PERIODIC POINTS AND MEASURES FOR AXIOM A
DIFFEOMORPHISMS

BY
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1. Introduction. We shall study the distribution of periodic points for a class of
diffeomorphisms defined by Smale [16, §1.6].

We recall some of the definitions. Let \( f: M \to M \) be a diffeomorphism of a
compact manifold. A point \( x \in M \) is wandering under \( f \) if it has a neighbourhood
\( U \) such that \( U \cap \bigcup_{n \neq 0} f^n(U) = \emptyset \); the set of other (i.e. nonwandering points)
forms the nonwandering set \( \Omega(f) \) which is closed and \( f \)-invariant. One sees that all
periodic points of \( f \) are in \( \Omega(f) \) and that any finite \( f \)-invariant measure on \( M \) has its
support in \( \Omega(f) \). A closed \( f \)-invariant subset \( \Lambda \) of \( M \) is hyperbolic under \( f \) if the
tangent bundle of \( M \) restricted to \( \Lambda \), \( T_\Lambda(M) \), has a continuous splitting \( T_\Lambda(M) = E^s + E^u \)
which is invariant under \( Df \) and such that \( Df: E^s \to E^s \) is contracting
and \( Df: E^u \to E^u \) is expanding (see [16, p. 758] for the meaning of these terms).

\( f \) satisfies Axiom A if

(Aa) \( \Omega(f) \) is hyperbolic and
(Ab) the periodic points of \( f \) are dense in \( \Omega(f) \).

Smale’s Spectral Decomposition Theorem [16, p. 777] states that for such an \( f \)
we can write \( \Omega(f) = \Omega_1 \cup \cdots \cup \Omega_r \) where the \( \Omega_i \) are disjoint closed \( f \)-invariant sets
and \( f|\Omega_i \) is topologically transitive (the \( \Omega_i \) are called basic sets). Our main result is
that the periodic points of \( f|\Omega_i \) have a definite limiting distribution as the period
becomes large; this distribution is given by a measure \( \mu_f \) on \( \Omega_i \). In the algebraic
case \( \mu_f \) turns out to be Haar measure.

We show that \( \mu_f \) is ergodic, positive on open sets and zero on points (unless \( \Omega_i \)
is finite). In a subsequent paper [7] it is shown that \( (f|\Omega_i, \mu_f) \) is a K-automorphism
in the C-dense case (in fact that it is isomorphic to a Markov chain) and that \( \mu_f \)
is the unique invariant normalized Borel measure on \( \Omega_i \) which maximizes entropy.

The Russian school has done much work on the measure theoretic aspects of
Anosov diffeomorphisms (i.e. all of \( M \) hyperbolic under \( f \)); as a sampling we refer
the reader to the papers [2], [14] and [15]. We also mention the papers [3], [9] and
[11] where various measures are constructed for expanding maps; our methods are
easily modified to give results along this direction also.

We now sketch our construction of \( \mu_f \). First we decompose \( \Omega_i = X_1 \cup \cdots \cup X_m \)
into disjoint closed pieces \( X_j \) such that \( f(X_j) = X_{j+1} \) and \( f^m|X_j: X_j \to X_j \) is C-dense
for all \( 1 \leq j \leq m \). We do not define C-density here but it implies topological mixing

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and the existence of periodic points of all sufficiently large periods; for Markov chains this is the well-known decomposition into transitive pieces.

One then restricts attention to the $C$-dense case; i.e. assume $f: \Omega_i \to \Omega_i$ is $C$-dense. What we want is a measure $\mu_f$ such that (letting $N_n(E)$ be the number of fixed points of $f^n$ lying in $E$)

$$N_n(E)/N_n(\Omega_i) \to \mu_f(E)$$

as $n \to \infty$ for many subsets $E$ of $\Omega_i$ (we save precision for later). A priori we do not know that such a limit exists; using a diagonalization process we can choose sequences of integers $\{n_k\}$ and measures $\mu_{f, (n_k)}$ such that

$$N_{n_k}(E)/N_{n_k}(\Omega_i) \to \mu_{f, (n_k)}(E)$$

for many $E \subseteq \Omega_i$. We then show that all these measures $\mu_{f, (n_k)}$ are ergodic and equivalent; the Radon-Nikodym theorem tells us that they are all equal. When enough subsequences converge to a common limit, the sequence itself converges. Thus we get our desired $N_n(E)/N_n(\Omega_i) \to \mu_f(E)$.

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2. **Axiom A* and C-density.** Let $g: M \to M$ be a diffeomorphism satisfying Smale’s Axiom A. Let $X = \Omega(g) \subseteq M$ and $f = g|X$. Define, for $x \in X = \Omega(g)$ and $\delta > 0$,

- $W^s_\delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta$ for all $n \geq 0\}$.
- $W^u_\delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta$ for all $n \leq 0\}$.
- $W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0$ as $n \to +\infty\}$.
- $W^u(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0$ as $n \to -\infty\}$.

Then (Smale [16, pp. 780–782] and Hirsch and Pugh [10]) the following are true:

A1. The periodic points of $f$ are dense in $X$.

A2. For each $\delta > 0$ there is an $\epsilon(\delta) > 0$ such that $W^s_\delta(x) \cap W^u_\delta(z) \neq \emptyset$ whenever $d(x, z) \leq \epsilon(\delta)$.

A3. There are $\delta^* > 0$, $0 < \lambda < 1$ and $c \geq 1$ such that for all $n \geq 0$,

$$d(f^n(x), f^n(y)) \leq c\lambda^nd(x, y) \quad \text{if } y \in W^s_\delta(x)$$

and

$$d(f^{-n}(x), f^{-n}(y)) \leq c\lambda^nd(x, y) \quad \text{if } y \in W^u_\delta(x).$$

The above statements are about $f$ and do not refer to $g$ or $M$. Any homeomorphism $f$ of a compact metric space $(X, d)$ we shall say satisfies Axiom A* provided that A1, A2, and A3 hold.

(2.1) **Standing hypothesis.** We shall assume throughout the remainder of the paper that $f: X \to X$ is a homeomorphism satisfying Axiom A*.
(2.2) Easy facts. (i) $f^n W^u(x) = W^u(f^n(x))$.
(ii) For $n \geq 0$, $f^{-n} W^s_\delta(x) \subseteq W^s_\delta(f^{-n}(x))$.
(iii) If $y \in W^s_\delta(x)$, then $W^s_\delta(y) \subseteq W^s_{\delta_1 + \delta_2}(x)$.
(iv) Let $f^m(x) = x$ and $\delta \leq \delta^*$. Then $f^{m(k+1)} W^s_\delta(x) \supseteq f^{mk} W^s_\delta(x)$ and (by A3)
\[ W^u(x) = \bigcup_{k=0}^{\infty} f^{mk} W^s_\delta(x). \]

The following fact is due to S. Smale and M. Shub:

(2.3) Lemma [6]. $\delta^*$ is an expansive constant for $f$ (i.e. if $x \neq y$, then $d(f^n(x), f^n(y)) > \delta^*$ for some $n \in \mathbb{Z}$).

(2.4) Lemma. For any $\varepsilon > 0$ there is a $D(\varepsilon)$ so that $d(x, y) < \varepsilon$ whenever $d(f^n(x), f^n(y)) \leq \delta^*$ for all $|n| \leq D(\varepsilon)$.

**Proof.** This is a property of expansive homeomorphisms [18].

(2.5) Periodic point construction. For any $\varepsilon > 0$ there are $\psi(\varepsilon) > 0$ and $R(\varepsilon)$ such that, if $m \geq R(\varepsilon)$ and $d(f^m(y), y) \leq \psi(\varepsilon)$, there is a point $z \in X$ with $f^m(z) = z$ and $d(f^k(z), f^k(y)) \leq \varepsilon$ for all $0 \leq k \leq m$.

**Proof.** This is a translation of [6, Proposition 3.5] using [6, 3.4(h)].

(2.6) Definition. $f$ (satisfying Axiom A*) is $C$-dense if $W^u(p)$ is dense in $X$ for every periodic point $p \in X$.

We permute ideas of Smale [16, pp. 780–782] to obtain

(2.7) $C$-DENSITY DECOMPOSITION THEOREM. $X = X_1 \cup \cdots \cup X_m$ where the $X_i$ are disjoint closed sets, $f(X_i) = X_{g(i)}$ where $g$ is a permutation of $(1, \ldots, m)$, and $f^*: X_i \to X_i$ is $C$-dense when $g^*(i) = i$.

**Proof.** For $p$ a periodic point let $X(p) = \text{Cl}(W^u(p))$.

(a) $X(p)$ is open.

**Proof.** Let $a = \varepsilon(\delta^*)$. We show that
\[ X(p) \supset B_a(X(p)) = \{y \in X : d(y, X(p)) < a\}. \]
Since $X(p)$ is closed, it suffices to show that periodic $q \in B_a(X(p))$ are in $X(p)$ because of A1. Let $x \in W^u(p)$ with $d(x, q) < a$ and set $M = \text{ord}_p \cdot \text{ord}_q$. By A2 choose $z \in W^s_\delta(x) \cap W^s_\delta(q)$. Then $z \in W^u(p)$ and
\[ d(f^M(z), q) = d(f^M(z), f^M(q)) \to 0 \quad \text{as} \ k \to +\infty. \]
Since $f^M W^u(p) \subseteq W^u(p)$, we get $q \in \text{Cl}(W^u(p)) = X(p)$. (Note: We use 2.1 without explicit mention.)

(b) $X(p) = X(q)$ or $X(p) \cap X(q) = \emptyset$.

**Proof.** Suppose $z \in X(p) \cap X(q)$. By (a) $X(p)$ is a neighborhood of $z$ and so there is a $w \in W^u(q) \cap X(p)$. Let $M = \text{ord}_q \cdot \text{ord}_q$. Then as $k \to +\infty$, $f^{-kM}(w) \to q$. But $f^{-kM} X(p) = X(p)$ since $f^{-kM} W^u(p) = W^u(p)$. Thus $q \in \text{Cl}(X(p)) = X(p)$. By (a) we have $X(p) \supset W^u_\delta(q)$. Since
\[ W^u(q) \subseteq \bigcup_{k=0}^{\infty} f^{kM} W^u_\delta(q), \]

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and \( f^n X(p) = X(p) \), we get \( W^n(q) \subseteq X(p) \). Hence \( X(q) \subseteq X(p) \). Symmetrically \( X(p) \subseteq X(q) \).

