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LIE-ADMISSIBLE, NODAL, NONCOMMUTATIVE JORDAN ALGEBRAS(1)

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Abstract. The main theorem is that if A is a central simple flexible algebra, with an identity, of arbitrary dimension over a field F of characteristic not 2, and if A is Lie-admissible and A^+ is associative, then ad $(A)'=[A, A]/F$ is a simple Lie algebra. It is shown that this theorem applies to simple nodal noncommutative Jordan algebras of arbitrary dimension, and hence that such an algebra A also has derived algebra ad $(A)'$ simple.

1. **Introduction.** An algebra A is said to be nodal in case every element can be written as $\alpha 1+n$ for α in the base field, 1 the identity of A, and n a nilpotent element, and if the set of nilpotent elements is not an ideal of A. A is called Lie-admissible if A^- (which has multiplication $(a, b)\to[a, b]=ab-ba$) is a Lie algebra. It was shown by R. H. Oehmke in [3] that if A is a finite-dimensional simple, Lie-admissible, nodal, noncommutative Jordan algebra of characteristic $p>2$, then ad $(A)'$ is a simple Lie algebra.

The main result here is to prove this theorem without any assumptions about dimensionality of the algebra A.

We first show, in §2, that if A is a simple nodal noncommutative Jordan algebra, then A^+ (which has the multiplication $(a, b)\to a\cdot b=\frac{1}{2}(ab+ba)$) is associative. Thus the above theorem turns out to be the characteristic $\neq 0$ of the following theorem:

Let A satisfy

(1) A is central simple, flexible, with an identity, over a field of characteristic $\neq 2$;

(2) A is Lie-admissible, and $[A, A]\neq 0$; and

(3) A^+ is associative.

Then ad $(A)'$ is a simple Lie algebra.

The inclusion of characteristic 0 seems to be nice; however, in §4 we show that there do not exist any such characteristic 0 algebras which are algebraic.

2. As mentioned in the introduction, we show here that if A is a simple nodal, noncommutative Jordan algebra over a field F, then A^+ is associative. Suppose A

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is such an algebra. Then a result of McCrimmon [2] implies (without the assumption of simplicity) that $A^+=Fl+N$, for N the nilradical of A^+. Moreover, as in [5, p. 145] it is true that A^+ is D-simple; i.e., A^+ has no ideals invariant under all derivations. That A^+ is associative now follows from the following proposition (which does not use the fact that A^+ is commutative or that N is nil).

PROPOSITION (DUE TO T. S. RAVISANKAR). *Any D-simple nonassociative algebra of the form* $A=F1+R$ (*where* 1 *is the identity of* A, R *is an ideal*) *is associative.*

Proof. In [4], Ravisankar has given the following simple proof for when A is finite dimensional. However, his proof is valid without this assumption. For completeness, we summarize the argument.

Let $P=$the set of associators of A. Then P is a D-subspace and $P\subseteq R$. But the ideal generated by P, say P^*, equals $P+PA+AP+A(AP)A+A(PA)+\cdots$, and it is clear from this that P^* is also D-invariant and contained in R. I.e., P^* is a proper D-ideal. Thus $P^*=0$ and $P=0$.

3. We shall let $\mathscr{A}$ denote the class of algebras satisfying

(1) A is central simple, flexible, with an identity, over a field of characteristic $\neq 2$;

(2) A is Lie-admissible, and $[A, A]\neq 0$; and

(3) A^+ is associative.

The proof that A in $\mathscr{A}$ implies that ad $(A)'$ is simple is modeled on Herstein's results [1] for the associative case and is based on the following sequence of lemmas and on Theorem 1, which we shall state later. By ad x, for x in A, we shall mean $[x,\ \ ]=L_x-R_x$. For A in $\mathscr{A}$, we have ad x is a derivation of A^-; also by [5, p. 146] ad x is a derivation of A^+. Therefore we have

LEMMA 1. *Let A be in $\mathscr{A}$, and x in A. Then* ad x *is a derivation of A.*

LEMMA 2. *Let A be in $\mathscr{A}$, and suppose the characteristic of the base field F is $p>2$. Then A is algebraic over F and N, the nilradical of A^+, equals the set of non-invertible elements of A^+.*

Proof. Let $T=\{x^p : x \text{ is in } A\}$. Then $[A, T]=0$, for given x, y in A, $[y, x^p]=px^{\cdot p-1}\cdot[y, x]=0$. But then associativity of A^+ implies directly that T is in the nucleus, whence the center, of A. That is, $T\subseteq F1$, which in turn proves the lemma.

