WEAKENING A THEOREM ON DIVIDED POWERS

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Abstract. We show that if a Hopf algebra has finite dimensional primitives and a
primitive lies in arbitrarily long finite sequences of divided powers then it lies in an
infinite sequence of divided powers.

Introduction. K. Newman has shown us a counter-example to [3, Theorem 2, p. 521]. The theorem states that if $H$ is a cocommutative Hopf algebra over a perfect field $k$ of characteristic $p > 0$ and $k$ is the unique simple subcoalgebra of $H$ then a primitive element $x$ of $H$ lies in a sequence of divided powers $1 = x, x, \ldots, x^{p^{n+1}-1}$ if and only if $x$ has coheight $n$, for $n = 0, 1, \ldots, \infty$. (The sequence should be considered infinite if $n = \infty$.) The proof given in [3, p. 524] correctly shows that the existence of the sequence of divided powers implies that $x$ has the desired coheight for all $n$. The proof there also correctly shows that for finite $n$ if $x$ has coheight $n$ then the desired sequence of divided powers exists. The error seems to be the
assertion $K^1 \cong H^1/(\bigcup \ker F^i)$ [3, p. 254, lines 23–24].

Newman’s example shows that $x$ may have infinite coheight—so lies in arbitrarily long finite sequences of divided powers—and yet $x$ lies in no infinite sequence of divided powers.

We show here that with the further assumption that the primitives of $H$ are
finite dimensional then $x$ having infinite coheight implies that $x$ lies in an infinite sequence of divided powers. This result is important because it plays a key role in the proof of Jacobson’s conjecture in [1]. We explain this in more detail at the end of the present paper.

1. Suppose $C$ is a cocommutative coalgebra with a unique simple subcoalgebra
which is one dimensional. We identify this simple subcoalgebra with $k$ and so consider $k \subset C$. Then the primitive elements of $C$, $P(C) = \{c \in C | \Delta c = 1 \otimes c + c \otimes 1\}$.

Heyneman’s Theorem. If $\dim P(C) < \infty$ then $C$ satisfies the minimum condition and descending chain condition for subcoalgebras.

Proof. For a vector space $U$ the coalgebra $Sh(U)$ is defined in [4, p. 244, p. 254]. By [4, p. 254, 12.1.1] there is a finite-dimensional space $U$ and an injective coalgebra
map $F: C \to Sh(U)$. In [4, p. 261] $B(U)$ is defined as the maximal cocommutative subcoalgebra of $Sh(U)$. Since $C$ is cocommutative $\text{Im } F \subseteq B(U)$ and we consider $F$ as a coalgebra injection $C \hookrightarrow B(U)$. Since $F$ is injective by applying $F$ to the coalgebras in a family of subcoalgebras of $C$ we see that it suffices to show that $B(U)$ has the minimal condition on subcoalgebras. Similarly for the descending chain condition.

By [4, p. 278, Example-Exercise] the linear dual $\text{Hom}_k(B(U), k)$ is a power-series ring in $\text{dim } U$ variables; hence, is Noetherian. By [4, p. 16, 1.4.3] if $D$ is a subcoalgebra of $B(U)$ then $D^1 = \{ f \in \text{Hom}_k(B(U), k) \mid f(D) = 0 \}$ is an ideal in $\text{Hom}_k(B(U), k)$. Thus the maximal condition and ascending chain condition for ideals in $\text{Hom}_k(B(U), k)$ implies the desired conditions for $B(U)$. Q.E.D.

Theorem 2. Let $k$ be a perfect field of characteristic $p > 0$ and $H$ a cocommutative Hopf algebra over $k$ where $k$ is the unique simple subcoalgebra of $H$. Suppose $^1x$ is a primitive element of $H$.

(i) If $l = 0x, 1x, 2x, \ldots, p^{n+1}_x = 1x$ is a sequence of divided powers in $H$ then $^1x$ has coheight $n$, for $n = 0, 1, \ldots, \infty$.

(ii) If $^1x$ has coheight $n$ then there is a sequence of divided powers in $H$.

(iii) If $\text{dim } P(H) < \infty$ and $^1x$ has infinite coheight then there is an infinite sequence of divided powers in $H$, $\{x\}^\infty_{i=0}$.

Proof. In [3, p. 524, proof of Theorem 2] statements 1 and 2 are correctly proved. In [3, p. 520, Theorem 1] the $V$ map $V: H \to H$ is defined. By [3, p. 521, Theorem 1] $V^n(V) = V \cdots V(H)$ (n-times) is a sub-Hopf algebra of $H$, thus a subcoalgebra. Assuming $\text{dim } P(H) < \infty$ the descending chain $H \supseteq V(H) \supseteq V^2(H) \cdots$ must stabilize say at $V^n(H)$. Thus $V^n(H)$. Since $^1x$ has infinite coheight it follows that $^1x \in V^n(H)$. Thus $l = 0x, 1x, \ldots, p^{n+1}_x$ may be extended to an infinite sequence of divided powers lying in $V^n(H)$ by [3, p. 522, Lemma 7]. Q.E.D.

The statement just above [1, p. 285, 3.3.4] that $(PH)_\infty = \{x \in PH \mid$ there is an infinite sequence of divided powers lying over $x\}$ is false (by Newman's example) unless we have $\text{dim } PH < \infty$, so Theorem 2 (this paper) applies. Thus the proof given for [1, p. 285, 3.3.4] is incorrect unless one assumes that both $PH$ and $PJ$ are finite dimensional. The use made of 3.3.4 is [1, p. 290, 3 lines above 3.5] in the proof of 3.5.3, Jacobson's conjecture, where both $PH$ and $PJ$ are finite dimensional. (Actually it can be shown by techniques developed by Newman that 3.3.4 is correct as it stands.)

Bibliography


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