Now by compactness, let \( X = X(p_1) \cup \cdots \cup X(p_n) \) with \( X(p_i) \neq X(p_j) \) for \( i \neq j \). Set \( X_i = X(p_i) \) and define \( g \) by \( f(p_i) \in X_{p_{i0}} \). That \( f \) is a homeomorphism and (c) below show that \( g \) is a permutation.

(c) \( f(X_i) = X_{p_{i0}} \).

**Proof.** As \( f \) is a homeomorphism, \( f X(p_i) = X(f(p_i)) \) follows from \( f W^u(p_i) = W^u(f(p_i)) \). Since \( f(p_i) \in X(f(p_i)) \), \( X(f(p_i)) = X(p_{i0}) \) by (b).

(d) If \( g(i) = i \), then \( f^e : X \to X \) is C-dense.

**Proof.** Suppose \( p \in X \) is periodic. It is an easy exercise to check that \( W^u_f(p) = W^u_f(r) \). Note that \( f^e : X \to X \) satisfies Axiom A* whenever \( f : X \to X \) does.

(2.8) LEMMA. Let \( f : X \to X \) be C-dense and \( \alpha > 0 \). Then there is an \( N \) such that \( f^n W^u_{\alpha}(x) \cap W^u_{\alpha}(y) \neq \emptyset \) whenever \( x, y \in X \) and \( m \geq N \).

**Proof.** Set \( \delta = \min \{ \delta^*, \frac{1}{4\alpha}, \frac{1}{4\epsilon(\alpha)} \} \) and choose \( p_1, \ldots, p_r \), periodic such that every \( x \in X \) is within \( \frac{1}{2\epsilon} \) of some \( p_\nu \). Let \( t_k \) be the period of \( p_k \). By 2.2 and \( \text{Cl}(W^u(p_i)) = X \), there is an \( m_k \) such that every \( y \in X \) is within \( \epsilon(\delta) \) of \( f^m W^u(p_k) \) for \( m \geq m_k \). Let \( N = (m_1 t_1) \cdots (m_r t_r) \). Then \( d(y, f^n W^u(p_k)) \leq \epsilon(\delta) \) for all \( k \) and all \( y \in X \).

Suppose \( x, y \in X \). Then \( d(x, p_j) < \frac{1}{4\epsilon(\alpha)} \) for some \( j \) and \( d(y, z) \leq \epsilon(\delta) \) for some \( z \in f^n W^u_{\alpha}(p_j) \). Let \( w \in W^u_{\alpha}(z) \cap W^u_{\alpha}(y) \). Then \( f^{-n}(w) \in W^u_{\alpha}(f^{-n}(z)) \subseteq W^u_{\alpha}(p_j) \) and \( d(f^{-n}(w), p_j) \leq \frac{1}{4\epsilon(\alpha)} \); thus \( d(f^{-n}(w), x) \leq \epsilon(\alpha) \) and there is a \( v \in W_{\alpha}(f^{-n}(w)) \cap W^u_{\alpha}(x) \). Then \( f^n(v) \in f^n W^u_\alpha(x) \) and \( f^n(v) \in W_{\alpha}(w) \subseteq W^u_{\alpha}(y) \). Therefore \( f^n W^u_{\alpha}(x) \cap W^u_{\alpha}(y) \neq \emptyset \), \( \forall x, y \in X \). If \( m \geq N \), then

\[
f^m W^u_{\alpha}(x) \cap W^u_{\alpha}(y) \supseteq f^m W^u_{\alpha}(f^{-n}(w)) \cap W^u_{\alpha}(x) \neq \emptyset.
\]

(2.9) DEFINITIONS. Let \( \text{Per}_n(U) = \{ x \in U : f^n(x) = x \} \), \( N_n(U) = \text{card} (\text{Per}_n(U)) \), and \( N_n(f) = N_n(X) \).

A \( G \)-time is a finite collection \( \tau = \{ I_1, \ldots, I_m \} \) of disjoint (finite) intervals of integers. We let \( \text{Tim}(\tau) = \bigcup_{I_i} I_i \), \( T(\tau) = \text{card}(\text{Tim}(\tau)) \), and \( L(\tau) \) be the length of the shortest interval containing \( \text{Tim}(\tau) \). A map \( P : \text{Tim}(\tau) \to X \) is \( (f, \tau) \)-admissible if \( f^{t_{I_i}} - 1 \), \( P(t_1) = P(t_2) \) whenever \( t_1, t_2 \in I \) (i.e. \( P(I) \) is part of an \( f \)-orbit). A specification is a pair \( \tau = (\tau, P) \) with \( \tau \) a \( G \)-time and \( P \) an \( (f, \tau) \)-admissible map; set \( L(\tau) = L(\tau) \) and \( \text{Tim}(\tau) = \text{Tim}(\tau) \); we also write sometimes \( \tau = \tau(s) \) or \( P = P_s \). For \( n \geq 0 \) we say that \( \tau \) is \( n \)-delayed if there is an interval of length at least \( n \) between every pair of intervals belonging to \( \tau \); \( s \) is \( n \)-delayed if \( \tau(s) \) is. Notice that while \( \text{Tim}(\tau) \) does not determine \( \tau \), it does if \( \tau \) is \( n \)-delayed with \( n > 0 \).

Finally, for \( \epsilon > 0 \), let

\[
U(\tau, \epsilon) = \{ x \in X : d(f^t(x), P_s(t) < \epsilon \text{ for all } t \in \text{Tim}(\tau) \}.
\]

(2.10) THEOREM. Suppose \( f : X \to X \) is C-dense and \( \epsilon > 0 \). There is an \( M(\epsilon) \) such that \( U(s, \epsilon) \neq \emptyset \) whenever \( s \) is an \( M(\epsilon) \)-delayed \( f \)-specification. In fact \( M(\epsilon) \) can be chosen so that \( \text{Per}_d U(s, \epsilon) \neq \emptyset \) for all \( d \geq M(\epsilon) + L(s) \).

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**Proof.** We tend $s$ to a new specification $s'$ as follows. Let $a_1$ be the smallest integer in $\text{Tim}(s)$. Set $\tau(s') = \tau(s) \cup \{a_1 + d\}$ and define $P_{s'}$ by $P_{s'}(a_1 + d) = P_{s}(a_1)$ and $P_{s'}|\text{Tim}(s) = P_{s}$.

Set $\beta = \frac{1}{2} \min \{\phi(\frac{1}{2}e), \varepsilon, \delta^*\}$ ($\psi$ defined in 2.5) and $\alpha = \beta/3c$; let $N$ be the integer given by 2.8 for this $\alpha$. Choose $M = M(e) \geq \max \{N, R(\frac{1}{2}e)\}$ ($R$ defined in 2.5) large enough so that $\sum_{n=0}^{\infty} \lambda^M < 2$. Assume $d \geq M(e) + L(s)$; then $s'$ is $M$-delayed.

Let $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots, I_m = [a_m, b_m] = \{a_1 + d\}$ be the members of $\tau(s')$ in their natural order. We set $z_1 = x_1$ and define $z_k$ (for $1 \leq k \leq m$) recursively as follows. Suppose $z_k$ has been chosen for some $1 \leq k < m$. As $s^1$ is $M$-delayed, $a_{k+1} - b_k > M \geq N$ and so by 2.8 there exists a point

$$v \in f^{a_{k+1} - b_k}W_{c\alpha}^n(f^{b_k}(z_k)) \cap W_{\alpha}^n(P_{s'}(a_{k+1})).$$

Set $z_{k+1} = f^{a_{k+1} - b_k}(v)$; then $f^{b_k}(z_{k+1}) \in W_{c\alpha}^n(f^{b_k}(z_k))$ and $f^{a_{k+1} - b_k}(z_{k+1}) \in W_{\alpha}^n(P_{s'}(a_{k+1})).$

By induction on $r$ we show that

$$f^{a_k}(z_{k+r+1}) \in W_{c\alpha + \cdots + c\alpha}^{M^r}(f^{b_k}(z_{k+r})).$$

(Here we use A3: If $x \in W_{\alpha}^n(y)$, then $d(f^{-n}(x), f^{-n}(y)) \leq c\alpha^n$ for $n \geq 0$ and so $f^{-n}(x) \in W_{2\alpha}^n(f^{-n}(y))$ for $m \geq 0$.) Applying (*) and our inductive hypothesis, it follows that (see 2.2(ii))

$$f^{b_k}(z_{k+r+1}) \in W_{c\alpha + \cdots + c\alpha}^{M^r}(f^{b_k}(z_k))$$

and so our induction is done.

Since $\sum_{n=0}^{\infty} \lambda^M < 2$ and $\alpha = \beta/3c$ we have $f^n(z_m) \in W_{2\alpha}^{M^r}(f^{b_k}(z_k))$ and $d(f^n(z_m), f^n(z_k)) < 2\beta/3$ for any $t \in I_k$ and any $k \in [1, m]$. Since $f^n(z_k) \in W_{\alpha}^{M}(P_{s'}(a_k))$ (by the definition of the $z_k$'s) we have

$$\beta/3 \geq \alpha \geq d(f^n(z_k), f^{a_k}(P_{s'}(a_k))) = d(f^n(z_k), P_{s'}(t))$$

for any $t \in I_k$. Combining inequalities,

$$d(f^n(z_m), P_{s'}(t)) < \beta \quad \text{for all } t \in \text{Tim}(s^1).$$

Thus $z_m \in U(s^1, \beta)$.

Now let $z^* = f^{a_1}(z_m)$. Then $z^*, f^d(z^*) \in B_{\beta}(P(a_1))$, and so $d(z^*, f^d(z^*)) \leq \psi(\frac{1}{2}e)$. Now $d > M(e) \geq R(\frac{1}{2}e)$ and by 2.5 there is a $z \in \text{Per}_d(X)$ with

$$d(f^t(z), f^t(z^*)) \leq \frac{1}{2}e \quad \text{for all } 0 \leq t \leq d.$$

Letting $z^t = f^{-a_1}(z)$ we get

$$d(f^t(z^t), f^t(z_m)) \leq \frac{1}{2}e \quad \text{for all } a_1 \leq t \leq a_1 + d.$$
Applying the triangle inequality to this and \( z_m \in U(s^1, \beta) \),

\[
z^1 \in U(s^1, \beta + \frac{1}{k} \varepsilon) \subseteq U(s^1, \varepsilon) \subseteq U(s, \varepsilon);
\]
also \( z^1 \in \text{Per}_d(X) \).