LEMMA 3. *Let A be in $\mathscr{A}$, and suppose the characteristic is* 0. *Then N, the nilradical of A^+, $=0$. More generally, if A is a flexible, Lie-admissible algebra over a field of characteristic* 0, *and if A^+ is associative, then N is an ideal of A.*

Proof. First suppose A is a flexible, Lie-admissible algebra over a field F of characteristic 0, and that A^+ is associative. That $(\text{ad } a)^j(x^k)$ is in $x\cdot A$ for $k=1, 2, \ldots$ and $0\leqq j<k$, and a, x arbitrary elements in A follows by induction on k and by the Leibnitz formula. Using this result, induction and Leibnitz again, it next follows that $(\text{ad } a)^j(x^j)=j!\,[a, x]^j \pmod{x\cdot A}$ for $j=1, 2, \ldots$. Now suppose x is in N, the nilradical of A^+. If $x^m=0$,

$$0 = (\text{ad } a)^m(x^m) = m!\,[a, x]^m$$

holds for arbitrary a in A. This forces $[a, x]^m=0$ and therefore $[a, x]$ to be in N. But N an ideal of A^- implies N is an ideal of A.

LEMMA 4. *Let A be flexible and Lie-admissible and U be an ideal of A^-. Define $T(U)=\{x \text{ in } A;\ [x, A]\subseteq U\}$. Then $T(U)$ is both a Lie ideal of A^- and a subalgebra of A. Moreover, $U\subseteq T(U)$.*

Proof. Let a, b be in $T(U)$, and r in A. By the Jacobi identity, $[[a, r], A]\subseteq [[a, A], r]+[a, [A, r]]\subseteq [U, r]+[a, A]\subseteq U$. Thus $[a, r]$ is in $T(U)$ and $[A, T(U)] \subseteq T(U)$, so $T(U)$ is a Lie ideal of A^-. Next $[a^2, r]=2a\cdot[a, r]=2[a, a\cdot r]$, which is in U because a is in $T(U)$; since r is arbitrary this implies a^2 is in $T(U)$. I.e., a in $T(U)$ implies a^2 is in $T(U)$; linearizing implies $ab+ba$ is in $T(U)$, and adding this to $ab-ba$ in $T(U)$ yields $2ab$ in $T(U)$. Thus $T(U)$ is a subalgebra of A.

LEMMA 5. *Let A be in $\mathscr{A}$ and U be an ideal of A^-. Then $[U, U]=0$ or $[A, A]\subseteq U$.*

Proof. Consider $T(U)$. If $T(U)=A$, $[A, A]=[A, T(U)]\subseteq U$ and we are done. Hence suppose $T(U)\subsetneq A$. By Lemma 1, for a, b in $T(U)$ and x in A, $[a, b]x= [a, bx]-b[a, x]$, and so Lemma 4 implies that $[a, b]x$ is in $T(U)$. Similarly $x[a, b]$ is in $T(U)$, so by multiplying in A^+, we have $[T(U), T(U)]\cdot A\subseteq T(U)\subsetneq A$. However $[T(U), T(U)]\cdot A$, an ideal of A^+ by associativity, is also an ideal of A^-. So $[T(U), T(U)]\cdot A$ is a proper ideal of A, and as such it must $=0$. But 1 in A now implies that $[T(U), T(U)]=0$, so $U\subseteq T(U)$ finishes the proof.

LEMMA 6. *Let A be in $\mathscr{A}$ and U be an ideal of A^- such that $[U, U]=0$. Then $[U, A]=0$.*

Proof. Let u be in U, a in A. Then $0=[u, [u, a^2]]=[u, a[u, a]]+[u, [u, a]a]$. But also $[u, a[u, a]]-[u, [u, a]a]=[u, [a, [u, a]]]=0$, and adding these two equations implies $[u, a[u, a]]=0$, or $0=[u, a][u, a]+a[u, [u, a]]$. But again $[u, [u, a]]$ is 0, implying finally $0=[u, a]^2$. Thus $[u, a]$ is in N, the nilradical of A^+, so in general $[U, A]\subseteq N$, and $[U, A]\cdot A\subseteq N$. But $[U, A]\cdot A$ is an ideal of both A^+ and A^- so is an ideal of A. As it is not all of A, it must be 0. Consequently $[U, A]=0$.

