If K is a $B[X, Y]$-valued set function which is convex with respect to length, $p\in P$, and σ is the partition of* [0, 1] *determined by p, then for $\sigma'>\sigma$,*

$$\sum_{\sigma'} [K((x_i, x_{i+1}])](\Delta_i p) = \sum_{\sigma} [K((x_i, x_{i+1}])](\Delta_i p).$$

Hence, p is v-integrable with respect to K and $v\int K\cdot df=\sum_\sigma [K((x_i, x_{i+1}])](\Delta_i p)$.

7.3. THEOREM. *Suppose K is a fundamentally bounded set function which is convex with respect to length and takes its values in $B[X, Y]$. Then the map $T(f)=v\int K\cdot df$ is a bounded linear operator from Γ into $\bar{Y}$, the completion of Y. Furthermore, $\|T\|=WK$.*

The proof follows as in 4.2.

7.4. THEOREM. *Suppose T is a bounded linear operator from Γ into Y. Then, there is a unique, fundamentally bounded set function K with values in $B[X, Y]$ which is convex with respect to length such that $T(f)=v\int K\cdot df$ for each $f\in\Gamma$. Furthermore, $\|T\|=WK$.*

Proof. Define the set function K on the set of intervals with values in $B[X, Y]$ by $K(H)x=T(\Psi(H)x)$. The remainder of the proof now follows as in 4.2.

In the event Y is complete, then the above two theorems yield the following corollary.

7.5. COROLLARY. *Suppose Y is complete. Then, a linear operator T from Γ into Y is bounded if and only if there is a unique, fundamentally bounded set function K with values in $B[X, Y]$ which is convex with respect to length, such that $T(f)=v\int K\cdot df$ for each $f\in\Gamma$. Furthermore, $\|T\|=WK$.*

An alternative way to define the variation of an X-valued function is $SV_0^1 f=\sup_\sigma\{\sup\{\|\sum_\sigma \alpha_i\Delta_i f\| : |\alpha_i|\leqq 1 \text{ for each } i\}\}$, which is called the semivariation of f. The space of functions of bounded semivariation which evaluate to θ_X at 0, and with norm equal to the semivariation is denoted by BSV. The notion of semi-absolute continuity is defined in a fashion analogous to that of bounded semivariation and SAC denotes the corresponding functions space. Let $S\Gamma$ denote the closure of P in the BSV norm. By a proof similar to that of 7.1 it follows that $S\Gamma\subset SAC$.

7.6. LEMMA. *Suppose $p, p'\in P$ and σ and σ' are the partitions of $[0, 1]$ determined by p and p' respectively. Then $\|p-p'\|_{BSV}=\sup\{\|\sum_{\sigma*\sigma'}\alpha_i\Delta_i(p-p')\| : |\alpha_i|\leqq 1\}$.*

Proof. Let $\{x_i\}=\sigma*\sigma'$. Then

$$\begin{aligned}\|p-p'\|_{BSV} &= \sup\left\{\sup\left\{\left\|\sum_\pi B_j\,\Delta_j(p-p')\right\| : |B_j|\leqq 1\right\} : \pi \text{ is a partition of } [0, 1]\right\}\\ &= \sup\left\{\sup\left\{\left\|\sum_\pi B_j[(x_{ij+1}-x_{ij})/(x_{i+1}-x_i)]\Delta_i(p-p')\right\| : |B_j|\leqq 1\right\} : \right.\\ &\qquad\qquad \left. \pi > \sigma*\sigma'\right\}\\ &= \sup\left\{\left\|\sum_{\sigma*\sigma'}\alpha_i\Delta_i(p-p')\right\| : |\alpha_i|\leqq 1\right\}.\end{aligned}$$

The last equality follows because $|\sum_j B_j[(x_{ij+1}-x_{ij})/(x_{i+1}-x_i)]|\leqq 1$ for $|B_j|\leqq 1$.

Lemma 7.6 allows us to observe that the analogies of 7.2, 7.3, 7.4 can be established in the semivariational setting by essentially the same proofs. The analogous theorem to Corollary 7.5 is stated below for comparison. Observe that for the special case $Y=R$, 7.5 and 7.7 imply that $\Gamma^*=(S\Gamma)^*$.

7.7. THEOREM. *Suppose Y is complete. Then, a linear operator T from $S\Gamma$ into Y is bounded if and only if there is a unique, fundamentally bounded set function K with values in $B[X, Y]$ which is convex with respect to length, such that $T(f)=v\int K\cdot df$ for each $f\in S\Gamma$. Furthermore, $\|T\|=WK$.*

Let $\Omega(B[X, Y])$ denote the space of set functions K which are fundamentally bounded and which are convex with respect to length and with norm $\|K\|=WK$. Let $\mathscr{L}(B[X, Y])$ denote the space of $B[X, Y]$-valued Lipschitz functions f on $[0, 1]$ (i.e., the difference quotients are uniformly bounded) such that $f(0)=\theta_{B[X,Y]}$ with norm $\|f\|_{BV}=L$, the Lipschitz constant of f.

The results obtained in 6.1–6.7 and 6.12–6.14 all carry over to the vector setting and we summarize them in the following theorems.

7.8. THEOREM. *The space $\Omega(B[X, Y])$ is isometrically isomorphic to the space $\mathscr{L}(B[X, Y])$ and the isometry is given by $G \leftrightarrow K_G$. Hence, Γ^* and $(S\Gamma)^*$ are each isometrically isomorphic to $\mathscr{L}(B[X, Y])$.*

7.9. THEOREM. *If $f \in BV$, and $\{K_n\}$ is a sequence of fundamentally bounded set functions which converge to K in the fundamental bound sense, and if f is v-integrable with respect to K_n for each n, then f is v-integrable with respect to K and $v \int K \cdot df = \lim_n v \int K_n \cdot df$.*

7.10. THEOREM. *If G is polygonal, then $v \int K_G \cdot df$ exists for every $f \in BSV$.*

7.11. COROLLARY. *A sufficient condition that $T(f) = v \int K_G \cdot df$ be a bounded linear operator from BV into $\overline{Y}$ is that G be in the closure of the polygonal functions in $\mathscr{L}(B[X, Y])$.*

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