(2.11) Remark. The above theorem is a statement about the freedom one has in specifying the approximate orbit of a periodic point. The remainder of this paper shall be derived from this freedom (together with expansiveness).

3. Counting. Throughout this section \( f : X \to X \) is a \( C\)-dense map.

(3.1) Definition. For \( \varepsilon > 0 \), \( E \subseteq X \) is an \( (n, \varepsilon) \)-separated set if for any distinct \( x, y \in E \) there is a \( t \) for which \( 0 \leq t < n \) and \( d(f^t(x), f^t(y)) > \varepsilon \). We let \( N(n, \varepsilon) \) denote the maximum cardinality of an \( (n, \varepsilon) \)-separated set.

(3.2) Lemma. (i) If \( \varepsilon \leq \delta^\ast \), then \( N(n, \varepsilon) \geq N(n, \delta) \).

(ii) If \( \varepsilon \leq \alpha \), then \( N(n, \alpha) \leq N(n, \varepsilon) \); for any \( \varepsilon > 0 \) there is a \( m_\varepsilon \) such that \( N(n, \varepsilon) \leq N(n + m_\varepsilon, \delta^\ast) \) for all \( n \geq 0 \).

(iii) \( N(\sum n_i, \varepsilon) \leq \prod N(n_i, \tfrac{\varepsilon}{2}) \).

Proof. (i) By 2.3 \( \varepsilon \) is an expansive constant; i.e. if \( p \neq q \), then \( d(f^t(p), f^t(q)) > \varepsilon \) for some \( t \). If \( p, q \in \text{Per}_n(X) \), then \( t \) can be chosen so that \( 0 \leq t < n \); i.e. \( \text{Per}_n(X) \) is \( (n, \varepsilon) \)-separated.

(ii) The first statement is obvious; if \( E \) is an \( (n, \varepsilon) \)-separated set, then \( f^{-D(\varepsilon)}E \) is an \( (n + 2D(\varepsilon), \delta^\ast) \)-separated set (use 2.4).

(iii) We prove the following stronger statement for later use: Suppose \( E \subseteq X \) and \( n_i, m_i (1 \leq i \leq s) \) are integers \( (n_i > 0) \) such that, when \( x, y \in E \) and \( x \neq y \), there is a \( t \in \bigcup_{i=1}^s [m_i, m_i + n_i) \) for which \( d(f^t(x), f^t(y)) > \varepsilon \); then \( \text{card}(E) \leq \prod N(n_i, \tfrac{\varepsilon}{2}) \).

Proof. Choose \( R_i \subset X \) so that \( f^m R_i \) is a maximal \( (m_i, \tfrac{\varepsilon}{2}) \)-separated set. Construct a map \( g = \prod g_i : E \to \prod R_i \) by requiring that \( d(f^t(x), f^t(g_i(x))) \leq \frac{\varepsilon}{2} \) for all \( t \in [m_i, m_i + n_i) \). Such a \( g_i(x) \) exists by the maximality of \( f^m R_i \)—otherwise \( f^m(R_i) \) would be an \( (n, \frac{\varepsilon}{2}) \)-separated set.

If \( g(x) = g(y) \) the triangle inequality would give us \( d(f^t(x), f^t(y)) \leq \varepsilon \) for all \( t \in \bigcup [m_i, m_i + n_i) \); thus \( g \) is injective and we are done.

Two specifications \( s \) and \( s^1 \) are \( p \)-separated if \( d(P_s(t), P_{s^1}(t)) > p \) for some \( t \in \text{Tim}(s) \cap \text{Tim}(s^1) \); a set of specifications is \( p \)-separated if every two members are. An \( S \)-set \( A \) is a set of specifications with the same \( G \)-time; let \( \tau(A) \) denote this common \( G \)-time, \( T(A) = T(\tau(A)) \), \( L(A) = L(\tau(A)) \), and \( U(A, \varepsilon) = \bigcup_{s \in A} U(s, \varepsilon) \).

3.3 Lemma. (i) If \( s \) and \( s^1 \) are \( p \)-separated, then \( U(s, \frac{1}{2}p) \cap U(s^1, \frac{1}{2}p) = \emptyset \).

(ii) If \( A \) is a \( 2\varepsilon \)-separated \( S \)-set, \( \tau(A) \) is \( M(\varepsilon) \)-delayed, and \( d \geq L(A) + M(\varepsilon) \), then \( N_d(U(A, \varepsilon)) \geq \text{card}(A) \).

Proof. (i) Trivial. (ii) Follows from (i) and 2.10.

Two specifications \( s \) and \( s^1 \) are disjoint if \( \text{Tim}(s) \cap \text{Tim}(s^1) = \emptyset \). In this case we define a new specification \( s \wedge s^1 \) by \( \tau(s \wedge s^1) = \tau(s) \cup \tau(s^1) \) and

\[
P_{s \wedge s^1}(t) = P_s(t) \quad \text{for} \ t \in \text{Tim}(s),
\]

\[
P_{s \wedge s^1}(t) = P_{s^1}(t) \quad \text{for} \ t \in \text{Tim}(s^1).
\]
Notice that \( U(s \land s^1, e) = U(s, e) \cap U(s^1, e) \). We call a \( G \)-time \( \tau \) an \( m \)-time if \( \text{card } \tau = m \); \( s \) is an \( m \)-specification if \( \tau(s) \) is an \( m \)-time.

(3.4) **Lemma.** If \( \tau \) is an \( n \)-delayed \( m \)-time and \( N \geq L(\tau) \), there is a \( \tau^1 \) such that

(a) \( \text{Tim } (\tau) \cap \text{Tim } (\tau^1) = \emptyset \),
(b) \( \tau \cup \tau^1 \) is \( n \)-delayed,
(c) \( L(\tau \cup \tau^1) \leq N \), and
(d) \( T(\tau^1) \geq N - 2mn - T(\tau) \).

**Proof.** Let \( a_i \) be the smallest integer in \( \text{Tim } (\tau) \). Set

\[
\text{Tim } (\tau^1) = \{ t \in [a_1, a_1 + N) : |t - r| > n \quad \text{for all } r \in \text{Tim } (\tau) \}.
\]

This determines a \( G \)-time \( \tau^1 \) which satisfies our condition.

(3.5) **Remark.** \( \tau^1 \) could be empty.

(3.6) **Lemma.** If \( \tau \) is a time specification and \( \varepsilon > 0 \), there is an \( \varepsilon \)-separated \( S \)-set \( A \) with \( \tau(A) = \tau \) and \( \text{card } (A) \geq N(T(\tau), 2\varepsilon) \).

**Proof.** Let \( \tau = \{I_1, \ldots, I_m\} \) and \( \tau_k = \{I_k\} \) for \( 1 \leq k \leq m \). Let \( A_k \) be an \( \varepsilon \)-separated \( S \)-set with \( \tau(A_k) = \tau_k \) and \( \text{card } (A_k) = N(T(\tau_k), \varepsilon) \). Then

\[
A = A_1 \land \cdots \land A_m = \{ s_1 \land \cdots \land s_m : s_k \in A_k, 1 \leq k \leq m \}
\]

is \( \varepsilon \)-separated with \( \tau(A) = \tau_1 \land \cdots \land \tau_m = \tau \) and \( \text{card } (A) = \prod N(T(\tau_k), \varepsilon) \geq N(\sum T(\tau_k), 2\varepsilon) = N(T(\tau), 2\varepsilon) \) by (3.2(iii)).

(3.7) **Theorem.** Suppose \( B \) is a \( 2\varepsilon \)-separated \( S \)-set with \( \tau(B) \) an \( M(\varepsilon) \)-delayed \( m \)-time. Then

\[
N_d(U(B, \varepsilon)) \geq \frac{K(m, \varepsilon) \text{card } (B) N(d, 8\varepsilon)}{N(T(\tau(B)), 4\varepsilon)}
\]

for all \( d \geq L(\tau(B)) + M(\varepsilon) \) where \( K(m, \varepsilon) > 0 \) depends only on \( m \) and \( \varepsilon > 0 \).

**Proof.** Let \( N = d - M(\varepsilon) \geq L(\tau(B)) \). Let \( \tau = \tau(B) \) and choose \( \tau^1 \) as in Lemma 3.4. By (3.3) let \( A \) be a \( 2\varepsilon \)-separated \( S \)-set with \( \tau(A) = \tau^1 \) and \( \text{card } (A) \geq N(T(\tau^1), 4\varepsilon) \).

Now \( A \land B \) is a \( 2\varepsilon \)-separated \( S \)-set with \( M(\varepsilon) \)-delayed time \( \tau \land \tau^1 \); \( d \geq N + M(\varepsilon) \geq L(\tau \land \tau^1) + M(\varepsilon) \). Hence, by (3.3(ii)), we have

\[
N_d(U(A \land B, \varepsilon)) \geq \text{card } (A) \text{card } (B).
\]

Since \( U(B, \varepsilon) \geq U(A \land B, \varepsilon) \),

\[
N_d(U(B, \varepsilon)) \geq \text{card } (A) \text{card } (B).
\]

Now \( T(\tau^1) \geq \max \{ 0, N - 2mM(\varepsilon) - T(\tau) \} \) (see Remark 3.5). Thus

\[
\text{card } A \geq \max \{ 1, N(N - 2mM(\varepsilon) - T(\tau), 4\varepsilon) \} = W
\]

(taking 1 in case \( N - 2mM(\varepsilon) - T(\tau) \leq 0 \)). Recalling that \( N = d - M(\varepsilon) \) and (3.2(iii)) we get

\[
N(d, 8\varepsilon) \leq W \cdot N((2m + 1)M(\varepsilon), 4\varepsilon) N(T(\tau), 4\varepsilon)
\]
(the inequality is good in the exceptional case we have been noting). Thus

\[ N_d(U(B, \varepsilon)) \geq \text{card } (B) \cdot W \]

\[ \geq \frac{K(m, \varepsilon) \text{card } (B) N(d, 4\varepsilon)}{N(T(\tau), 4\varepsilon)} \]

where \( K(m, \varepsilon) = N((2m + 1)M(\varepsilon), 4\varepsilon)^{-1} \).