These six lemmas, together with the fact that $[U, A]=0 \Rightarrow U\subseteq F1$, imply

THEOREM 1. *Let A be in $\mathscr{A}$ and U be an ideal of A^-. Then $[A, A]\subseteq U$ of $U\subseteq F1$.*

LEMMA 7. *Let A be in $\mathscr{A}$ and U be an ideal of $[A, A]^-$. If $U\subseteq M$, for M a primary ideal of A^+, then $U\subseteq F1$. In particular if $U\subseteq N$, the nilradical of A^+, then $U=0$.*

Proof. Let u be in U, a in A. Then $[u, a]^2=[u, a]^{\cdot 2}=[u, a\cdot[u, a]]-a\cdot[u, [u, a]] =[u, [a\cdot u, a]]-a\cdot[u, [u, a]]$. But this last expression is contained in $U+A\cdot U\subseteq M$. Thus the assumption that M be primary implies that already $[u, a]$ is in M. That is $U\subseteq M$ implies $[U, A]\subseteq M$. But the Jacobi identity implies that $[U, A]$ is also an ideal of $[A, A]^-$, so the argument may be repeated to get $[[U, A], A]\subseteq M$, and by induction, then for $n=1, 2, \ldots$, we obtain $B_n=[[\ldots[[U, A], A], \ldots], A]\subseteq M$, where there are n A's in B_n. Therefore also $B=\sum_{n=0}^{\infty} B_n\subseteq M$. But clearly $[B_n, A] \subseteq B_{n+1}$ holds for all n, so actually B is an ideal of A^-. We may assume $B\not\supseteq[A, A]$, for if $[A, A]\subseteq B$ occurred we would have M an ideal of A ($[M, A]\subseteq[A, A]\subseteq B\subseteq M$ means M is an ideal of A^-) forcing $M=0$ and the lemma to be trivial. That is, B is a Lie ideal of A^- which does not contain $[A, A]$ so by Theorem 1, $B\subseteq F1$. Since $U\subseteq B$ by definition we have $U\subseteq F1$. Finally if $M=N$ is nil, since N is primary, $U\subseteq F1 \cap N=0$.

LEMMA 8. *Let A be in $\mathscr{A}$ and U be an ideal in $[A, A]^-$ such that $[U, U]=0$. Then $U\subseteq F1$.*

Proof. Let u, v, w be in U, x in $[A, A]$, and z be in A. Then

$$\begin{aligned} 0 &= [u, [v, [x, zw]]] \\ &= [u, [v, z[x, w]]]+[u, [v, [x, z]w]] \\ &= [u, [v, z][x, w]]+[u, z[v, [x, w]]]+[u, [v, [x, z]]w]+[u, [x, z][v, w]] \\ &= [u, [v, z][x, w]]+[u, [v, [x, z]]w] \\ &= [v, z][u, [x, w]]+[u, [v, z]][x, w]+[v, [x, z]][u, w]+[u, [v, [x, z]]]w \\ &= [u, [v, z]][x, w]. \end{aligned}$$

Thus $[U, [U, A]][[A, A], U]=0$. Then for $V=[U, [U, A]]$ we have that V is an ideal of $[A, A]^-$ and that $V^2=0$, as $V^2\subseteq[U, [U, A]][U, [A, A]]=0$. In particular, V is contained in N, so Lemma 7 implies $V=0$.

Now consider u in U, a in A; using $V=0$, we obtain $[u, a]^2=[u, a][u, a] =[u, a[u, a]]-a[u, [u, a]]=[u, a[u, a]]=[u, [u, a^2]]-[u, [u, a]a]=-[u, [u, a]a] =-[u, a]^2$. Therefore, characteristic not 2 implies $[u, a]^2=0$, so in particular $[u, a]$ is in N, and therefore $[U, A]\subseteq N$. But as $[U, A]$ is a Lie ideal of $[A, A]$, Lemma 7 implies $[U, A]=0$. Finally, this implies $U\subseteq F1$, which was to be proved.

LEMMA 9. *Let A be in $\mathscr{A}$ and U be an ideal of $[A, A]^-$ such that $[U, U]\subseteq F1$. Then $U\subseteq F1$.*

Proof. Let v be in U and suppose there exists a u in U such that $[u, v]=\alpha\neq 0$ in $F1$. Let $x=[y, v]$ so x, $xv=[y, v]v=[yv, v]$ and $(xv)v=[yv, v]v=[(yv)v, v]$ are all in $[A, A]$. Then $F1$ contains

$$\gamma = [u, [x, v]],$$

$$\begin{aligned} \beta &= [u, [xv, v]] \\ &= [u, [x, v]v] \\ &= [x, v][u, v]+[u, [x, v]]v \\ &= \alpha[x, v]+\gamma v, \end{aligned}$$

and thirdly,

$$\begin{aligned}\delta &= [u, [(xv)v, v]] \\ &= [u, [xv, v]v] \\ &= [xv, v][u, v]+[u, [xv, v]]v \\ &= \alpha[x, v]v+\beta v.\end{aligned}$$

Thus we have

(1) $\beta v+\alpha[x, v]v=\delta$,

(2) $\gamma v+\alpha[x, v]=\beta$, with α, β, γ and δ in $F1$.

Then v times equation (2) minus equation (1) yields $\gamma v^2-2\beta v+\delta=0$. If $\gamma\neq 0$ for some choice of y, then this equation implies $v^2=(2\beta/\gamma)v-(\delta/\gamma)$, so $[u, v^2]=(2\beta/\gamma)\alpha$, which is in $F1$. However $[u, v^2]=[u, v]v+v[u, v]=2\alpha v$, and equating gives $2\alpha v=(2\beta/\gamma)\alpha$ in $F1$, and hence the assumption that $\alpha\neq 0$ implies v is in $F1$, a contradiction. This means that we may assume $\gamma=0$ no matter which x is chosen. In particular, for $x=[y, v]v$ we have $[u, [[y, v]v, v]]=0$, and for $x=[y, v]$, we have $[u, [[y, v]v]]=0$. Thus

$$\begin{aligned}0 &= [u, [[y, v]v, v]] = [u, [[y, v], v]v]+[u, [y, v][v, v]] = [u, [[y, v], v]v] \\ &= [[y, v], v][u, v]+[u, [[y, v], v]]v = \alpha[[y, v], v].\end{aligned}$$

Again we apply the assumption that $\alpha\neq 0$, this time to get $[[y, v], v]=0$. This holds for all y in A, and any v for which there exists a u in U with $[v, u]\neq 0$. In other words $[v, U]\neq 0$ implies $[[A, v], v]=0$. Next consider $U^*=\{u \text{ in } U;\ [u, U]=0\}$. We note that $[[U^*, [A, A]], U]\subseteq[[U^*, U], [A, A]]+[U^*, [[A, A], U]]=0$, so $[U^*, [A, A]]\subseteq U^*$. Therefore U^* is a Lie ideal of $[A, A]^-$ satisfying $[U^*, U^*]=0$, and so Lemma 8 implies that $U^*\subseteq F1$. In particular v in U^* implies $[v, y]=0$, so also $[[y, v], v]=0$ trivially. Thus we now have that for any v in U (v in U^* or not) we have $[[y, v], v]=0$ for all y in A. By linearizing this in the subspace U, we get

$$[[y, u], v]+[[y, v], u] = 0 \quad \text{for all } u, v \text{ in } U, y \text{ in } A.$$

But

$$[[y, u], v]+[[v, y], u] = [[v, u], y] = 0 \quad \text{by the Jacobi identity,}$$

and adding these equations yields $[[y, u], v]=0$ for all u, v in U, y in A. That is $[[A, U], U]=0$, which by the proof of Lemma 8 already implies $U\subseteq F1$, the desired conclusion.

REMARK. The last two lemmas imply that if U is an ideal of $[A, A]^-$, and if U is solvable, then $U\subseteq F1$. The next two lemmas will imply that any proper ideal of $[A, A]^-$ is solvable, and therefore is contained in *Fl.*

LEMMA 10. *Let U be an ideal of* $[A, A]^-$ *for A Lie-admissible and flexible. Define* $T(U)=\{x \text{ in } A : [x, A]\subseteq U\}$. *Then*

(1) $[U, U]\subseteq T(U)$,

(2) $[U, T(U)]\subseteq T(U)$,

(3) $[[A, T(U)], T(U)] \subseteq T(U) \cap U$,
(4) $[[T(U), T(U)], A] \subseteq T(U) \cap U$,
(5) $[T(U), T(U)] \subseteq T(U)$,
(6) $T(U)$ *is a subalgebra of* A.

Proof. We shall only include the proof of (6). For a in $T(U)$ and r in A we have $[a^2, r] = 2[a, a \cdot r]$ is in U, and thus a in $T(U)$ implies a^2 is in $T(U)$. Linearizing and using part (5) then gives that $T(U)$ is a subalgebra.

LEMMA 11. *Let A be in $\mathscr{A}$ and U be an ideal of* $[A, A]^-$. *Then* $U = [A, A]$ *or* $U^{(3)} = [[[U, U], [U, U]], [[U, U], [U, U]]] \subseteq F1$.