(3.8) DEFINITION. For \( U \subset X \) let

\[ \varphi(U) = \liminf_{n \to \infty} \frac{N_d(U)}{N_d(f)} \quad \text{and} \quad \theta(U) = \limsup_{n \to \infty} \frac{N_d(U)}{N_d(f)}. \]

(3.9) COROLLARY. (i) For any \( \alpha > 0 \)

\[ \liminf_{d \to \infty} \frac{N_d(f)}{N(d, \alpha)} > 0. \]

(ii) \( \varphi(V) > 0 \) when \( V \neq \emptyset \) is open.

(iii) There is a \( K^* > 0 \) such that \( \varphi(U) \geq K^* \theta(V) \) whenever \( U \) and \( V \) are open in \( X \) and \( U \supset V \).

(iv) There are \( m_0 \) and \( S > 0 \) such that \( N_m(f) \geq SN(m, \delta^*)N(n, \delta^*), \) for all \( 0 < k < m \), provided that \( m \geq m_0 \).

(v) There are \( m_0 \) and \( S > 0 \) such that, if \( m \geq m_0 \) and \( U \subset X \) satisfies \( \text{diam } f^n(U) \leq \delta^* \) for all \( 0 < k < m \), then \( \theta(U) \leq 1/SN_m(f) \).

Proof. (i) and (ii). Let \( x \in V \) and choose \( \varepsilon > 0 \) so small that \( B_\varepsilon(x) \subset V \) and \( 8 \varepsilon \leq \min \{ \alpha, \delta^* \} \). Let \( s \) be given by \( \tau(s) = \{0\} \) and \( P_0(0) = x; B = \{s\} \). Then \( V \supset U(s, \varepsilon) \) and by the theorem

\[ N_d(f) \geq N_d(V) \geq K(1, \varepsilon) N(d, 8\varepsilon)/N(1, 4\varepsilon) \]

for \( d \geq 1 + M(\varepsilon) \). As \( N(d, 8\varepsilon) \geq N(d, \varepsilon) \), (i) follows immediately. As \( N(d, 8\varepsilon) \geq N(d, \delta^*) \geq N_d(f) \), so does (ii).

(iii) Choose \( \varepsilon > 0 \) so that \( U \supset B_\varepsilon(V) \) and let \( D(\varepsilon) \) be given as in 2.4. Consider \( n > 2D(\varepsilon) \). For each \( p \in \text{Per}_n(V) \) form the 1-specification \( s(p) \) with \( \tau(s(p)) = \{ -D(\varepsilon), n - D(\varepsilon) \} \) and \( P_{s,p}(f) = f^n(p) \). \( B_n = \{s(p) : p \in \text{Per}_n(V)\} \) is \( \delta^* \)-separated (see the proof of 3.2(iii)). By the definition of \( \varepsilon \) and \( D(\varepsilon) \) we have \( U(B_n, \delta^*) \subset U \).

Trivially, \( U(B_n, 1\delta^*) \subset U \); so by the theorem

\[ N_d(U) \geq K(1, 1\delta^*)N_n(V)N(d, \delta^*)/N(n, 1\delta^*) \]

for \( d \geq n + M(1\delta^*) \). By (i) above there is an \( n_0 \) and a \( K_1 \) such that \( N(n, 1\delta^*) \leq K_1 N_n(f) \) when \( n \geq n_0 \); also \( N(d, \delta^*) \geq N_d(f) \). Thus for \( n \geq n_0 \) and \( d \geq n + M(1\delta^*) \) we have

\[ N_d(U)/N_d(f) \geq K^* N_n(V)/N_n(f) \]

where \( K^* = K(1, 1\delta^*)/K_1 > 0 \). Then \( \varphi(U) \geq K^* \theta(V) \).

(iv) Set \( m_0 = 2M(1\delta^*) \). Let \( A \) be a \( 1\delta^* \)-separated \( S \)-set with \( \tau(A) = \{0, n\} \) and \( \text{card } A = N(n, 1\delta^*) \); \( B \) a \( 1\delta^* \)-separated \( S \)-set with \( \tau(B) = \{n + M(1\delta^*), n + m \}

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and \( \text{card } B = N(m-m_0, \frac{1}{2}\delta^*) \). Now \( A \cap B \) is \( \frac{1}{2}\delta^* \)-separated with \( M(\frac{1}{2}\delta^*) \)-delayed time.

By Proposition 3.2(iii) we have
\[ N_{n+m}(f) \geq \text{card } (A \cap B) = N(n, \frac{1}{2}\delta^*)N(m-m_0, \frac{1}{2}\delta^*). \]

By Proposition 3.2(iii) we have
\[ N(m, \delta^*) \leq N(m-m_0, \frac{1}{2}\delta^*)N(m_0, \frac{1}{2}\delta^*). \]

Taking \( S = N(m_0, \frac{1}{2}\delta^*)^{-1} \), we have \( N_{n+m}(f) \geq SN(n, \delta^*)N(m, \delta^*) \).

(iii) Let \( m_0 \) and \( S \) be as above. Since \( \text{Per}_{n+m}(U) \) is an \((n+m, \delta^*)\)-separated set and \( \text{diam } f^n(U) \leq \delta^* \) for \( 0 \leq k < m, f^m \text{ Per}_{n+m}(U) \) is an \((n, \delta^*)\)-separated set; thus \( N_{n+m}(U) \leq N(n, \delta^*) \). By (iv) we have, since \( m \geq m_0 \), \( N_{n+m}(f) \geq SN(n, \delta^*)N(m, \delta^*) \) and so
\[ N_{n+m}(U)/N_{n+m}(f) \leq 1/\text{SN}_n(f). \]

Letting \( n \rightarrow \infty \), \( \theta(U) \leq 1/\text{SN}_n(f) \).

(3.10) Definition. For \( A \subseteq X \) let \( N(n, \varepsilon, A) \) be the largest cardinality of an \((n, \varepsilon)\)-separated set contained in \( A \).

(3.11) Proposition. For each \( \varepsilon \) with \( 0 < \varepsilon < \frac{1}{2}\delta^* \) there are constants \( c_\varepsilon > 0 \) and \( 0 < \tau_\varepsilon < 1 \) for which the following holds. If \( A \subseteq X \), \( 0 \leq k_1 < k_2 < \cdots < k_m \), are integers and \( w_{k_1}, \ldots, w_{k_m} \in X \) satisfy \( f^n(A) \cap B_\varepsilon(w_{k_r}) = \emptyset \) for \( r = 1, \ldots, m \), then \( N(n, \varepsilon, A) \leq c_\varepsilon \tau^m N(n, \varepsilon) \) for all \( n > k_m \).

Proof. Let \( M = M(\frac{1}{2}\delta^*) \) as in 2.10. Let \( j_1 < j_2 < \cdots < j_q \) be a subsequence of \( k_1 < \cdots < k_m \) such that \( j_{i+1} - j_i \geq 2M \) and \( q \geq m/(2M+1) \). Let \( n > k_m \) and \( E_n \subseteq A \) be a \((n, \varepsilon)\)-separated set. For each \( I \subseteq J = \{j_1, \ldots, j_q\} \) and each \( x \in E_n \) we define the specification \( s(x, I) \) by requiring that it be an \( M \)-delayed specification with
\[ \text{Tim } s(x, I) = (0, n) \setminus \bigcup_{j \in I} [j - M, j + M] \cup J, \]

\[ P_{s(x, I)}(t) = f^x(t) \quad \text{for } t \notin I \quad \text{and} \quad P_{s(x, I)}(j) = w_{j_i} \quad \text{for } j_i \in I. \]

Set \( d = n + m \). By Theorem 2.10 choose
\[ p(x, I) \in U(s(x, I), \frac{1}{2}\varepsilon) \cap \text{Per}_d(X). \]

Let \( F_1 = \{p(x, I) : x \in E_n\} \). If \( I_1 \neq I_2 \) and \( x, y \in E_n \), then \( s(x, I_1) \) and \( s(y, I_2) \) are \( \varepsilon \)-separated; for if \( j_1 \in I_1 \setminus I_2 \), then \( j_1 \in \text{Tim } s(x, I_1) \cap \text{Tim } s(y, I_2) \) and
\[ d(P_{s(x, I_1)}(j_1), P_{s(y, I_2)}(j_1)) = d(w_{j_1}, f^1(y)) > \varepsilon. \]

By lemma (i) we have \( p(x, I_1) \neq p(y, I_2) \); thus \( I_1 \neq I_2 \) implies \( F_{I_1} \cap F_{I_2} = \emptyset \).

Suppose \( z = p(x, I) = p(y, I) \) and \( x \neq y \). For \( t \in \text{Tim } s(x, I) \cap I \), we have \( P_{s(x, I)}(t) = f^x(t) \) and \( P_{s(y, I)}(t) = f^y(t) \); so \( d(f^x(z), f^x(y)) < \frac{1}{2}\varepsilon \) and \( d(f^y(z), f^y(y)) < \frac{1}{2}\varepsilon \), hence \( d(f^x(z), f^y(y)) < \varepsilon \). Since \( x, y \in E_n \), an \((n, \varepsilon)\)-separated set, we must have
\[ d(f^x(z), f^y(y)) > \varepsilon \quad \text{for some} \quad t \in (0, n)(\text{Tim } s(x, I) \cap I) = \bigcup_{j \in I} [j - M, j + M]. \]
By the proof of 3.2(iii), \( \{x \in E_n : p(x, I) = z\} \) has at most \( g^{\text{card} I} \) elements where \( g = N(2M + 1, \frac{1}{8}e) \). Thus \( F_I \) has at least \( \text{card} E_n \cdot g^{\text{card} I} \) elements.

As the \( F_I \)'s are disjoint

\[
N_d(f) \geq \sum_{I \subseteq J} \text{card} F_I \geq \sum_{I \subseteq J} \frac{1}{g^{\text{card} I}} \text{card} E_n
\]

\[
\geq \sum_{r=0}^{\text{card} J} \binom{\text{card} J}{r} \frac{1}{g} \text{card} E_n = \left(1 + \frac{1}{g}\right)^{\text{card} J} \text{card} E.
\]

Since \( 2e < \delta^* \), by 3.2(i) and 3.2(iii)

\[
N_d(f) = N_{n+m}(f) \leq N(n + M, 2e) \leq N(n, e)N(M, e).
\]

Also \( \text{card} J = q \geq m/(2M + 1) \). Thus

\[
N(n, e, A) = \text{card} E_n \leq \frac{N(M, e)}{(1 + 1/g)^{1/2M+1}} N(n, e).
\]

4. **Topological entropy.** Suppose \( \mathcal{A} \) is a finite open cover of \( X \). \( \mathcal{E} = \mathcal{A} \times \cdots \times \mathcal{A} \) (\( n \)-times) is an \( n \)-cover for \( (f, \mathcal{A}) \) if for every \( z \in X \) there is an \((A_0, \ldots, A_{n-1}) \in E\) such that \( f^k(x) \in A_k \) for all \( 0 \leq k < n \). Let \( M_n(f, \mathcal{A}) \) denote the minimum cardinality of an \( n \)-cover for \( (f, \mathcal{A}) \). Then (see Adler, Konheim and McAndrew [1]) the limit

\[
h(f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log M_n(f, \mathcal{A})
\]

exists and the **topological entropy** of \( f \) is defined by

\[
h(f) = \sup_{\mathcal{A}} h(f, \mathcal{A}).
\]

(The above definitions and 4.1 and 4.2 below do not depend on our standing hypothesis that \( f \) satisfies Axiom A*; they work for any continuous map of a compact Hausdorff space.)

(4.1) **DEFINITION.** \( f : X \to X \) has completely positive topological entropy (c.p.t.e.) if \( h(f, \{C, D\}) > 0 \) whenever \( \{C, D\} \) is an open cover of \( X \) with \( \overline{C} \neq \overline{D} \).

(4.2) **PROPOSITION.** Suppose \( f : X \to X \) has c.p.t.e. Then \( h(f) > 0 \) unless \( X \) is a single point, and it is topologically transitive. If \( g : Y \to Y \) and \( h : X \to Y \) are continuous maps with \( h \) surjective and \( g \circ h = h \circ f \), then \( g \) has c.p.t.e.

**Proof.** Unless \( X \) is a single point an open cover \( \{C, D\} \) as in 4.1 can be found and so \( h(f) > 0 \).

If \( f \) is not transitive, then there is an open set \( C \neq \emptyset \) with \( f^{-1}(C) \subset C \) and \( \overline{C} \neq X \). Let \( B \neq \emptyset \) be open with \( \overline{B} \subset C \) and set \( D = X \setminus \overline{B} \). Then \( \{C, D\} \) is as above. Let

\[
E_n = \{(C, \ldots, C, D, \ldots, D) : i + j = n, i, j \geq 0\}.
\]

We claim \( E_n \) is an \( n \)-cover for \( \{f, \{C, D\}\} \). For, if \( x \in X \), then either \( f^k(x) \in D \) for all \( 0 \leq k < n \) or there is a largest \( k \), denoted \( k(x) \), such that \( 0 \leq k < n \) and \( f^k(x) \notin D \).
In the latter case \( f^{x(k)}(x) \in C \) and so \( f^{m}(x) \in C \) for all \( m \leq k(x) \) as \( f^{-1}(C) \subseteq C \); \( f^{m}(x) \in D \) for \( m > k(x) \). As \( \text{card } E_{n} = n + 1 \), \( M_{n}(f, \{C, D\}) \leq n + 1 \) and \( h(C, D) = 0 \) — a contradiction.

Suppose \( \{C, D\} \) is an open cover of \( Y \) with \( \bar{C} \neq \bar{Y} \neq D \). Then \( \{h^{-1}(C), h^{-1}(D)\} \) satisfies the condition of 4.1 also. \( h \) and \( h^{-1} \) induce a bijection between \( n \)-covers for \( (f, \{h^{-1}(C), h^{-1}(D)\}) \) and \( (g, \{C, D\}) = h(f_{i}(h^{-1}(C), h^{-1}(D))) > 0 \).

(4.3) Theorem. If \( f: X \to X \) is C-dense, then \( f \) has c.p.t.e.

Proof. Let \( \{C, D\} \) be a cover as in 4.1. Choose \( \epsilon > 0 \) and \( p, q \in X \) such that \( B_{\epsilon}(p) \subseteq C \setminus D \) and \( B_{\epsilon}(q) \subseteq D \setminus C \). Let \( M(\epsilon) \) be the integer given by 2.10; set \( N = M(\epsilon) + 1 \). Then \( \tau_{n} = \{[k,n] : 0 \leq k < n\} \) is \( M(\epsilon) \)-delayed.

For \( (a_{0}, \ldots, a_{n-1}) \in \prod_{k=0}^{n-1} \{p, q\} \) define a specification \( s = s_{n}(a_{0}, \ldots, a_{n-1}) \) by \( \tau(s) = \tau_{n} \) and \( P_{n}(kN) = a_{k} \). By 2.10 choose points \( x_{n}(a_{0}, \ldots, a_{n-1}) \in U(s_{n}(a_{0}, \ldots, a_{n-1}), \epsilon) \).

Let \( E_{n} \) be an \( nN \)-cover for \( (f, \{C, D\}) \); for \( x \in X \) let \( F_{n}(x) = (F_{n}^{0}(x), \ldots, F_{n}^{nN-1}(x)) \in E_{n} \) be such that \( f(x) \in F_{n}(x) \) for \( 0 \leq j < nN \). Suppose \( (a_{0}, \ldots, a_{n-1}) \neq (b_{0}, \ldots, b_{n-1}) \); say \( a_{k} = p \) and \( b_{k} = q \). Then

\[
F_{n}^{kN}(x_{n}(a_{0}, \ldots, a_{n-1})) \in B_{\epsilon}(p) \subseteq C \setminus D,
\]

and so \( F_{n}^{kN}(x_{n}(a_{0}, \ldots, a_{n-1})) = C \); similarly \( F_{n}^{kN}(x_{n}(b_{0}, \ldots, b_{n-1})) = D \) and so \( F_{n}(x_{n}(a_{0}, \ldots, a_{n-1})) \neq F_{n}(x_{n}(b_{0}, \ldots, b_{n-1})) \). It follows that \( \text{card } E_{n} \geq 2^{n} \) and \( M_{n}(f, \{C, D\}) \geq 2^{n} \); thus

\[
h(f, \{C, D\}) \geq \lim_{n \to \infty} \frac{1}{nN} \log 2^{n} = \frac{1}{N} \log 2 > 0.
\]

(4.4) Remark. Now \( f: X \to X \) satisfying Axiom A* could not be topologically transitive unless the permutation \( g \) in its C-dense decomposition (2.7) is a cycle, i.e. if the decomposition \( X = X_{1} \cup \cdots \cup X_{m} \) satisfies \( X = \bigcup f^{k}X_{1} \); with 4.2 and 4.3 one sees that this is a sufficient condition for transitivity. It is now clear how 2.7 is just another version of Smale’s Spectral Decomposition [16, p. 777]. We also see that \( h(f) > 0 \) unless \( X \) is finite; this result was proved before in [6]. The following is an improvement of the main result of [6].

(4.5) Theorem. If \( f: X \to X \) is C-dense, then

\[
h(f) = \lim_{n \to \infty} \frac{1}{n} \log N_{n}(f).
\]

Proof. Let \( \mathcal{A} \) be a finite open cover of \( X \) with diam \( (A) < \delta^{*} \) for all \( A \in \mathcal{A} \) and let \( \beta > 0 \) be a Lebesgue number for \( \mathcal{A} \) (i.e. every closed \( \beta \)-ball \( B_\beta(x) \) lies inside some member of \( \mathcal{A} \)).

Let \( Q \) be a maximal \( (n, \beta) \)-separated set. For \( z \in Q \) choose \( B(z) = (A_{0}(z), \ldots, A_{n-1}(z)) \) with \( A_{k}(z) \in \mathcal{A} \) and

\[
A_{k}(z) = \text{Cl}(B_{\beta}(f^{k}(z))) \quad \text{for all } 0 \leq k < n.
\]
We claim $E_n = \{ B(z) : z \in Q \}$ is an $n$-cover for $(f, \mathcal{A})$. For each $x \in X$ there is a $z_x \in Q$ for which $d(f^k(x), f^k(z_x)) \leq \beta$ for all $0 \leq k < n$; otherwise $Q \cup \{ x \}$ would be an $(n, \beta)$-separated set bigger than $Q$. Since $f^k(x) \in A_k(z_x)$, $E_n$ is an $n$-cover. We have shown $M_n(f, \mathcal{A}) \leq N(n, \beta)$.

Let $E$ be an $n$-cover for $(f, \mathcal{A})$ and $R$ an $(n, \delta^*)$-set. For $x \in R$ choose $g(x) = (A_0(x), \ldots, A_{n-1}(x)) \in E$ such that $f^k(x) \in A_k(x)$ for all $0 \leq k < n$. If $g(x) = g(y)$, then $A_k(x) = A_k(y)$ and $d(f^k(x), f^k(y)) \leq \diam A_k(x) \leq \delta^*$ for $0 \leq k < n$; $x = y$ as $R$ is an $(n, \delta^*)$-separated set. As $g : R \to E$ is injective, $\card E \geq \card R$ and $M_n(f, \mathcal{A}) \geq N(n, \delta^*) \geq N_n(f)$.

By 3.9(i) there is an $S > 0$ and $n_0$ such that $N_n(f) \geq SN(n, \beta)$ for $n \geq n_0$. Hence $SM_n(f, \mathcal{A}) \leq N_n(f) \leq M_n(f, \mathcal{A})$ for all $n \geq n_0$. Since $(1/n) \log M_n(f, \mathcal{A})$ approaches the limit $h(f, \mathcal{A})$, so does $(1/n) \log N_n(f)$. As this is true for every $\mathcal{A}$ with $\diam \mathcal{A} < \delta^*$ and in calculating $h(f)$ we need only consider $h(f, \mathcal{A})$ with $\mathcal{A}$ having small diameter,

$$h(f) = h(f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log N_n(f).$$

(4.6) REMARK. Let

$$\gamma_f(\epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon).$$

The proof above shows that, for any map $f$ a compact metric space, $h(f) = \lim_{\epsilon \to 0} \gamma_f(\epsilon)$. Suppose $f$ is a homeomorphism and $\delta$ is an expansive constant; if $\epsilon \leq \delta$, then 3.2(ii) goes through, i.e.

$$N(n, \delta) \leq N(n, \epsilon) \leq N(n + m_\epsilon, \delta)$$

for some $m_\epsilon$, and so $\gamma_f(\epsilon) = \gamma_f(\delta)$. In this case we have $\gamma_f(\delta) = h(f)$.