Proof. Let $T(U)$ be as in Lemma 10. $T(U) = A$ implies $[A, A] \subseteq U$ or $[A, A] = U$, so we may suppose $T(U) \subsetneq A$. Define $B = [T(U), T(U)]$, and let a, b be in B, and r in A. Then $[b, a]r = -a[b, r] + [b, ar]$, which is contained in $T(U)$ by Lemma 10. Similarly $r[a, b]$ is in $T(U)$. Clearly then $A \cdot [B, B] \subseteq T(U) \subsetneq A$. Since $A \cdot [B, B]$ is thus a proper ideal of A^+ containing $[B, B]$, we may suppose by a Zorn's lemma argument that $[B, B] \subseteq M$ for M a maximal (hence primary) ideal of A^+. Also $[[T(U), [A, A]], A] \subseteq U$ implies $[T(U), [A, A]] \subseteq T(U)$. Thus $T(U)$, whence also $[T(U), T(U)] = B$, and finally $[B, B]$ are Lie ideals of $[A, A]^-$. Therefore, by Lemma 7, $[B, B] \subseteq F1$, so $[[T(U), T(U)], [T(U), T(U)]] \subseteq F1$, so that $[u, u] \subseteq T(U)$ completes the proof.

THEOREM 2. *Let A be in $\mathscr{A}$. Then* ad $(A)'$ *is a simple Lie algebra.*

Proof. Three things must be shown:
(i) $[A, A] \not\subseteq F1$, so that ad $(A)' = [A, A]/F1$ will be nontrivial;
(ii) $[A, A] = [[A, A], [A, A]]$; and
(iii) if U is an ideal of $[A, A]^-$, then $U = [A, A]$ or $U \subseteq F1$.

(iii) is now clear by the last three lemmas, for suppose U is a proper Lie ideal of $[A, A]^-$. Lemma 11 implies that $U^{(3)} \subseteq F1$, so $U^{(2)} = [[U, U], [U, U]]$ is a Lie ideal of $[A, A]^-$ satisfying $[U^{(2)}, U^{(2)}] = U^{(3)} \subseteq F1$. Thus Lemma 9 first implies $U^{(2)} \subseteq F1$, then applied to $U' = [U, U]$ it implies $U' \subseteq F1$ and finally applied to U it implies $U \subseteq F1$.

Now (iii) implies that if $[[A, A], [A, A]] \subsetneq [A, A]$, then we must already have $[[A, A], [A, A]] \subseteq F1$, whence Lemma 9 implies $[A, A] \subseteq F1$. Thus it suffices to prove $[A, A] \subseteq F1$ cannot happen, or that indeed $[A, A] \not\subseteq F1$. Now, if this fails to happen, there would exist x, y in A with $[x, y] = \alpha 1$ in $F1$ and $\alpha \neq 0$. But $\alpha x = [x, y]x = [x, yx]$ in $F1$ forces x to be in Fl and $[x, y] = 0$, a contradiction.

COROLLARY. *If A is a simple, nodal, noncommutative Jordan algebra which is Lie-admissible and of characteristic $p > 2$, then* ad $(A)'$ *is a simple Lie algebra.*

4. We included characteristic 0 in the previous section; however, we have the following

PROPOSITION. *There do not exist any algebras in the class of characteristic* 0 *which are algebraic.*

Proof. Let A be in $\mathscr{A}$ and first assume that the nilradical, N, of A^+ equals the set of noninvertible elements of A^+. Then by Lemma 3, A^+ is actually a field (since $N=0$). Hence for arbitrary (algebraic) elements x, y, z in A, the subalgebra of A^+ generated by x, y, z is a finite field extension of F. Such an extension (of characteristic 0) has a primitive element, say w, so that x, y, z are all polynomials in w. But as powers in A^+ and A coincide this holds in A also, so finally power associativity implies that x, y, z both commute and associate in A. That is, A is commutative and associative, A = the center of $A=F1$, and A is trivial. We now complete the proof of the proposition by proving the

LEMMA. *If A is in $\mathscr{A}$ and is algebraic over the arbitrary field F of characteristic* 0, *then N, the nilradical of A^+, equals the set of noninvertible elements of A^+.*

Proof. First, A is of (idempotent) degree 1, for suppose $e\neq 0$ is an idempotent of A. Then the Peirce decomposition of A^+ is simply $A^+=A_e^+(1)+A_e^+(0)$ and so $A=A_e(1)+A_e(0)$ is also the Peirce decomposition of A. This implies $A_e(1)$ is actually an ideal of A, whence $A_e(1)=A$ and finally $e=1$. Now because A is algebraic, the subalgebra generated by x, $F[x]$, is a finite-dimensional commutative associative algebra. Hence, if x is not in N, $F[x]$ is not nilpotent, so $F[x]$ contains an idempotent e, which by the first paragraph must equal 1. That is, 1 is in $F[x]$ and x is invertible. Hence any nonnilpotent element is invertible. As the converse is obvious, the lemma is proved.

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