(4.7) THEOREM. Suppose $f : X \to X$ is $C$-dense and $A \subset X$ is closed with $\emptyset \neq A \neq X$ and $f(A) = A$. Then $h(f|A) < h(f)$.

Proof. By the remark above, $h(f|A) = \gamma_{f|A}(\epsilon)$ for $\epsilon \leq \delta^*$. Choose $w \in X \setminus A$ and $\epsilon > 0$ so small that $A \cap B_\epsilon(w) = \emptyset$. Recall 3.11, $N(n, \epsilon, A) \leq c_\tau \tau_{n\epsilon}^{n\epsilon}$, for $n > m$ where $\tau_\epsilon < 1$. Then

$$\gamma_{f|A}(\epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, A)$$

$$\leq \lim_{n \to \infty} \frac{1}{n} \log c_\tau \tau_{n\epsilon}^{n\epsilon} \log N(n, \epsilon)$$

$$\leq \log \tau_\epsilon + \gamma_f(\epsilon) = \log \tau_\epsilon + h(f) < h(f).$$

5. CONSTRUCTION OF A MEASURE. Let $\psi$ be a countable base for the topology of $X$ which is closed under finite union. Assume $\omega : \psi \to R$ satisfies, for $B \in \psi$,

$$\omega(B) \geq 0, \quad \omega(X) = 1,$$

$$\omega(B_1) \geq \omega(B_2) \text{ when } B_1 \supset B_2,$$

$$\omega(B_1 \cup \cdots \cup B_n) \leq \sum \omega(B_i).$$
and
\[ \omega(B_1 \cup B_2) = \omega(B_1) + \omega(B_2) \quad \text{when } \overline{B}_1 \cap \overline{B}_2 = \emptyset. \]

For \( U \) open in \( X \) define \( m(U) = \sup \{ \omega(B) : \overline{B} \subset U \text{ and } B \in \psi \} \).

(5.1) Lemma. If \( U \subset \bigcup_{i=1}^{r} U_i \), then \( m(U) \leq \sum m(U_i) \). If \( U \cap V = \emptyset \), then \( m(U \cup V) = m(U) + m(V) \).

Proof. Let \( B \in \psi \) with \( \overline{B} \subset U \). By compactness let \( U_1, \ldots, U_n \) cover \( B \). For \( x \in \overline{B} \) choose \( B_x \in \psi \) so that \( \overline{B}_x \subset U_i \) for some \( i \) satisfying \( 1 \leq i \leq n \). Let \( B_{x_1}, \ldots, B_{x_r} \) cover \( \overline{B} \) and set \( A_i = \bigcup \{ B_{x_i} : \overline{B}_x \subset U \} \). Then
\[ \omega(B) \leq \omega \left( \bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} \omega(A_i) \leq \sum_{i=1}^{n} m(U_i). \]

Now vary \( B \).

By the first part of the lemma, \( m(U \cup V) \leq m(U) + m(V) \). Suppose \( B_1, B_2 \in \psi \) with \( \overline{B}_1 \subset U \) and \( \overline{B}_2 \subset V \). Then \( \text{Cl} (B_1 \cup B_2) \subset U \cup V \) and \( \overline{B}_1 \cap \overline{B}_2 = \emptyset \); so
\[ m(U \cup V) \geq \omega(B_1 \cup B_2) = \omega(B_1) + \omega(B_2). \]

Varying the \( B_i \) we obtain \( m(U \cup V) \geq m(U) + m(V) \).

For any \( E \subset X \) we define
\[ m(E) = \inf \{ m(U) : U \supseteq E, \ U \text{ open} \}. \]

One sees easily that this definition agrees with the earlier one on open sets and that \( m(K) = \inf \{ \omega(B) : B \supseteq K, B \in \psi \} \) when \( K \) is closed. We let
\[ \mathcal{M} = \{ E \subset X : m(E) = \sup \{ m(K) : K \subset E, \ K \text{ closed} \} \}. \]

With standard arguments we get

(5.2) Proposition. \( \mathcal{M} = \mathcal{M}_{\psi, \omega} \) is a \( \sigma \)-field containing the Borel sets of \( X \) and \( m = m_{\psi, \omega} \) is a complete normalized regular measure on \( \mathcal{M} \).

Proof. One can, for example, use 5.1 and imitate the proof of the Riesz Representation Theorem given in Rudin [19, p. 40].

(5.3) Lemma. If \( \omega_1 : \psi_1 \to R \) and \( \omega_2 : \psi_2 \to R \) are as above and there is a \( K > 0 \) such that \( \omega_2 (B_2) \leq K \omega_1 (B_1) \) when \( B_2 \supseteq B_1 \) and \( \omega_1 (B_1) \leq K \omega_2 (B_2) \) when \( B_1 \supseteq \overline{B}_2 \), then \( \mathcal{M}_{\psi_1, \omega_1} = \mathcal{M}_{\psi_2, \omega_2} \) and \( K m_{\psi_1, \omega_1} (U) \geq m_{\psi_2, \omega_2} (U) \). Similarly \( m_{\psi_1, \omega_1} (U) \geq K m_{\psi_2, \omega_2} (U) \). These inequalities extend to any \( E \subset X \).

Suppose \( E \in \mathcal{M}_{\psi_1, \omega_1} \). Letting \( K_n \subset E \) be compact with \( m_{\psi_1, \omega_1} (K_n) \leq m_{\psi_1, \omega_1} (E) - 1/n \) we see that \( E = E_1 \cup \bigcup_{n=1}^{\infty} K_n \) where \( E_1 \subset F \) for some Borel set \( F \) with \( m_{\psi_1, \omega_1} (F) = 0 \). Then \( m_{\psi_1, \omega_1} (F) = 0 \) also and \( E_1 \in \mathcal{M}_{\psi_2, \omega_2} \) since \( m_{\psi_2, \omega_2} \) is complete. As \( \psi_2, \omega_2 \)
contains Borel sets, we finally see that $E \in \mathcal{M}_{\psi_2, \omega_2}$. The proof of $\mathcal{M}_{\psi_1, \omega_1} \subset \mathcal{M}_{\psi_2, \omega_2}$ is the same.

We will now see how to define some $\omega$'s when we are given a homeomorphism $f: X \to X$ which is $C$-dense. Let $\psi$ be any base as above. By diagonalization we can find increasing sequences of integers $\{n_k\}$ such that

$$\omega(B) = \alpha_{\{n_k\}}(B) = \lim_{k} \frac{N_{n_k}(B)}{N_{n_k}(f)}$$

exists for every $B \in \psi$. The measure we obtain we denote by $\mu_{f, \{n_k\}}$. Lemma 5.3 (with $K=1$) shows us that the measure does not depend on the base used.

Let $\mu_n$ be the measure obtained by giving each point of $\text{Per}_n(X)$ measure $1/N_n(f)$. Then $\mu_{n_k} \Rightarrow \mu_{f, \{n_k\}}$ weakly (see Corollary 6.7).

(5.4) Theorem. Suppose $f: X \to X$ is $C$-dense. The measures $\mu_{f, \{n_k\}}$ are all equivalent in the sense of 5.3. They are positive on nonempty open sets and $\mu_{f, \{n_k\}}(\{x\})=0$ unless $X=\{x\}$. $f$ is an automorphism of $(\mathcal{M}, \mu_{f, \{n_k\}})$.

Proof. Let $\mu_{f, \{n_k\}}$ and $\mu_{f, \{m_k\}}$ be defined using bases $\Psi_1$ and $\Psi_2$ respectively. By 3.9(iii) there is a $K^*>0$ such that, if $B_1 \supseteq B_2$, then

$$\alpha_{\{n_k\}}(B_1) \geq \varphi(B_1) \geq K^* \theta(B_2) \geq \alpha_{\{m_k\}}(B_2).$$

5.3 gives equivalence.

If $U \neq \emptyset$ is open, then $U \supseteq B \neq \emptyset$ for some $B \in \Psi$. Then, using 3.9(ii), $\mu_{f, \{n_k\}}(U) \geq \alpha_{\{n_k\}}(U) \geq \varphi(B) > 0$. Suppose $x \in X$ but $X\neq \{x\}$. Let

$$U_m = \{y \in X : d(f^k(y), f^k(x)) < \frac{1}{2}\delta^* \text{ for } 0 \leq k < m\}.$$

Let $B_m \in \Psi$ with $x \in B_m \subset U_m$. Then $\mu_{f, \{n_k\}}(\{x\}) \leq \alpha_{\{n_k\}}(B_m) \leq \theta(U_m)$. By 3.9(b) there are $m_0$ and $S>0$ with $\theta(U_m) \leq 1/sN_m(f)$ for all $m \geq m_0$. By 4.3 and 4.2

$$h(f) = \lim_{m} \frac{1}{m} \log N_m(f) > 0.$$

Thus $N_m(f) \to \infty$, $\theta(U_m) \to 0$ and $\mu_{f, \{n_k\}}(\{x\})=0$.

Now $\Psi$, $\alpha_{\{n_k\}}$ and $f\Psi$, $\alpha_{\{n_k\}}$ clearly satisfy the hypotheses of 5.3 with $K=1$ (by the obvious and crucial fact that $f$ permutes $\text{Per}_n(X)$). Hence

$$f\mu_{f, \{n_k\}} = f\alpha_{\{n_k\}} = m_{f, \Psi, \alpha_{\{n_k\}}} = m_{\Psi, \alpha_{\{n_k\}}} = \mu_{f, \{n_k\}}.$$

(5.5) Remark. Above we assumed $f: X \to X$ is $C$-dense. Suppose $f: X \to X$ satisfying Axiom A* is only assumed to be topologically transitive. Then $X = X_1 \cup \cdots \cup X_m$ with $f(X_1) = X_{m+1}$ ($X_{m+1} = X_1$) and $f^m: X_1 \to X_1$ $C$-dense. From an invariant measure $\mu$ for $f^m: X_1 \to X_1$ we get one $\mu'$ for $f: X \to X$ by defining $\mu'(f^mE) = \mu(E)/m$ for $E \subset X_1$ measurable. This gives a bijection between invariant Borel measures for $f^m: X_1 \to X_1$ and $f: X \to X$. One sees that $\mu'$ is ergodic if and only if $\mu$ is, $h(f^m|X_1) = mh(f)$ and $h_{\mu}(f^m|X_1) = mh_{\mu}(f)$. The measures defined above,
in terms of periodic points of $f^m|X$, correspond to measures on $X$ defined in terms of periodic points of $f: X \to X$. We shall study the $C$-dense case and this will give us results also for the general transitive case.


(6.1) Definition. $f$ is said to be partially mixing with respect to the $f$-invariant measure $\mu$ if there is an $R > 0$ such that for any $E, F \in \mathcal{M}$,

$$\liminf_{n \to \infty} \mu(E \cap f^{-n}F) \geq R \mu(E) \mu(F).$$

If $c_1 < c_2 < \cdots < c_r$ are integers, set $I(c_1, \ldots, c_r) = \min_i (c_i + 1 - c_i)$. $f$ is partially mixing in order $r$ if there is an $R_r > 0$ such that, if $E_1, \ldots, E_r \in \mathcal{M}$ and $I(c_1, \ldots, c_r) \to \infty$ as $n \to \infty$, then

$$\liminf_{n \to \infty} \mu(f^{-c_1}E_1 \cap \cdots \cap f^{-c_r}E_r) \geq R_r \mu(E_1) \cdots \mu(E_r).$$

Notice that partially mixing is a stronger condition than ergodicity or weak mixing.

(6.2) Theorem. If $f: X \to X$ is $C$-dense, then $f$ is partially mixing in all orders with respect to each $\mu = \mu_{r, (n_k)}$.

Proof. Let $I(c_1, \ldots, c_r) \to \infty$. Let $\alpha = \frac{1}{4} \delta^*$; by 3.9(i) choose $n_0$ and $S > 0$ so that $N_n(f) \geq SN(n, 2\alpha)$ for all $n \geq n_0$.

Suppose $E_1, \ldots, E_r$ are closed and $V_i \supseteq E_i$ with $V_i \in \Psi$. Choose $\epsilon > 0$ so that $B_\epsilon(E_i) \subseteq V_i$. Choose $k$ large enough so that $n_k > 2D(\epsilon)$ (see 2.4) and $n$ so that $I(c_1^\alpha, \ldots, c_r^\alpha) > M(\alpha) + n_k$. Let $\tau_i = [c_i^\alpha - D(\epsilon), c_i^\alpha + n_k - D(\epsilon)]$ and for $x \in \text{Per}_{n_k}(V_i)$ define the specification $s_x$ by $\tau_i(s_x) = \tau_i$ and $P_x = f^{t - \tau_i}(x)$; let $A_i = \{s_x \subseteq x \in \text{Per}_{n_k}(V_i)\}$. One notes now that $B = A_1 \land \cdots \land A_r$ is an $8\alpha$-separated s-set which is $M(\alpha)$-delayed. Also, by 2.4, we get

$$U(B, \alpha) \subseteq \bigcap_{i=1}^r f^{-c_i^\alpha}B_k(E_i) \subseteq \bigcap_{i=1}^r f^{-c_i^\alpha}V_i.$$  

By 3.7, we get

$$N_d(\bigcap_{i=1}^r f^{-c_i^\alpha}V_i) \geq N_d(U(B, \alpha)) \geq \frac{K(r, \alpha) \text{card (B)} N(d, \delta^*)}{N(n_k, \frac{1}{2} \delta^*)}$$

for $d$ sufficiently large. Now

$$N(d, \delta^*) \geq N_d(f), \quad \text{card (B)} = \prod \text{N}(n_k(V_i))$$

and, using 3.2(iii),

$$N(n_k, \frac{1}{2} \delta^*) \leq N(n_k, \frac{1}{4} \delta^*) \leq N(n_k(f))^{\gamma}.$$  

Combining all these,

$$\frac{N_d(\bigcap_{i=1}^r f^{-c_i^\alpha}V_i)}{N_d(f)} \geq R \prod \frac{N(n_k(V_i))}{N(n_k(f))}$$

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where \( R_r = K(r, \alpha)S^r > 0 \). Letting \( d \to \infty \),

\[
\varphi(\bigcap f^{-c} V_i) = \liminf_{d \to \infty} \frac{N_d(\bigcap f^{-c} V_i)}{N_d(f)} \geq R_r \prod_n \frac{N_{n_k}(V_i)}{N_{n_k}(f)}.
\]

This being true for all big \( n \),

\[
\liminf_{n \to \infty} \varphi(\bigcap f^{-c} V_i) \geq R_r \prod_n \alpha_{(n_k)}(V_i) \geq R_r \prod \mu(E_i).
\]

Letting \( n_k \to \infty \),

\[
\liminf_{n \to \infty} \varphi(\bigcap f^{-c} V_i) \geq R_r \prod \alpha_{(n_k)}(V_i) \geq R_r \prod \mu(E_i).
\]

Now suppose \( V_i \supseteq E_i \) open and choose the \( V_i \) above so that \( V_i \supseteq \overline{V_i} \). Then

\[
\bigcap f^{-c} V_i \supseteq \overline{\bigcap f^{-c} V_i}.
\]

Choose \( B \in \Psi \) so that

\[
\bigcap f^{-c} V_i \supseteq B \supseteq \bigcap f^{-c} V_i.
\]

Then

\[
\mu(\bigcap f^{-c} V_i) \geq \alpha_{(n_k)}(B) \geq \varphi(\bigcap f^{-c} V_i)
\]

and

\[
\liminf_{n \to \infty} \mu(\bigcap f^{-c} V_i) \geq R_r \prod \mu(E_i).
\]

Now

\[
\mu(\bigcap f^{-c} E_i) \geq \mu(\bigcap f^{-c} V_i) - \sum \mu(V_i \backslash E_i).
\]

Letting \( \mu(V_i \backslash E_i) \to 0 \) we get

\[
\liminf_{n \to \infty} \mu(\bigcap f^{-c} E_i) \geq R_r \prod \mu(E_i).
\]

For any \( E_i \in \mathcal{M} \) consider \( E_i \in E_i^* \) closed. Then

\[
\liminf_{n \to \infty} \mu(\bigcap f^{-c} E_i^*) \geq \liminf_{n \to \infty} \mu(\bigcap f^{-c} E_i) \geq R_r \prod \mu(E_i).
\]

Now let \( \mu(E_i) \to \mu(E_i^*) \).

(6.3) COROLLARY. Suppose \( f: X \to X \) satisfying Axiom A* is topologically transitive. Then the measure \( \mu^* \) on \( X \) corresponding to \( \mu_{\tau_{n_k}}(E_i) \) on one of its C-dense factors is ergodic under \( f \).

Proof. See Remark 5.5.

The following standard fact was pointed out to us by W. Parry.

(6.4) LEMMA. Suppose \( f: X \to X \) is an ergodic automorphism of two equivalent normalised Borel measures \( m_1 \) and \( m_2 \). Then \( m_1 = m_2 \).

Proof. Let \( dm_1/dm_2 \) denote the Radon-Nikodym derivative. It is \( f \)-invariant, hence a constant (clearly 1) by ergodicity.
(6.5) \textsc{Theorem.} Let $f: X \to X$ be $C$-dense. Then all the $\mu_{f,(n_k)}$ have a common value $\mu_f$.

\textbf{Proof.} 5.4, 6.2, and 6.4.

(6.6) \textsc{Theorem.} Let $f: X \to X$ be $C$-dense. If $K$ is closed and $\mu_f(K)=0$, then

\[ \lim_{n \to \infty} \left( \frac{N_n(K)}{N_n(f)} \right) = 0. \]

If $U$ is open with $\mu_f(\partial U) = 0$, then $\lim \left( \frac{N_n(U)}{N_n(f)} \right) = \mu_f(U)$.

\textbf{Proof.} Suppose $\{m_i\}$ is an increasing sequence of integers so that either

\[ \frac{N_{m_i}(K)}{N_{m_i}(f)} \to a > 0 \quad \text{or} \quad \frac{N_{m_i}(U)}{N_{m_i}(f)} \to b \neq \mu_f(U). \]

Let $\psi$ be a countable base closed under finite union and $\{n_k\}$ a subsequence of $\{m_i\}$ so that $\mu_{f,(n_k)}$ is defined.

Suppose $N_{n_k}(K)/N_{n_k}(f) \to a > 0$. If $B \supseteq K$, $B \in \psi$, then

\[ \alpha_{(n_k)}(B) = \lim_{N_{m_k}(f)} \frac{N_{n_k}(B)}{N_{n_k}(f)} \geq \lim_{N_{m_k}(f)} \frac{N_{n_k}(K)}{N_{n_k}(f)} = a. \]

It follows that $\mu_f(K) = \inf \alpha_{(n_k)}(B) \geq a > 0$, a contradiction. Suppose $N_{m_i}(U)/N_{m_i}(f) \to b \neq \mu_f(U)$. For $B \supseteq \overline{U}$, $B \in \psi$ we have $\alpha_{(n_k)}(B) \geq b$; hence $\mu_f(\overline{U}) = \mu_{f,(n_k)}(\overline{U}) \geq b$.

For $B \subseteq U$, $B \in \psi$, we have $\alpha_{(n_k)}(B) \leq b$; hence $\mu_f(U) \leq b$. As $\mu_f(\partial U) = 0$, $b \geq \mu_f(U) = \mu_f(\overline{U}) = b$ and so $\mu_f(U) = b$, a contradiction.

(6.7) \textsc{Corollary.} Let $f: X \to X$ be $C$-dense. Then, for any $F \in C(X)$,

\[ \frac{1}{N_n(f)} \sum_{x \in \text{Per}_n(f)} F(x) \to \int F \, d\mu_f \]

as $n \to \infty$. (We say that $\mu_f$ is derived from $f$ by periodic points to mean the above statement.)

\textbf{Proof.} Choose $b$ such that $-b < F(x) < b$ for all $x \in X$. Let $\varepsilon > 0$. Choose $-b = a_0 < a_1 < \cdots < a_r = b$ with $a_{i+1} - a_i < \varepsilon$, $\mu_f(\{x : F(x) = a_i\}) = 0$ and $F(x) = a_i$ for no periodic point $x$.

Let $U_i = \{x : a_{i-1} < F(x) < a_i\}$. Choose $N(\varepsilon)$ so big that

\[ |(N_n(U_i)/N_n(f)) - \mu_f(U_i)| < \varepsilon/b \]

for all $n \geq N(\varepsilon)$ and each $i$. This is possible since $F(\partial U_i) \subseteq \{a_{i-1}, a_i\}$ and so $\mu_f(\partial U_i) = 0$ by construction; hence 6.6 applies to $U_i$. We also have

\[ \left| \frac{N_n(f)^{-1}}{x \in \text{Per}_n(f)} \sum_{x \in \text{Per}_n(f)} F(x) - \sum_{i=1}^r a_i (N_n(U_i)/N_n(f)) \right| \leq \varepsilon. \]

Putting our above two inequalities together one sees that

\[ \left| \frac{N_n(f)^{-1}}{x \in \text{Per}_n(f)} \sum_{x \in \text{Per}_n(f)} F(x) - \sum a_i \mu_f(U_i) \right| \leq 2\varepsilon. \]
Since $|\int F d\mu_f - \sum a_i \mu_f(U_i)| \leq \varepsilon$, we finally get

$$\left| \int F d\mu_f - N_n(f)^{-1} \sum_{x \in \text{Per}_n(f)} F(x) \right| \leq 3\varepsilon$$

for all $n \geq N(\varepsilon)$.

7. **The algebraic case.** Suppose $f: G \to G$ is an automorphism of an $n$-dimensional torus $G$. $f$ is a **hyperbolic** if $Df: T_e G \to T_e G$ has no eigenvalues on the unit circle. Then (see [16]) $f$ satisfies Axiom A* and is C-dense because $G$ is connected (using 2.7). $f$ of course preserves the normalized Haar measure $m$ on $G$.

(7.1) **Proposition.** If $f$ is a hyperbolic automorphism of a torus, then $\mu_f = m$.

**Proof.** Suppose $g \in G$ and $E \subset G$ is closed. Let $\mu_f = \mu_{f, (n_0)}$ be defined via the base $\Psi$. Consider $B \in \Psi$ with $B \supset E + g$. There are $B^1 \in \Psi$ and open $V$ such that $B^1 \supset E$, $g \in V$ and $B^1 + V \subset B$. By 3.9(ii) there is an $N$ such that $N_n(V) > 0$ for all $n \geq N$. For $n_k \geq N$ and $g_{nk} \in \text{Per}_{nk}(V)$ we have $g_{nk} + \text{Per}_{nk}(B^1) \subset B$. If $x \in \text{Per}_{nk}(B^1)$, then as $f$ is a group automorphism $f^{n_k}(g + x) = f^{n_k}(g) + f^{n_k}(x) = g + x$; so $g_{nk} + x \in \text{Per}_{nk}(B)$.

Thus $N_{n_k}(B) \geq N_{n_k}(B^1)$ for $n_k \geq N$ and $\alpha_{(n_k)}(B) \geq \alpha_{(n_k)}(B^1) \geq \mu_{f, (n_k)}(E)$. Varying $B$, $\mu_{f, (n_k)}(g + E) \geq \mu_{f, (n_k)}(E)$. Using $-g$ instead of $g$, $\mu_{f, (n_k)}(g + E) \leq \mu_{f, (n_k)}(E)$. Thus $\mu_f(E) = \mu_f(g + E)$ for all $g \in G$ and $E$ closed; it follows that $\mu_f$ is Haar measure.

Now let $G$ be a torus acting freely on a compact metric space $X$ (i.e. $g_1 x = g_2 x$ implies $g_1 = g_2$) and let $\mu$ be normalized Haar measure on $G$. Let $\pi: X \to X_G = X/G$ be the projection map. Now suppose $X_G$ has a normalized Borel measure $m_G$. Suppose $f \in C(X)$. If $\pi(x_1) = \pi(x_2) = y$, then

$$\int_y F(g x_1) \, d\mu = \int_y F(g x_2) \, d\mu$$

for $x_1 = g_1 x_2$ for some $g_1 \in G$ and then $F(g x_1) = F(g_1 g x_2)$ is obtained from $F(g x_2)$ (as a function on $G$) by translating the variable. Denote this common value by $H_{f,y}$; $H_f \in C(X_G)$. Define a measure $m$ on $X$ by

$$\int_X F \, dm = \int_{X_G} H_f \, dm_G.$$ 

Now suppose $S: X \to X$ is a homeomorphism and $\sigma: G \to G$ an automorphism such that $S(g x) = \sigma(g) S(x)$. Then $S$ induces a homeomorphism $S_G$ of $X_G$ such that $\pi \circ S = S_G \circ \pi$. If $S_G$ preserves $m_G$, then $S$ preserves $m$ and we say $(S, m)$ is a $\sigma$-extension of $(S_G, m_G)$.

(7.2) **Proposition.** Let $(S, m)$ be a $\sigma$-extension of $(S_G, m_G)$ with $\sigma$ a hyperbolic automorphism of the torus. If $m_G$ is derived from $S_G$ by periodic points, then $m$ is derived from $S$ by periodic points.

**Proof.** Let $F \in C(X)$ and $\varepsilon > 0$. Choose $x_1, \ldots , x_5 \in X$ such that for each $x \in X$
there is an \( x_i \) such that \( |F(gx) - F(gx_i)| \leq \varepsilon/3 \) for all \( g \in G \). Since \( \mu \) is derived from \( \sigma \) by periodic points (see 6.7), there is an \( N(\varepsilon) \) such that
\[
\left| N_n(\sigma)^{-1} \sum_{g \in \text{Per}_n(\sigma)} F(gx) - \int_G F(gx) \, d\mu \right| \leq \varepsilon/3
\]
for any \( n \geq N(\varepsilon) \). Combining the above inequalities we get
\[
\left| N_n(\sigma)^{-1} \sum_{g \in \text{Per}_n(\sigma)} F(gx) - \int_G F(gx) \, d\mu \right| \leq \varepsilon
\]
for any \( x \in X \) and any \( n \geq N(\varepsilon) \).

Recall that \( \int_X F \, dm = \int_{X_0} H_F \, dm_0 \) where \( H_F(\pi(x)) = \int_G F(gx) \, d\mu \). As \( m_0 \) is derived from \( S_0 \) by periodic points there is an \( M \geq N(\varepsilon) \) such that
\[
\left| \int_{X_0} H_F \, dm_0 - N_n(S_0)^{-1} \sum_{y \in \text{Per}_n(S_0)} H_F(y) \right| \leq \varepsilon
\]
for any \( n \geq M \). At this stage of the proof we need the following.

**Lemma.** If \( S_0^n(y) = y \) then \( S^n(x) = x \) for some \( x \in \pi^{-1}(y) \).

**Proof.** Let \( z \in \pi^{-1}(y) \). Then \( S^n(z) = g_1 z \) for some \( g_1 \in G \), \( S^n(gz) = \sigma^n(g)g_1 z \). We want to solve \( S^n(gz) = gz \) or \( g = \sigma^n(g)g_1 \). In additive notation \( (\sigma^n - I)g = -g_1 \). Since \( \sigma^n \) is hyperbolic, there is such a \( g \). Let \( x = gz \). By this lemma for \( y \in \text{Per}_n(S_0) \) choose \( x_y \in \pi^{-1}(y) \cap \text{Per}_n(S) \). Then
\[
N_n(S_0)^{-1} \sum_{y \in \text{Per}_n(S_0)} F(gx_y) = N_n(S)^{-1} \sum_{x \in \text{Per}_n(S)} F(z).
\]

Hence, as \( \int_X F \, dm = \int_{X_0} H_F \, dm_0 \), we have
\[
\left| \int_X F \, dm - N_n(S)^{-1} \sum_{x \in \text{Per}_n(S)} F(z) \right| \leq 2\varepsilon
\]
for all \( n \geq M \).

Suppose \( f: N/\Gamma \to N/\Gamma \) is a hyperbolic automorphism of a nilmanifold (one can see [13] or [16] for the definition). Then \( N/\Gamma \) has a unique normalized Borel measure \( m \) which is invariant under the action of \( N \); \( m \) is \( f \)-invariant. It is well known that \( (f, m) \) is obtained through a succession of extensions via hyperbolic toral automorphisms with a single point as the initial base space. By 7.2 we have that \( m \) is derived from \( f \) by periodic points.
THEOREM. If \( f \) is a hyperbolic automorphism of a nilmanifold, then \( \mu_f = m \).

Proof. \( f \) satisfies Axiom A* and is \( C \)-dense since \( N/\Gamma \) is connected (by 2.7). 6.7 says that \( \mu_f \) is derived from \( f \) by periodic points. At most one measure can be derived from \( f \) by periodic points.

REMARK. Conversations with W. Parry, S. Smale, and P. Walters were helpful in finding a proof for 7.3. Parry in particular pointed out how the periodic points of \( S \) are related to those of \( S_G \) and \( \sigma \). Hyperbolic automorphisms of nilmanifolds thus distribute their periodic points uniformly with respect to the usual measure. For this particular case §§6 and 8 yield already known facts (see [2] or [13] for example).

8. The entropy of \( \mu_f \). We refer the reader to [5] for a definition of measure theoretic entropy.

(8.1) Suppose \( f: X \to X \) satisfying Axiom A* is topologically transitive. Then \( h_{\mu_f}(f) = h(f) \).

Proof. By 5.5 we may assume \( f \) is \( C \)-dense. Cover \( X \) by open sets \( U_1, \ldots, U_r \) with \( \text{diam } U_i < \delta^* \). Choose disjoint Borel sets \( A_1, \ldots, A_r \) such that \( U_i \supset A_i \) and \( X = \bigcup_{i=1}^r A_i \). In [8] L. Goodwyn shows that for any \( f \)-invariant normalized Borel measure \( \rho \) on \( X \) (and \( f: X \to X \) any continuous map) we have \( h_{\rho}(f) \leq h(f) \). We complete our proof by showing the partition \( \beta = \{A_1, \ldots, A_r\} \) satisfies \( h_{\mu_f}(f, \beta) \geq h(f) \).

By 3.9(v) there are \( m_0 \) and \( S > 0 \) such that \( \theta(V) \leq 1/SN_m(f) \) for all \( m \geq m_0 \). Then \( \mu_f(D) \leq \theta(V) \leq 1/SN_m(f) \). Define the function

\[
\int_{h_m} d\mu_f \to h_{\mu_f}(f; \beta)
\]

as \( n \to \infty \). Hence, using 4.5,

\[
h_{\mu_f}(f; \beta) \geq \lim_{m} \frac{1}{m} [\log N_m(f) + \log S] = h(f).
\]

REFERENCES


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