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TOPOLOGIES FOR 2^X; SET-VALUED FUNCTIONS AND THEIR GRAPHS(1)

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Abstract. We consider the problem of topologizing 2^X, the set of all closed subsets of a topological space X, in such a way as to make continuous functions from a space Y into 2^X precisely those functions with closed graphs. We show there is at most one topology with this property, and if X is a regular space, the existence of such a topology implies that X is locally compact. We then define the compact-open topology for 2^X, which has the desired property for locally compact Hausdorff X. The space 2^X with this topology is shown to be homeomorphic to a space of continuous functions with the well-known compact-open topology. Finally, some additional properties of this topology are discussed.

1. **Introduction.** Given a topological space X, let 2^X denote the set of all closed subsets of X (including the empty set $\varnothing$). We will consider the problem of putting a topology on the set 2^X which has a natural relationship to the given topology on X. In particular we choose conditions for the topology on 2^X which will identify continuous maps from spaces Y into 2^X in terms of the topologies of Y and X.

Let Y be a topological space, and $F\colon Y \to 2^X$. Define the *graph* of F to be the set $G_F = \{(x, y) \in X \times Y \mid x \in F(y)\}$. We say a topology on 2^X is *admissible* if for any space Y and any continuous map $F\colon Y \to 2^X$, G_F is a closed subset of $X \times Y$. Conversely, we say a topology on 2^X is *proper* if for any space Y and function $F\colon Y \to 2^X$, if G_F is closed then F is continuous. We will consider the problem of giving 2^X a topology which is both admissible and proper, i.e., one for which continuity of functions is equivalent to their having closed graphs. (Recall that for *point-valued* functions into a compact Hausdorff space X it is in fact true that the properties of continuity and closed graph are equivalent.)

In §2 we consider some properties of proper and admissible topologies for 2^X. In particular, we will show that there is at most one topology for 2^X which is both proper and admissible. In addition if X is a regular space (T_1, and points and closed

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sets can be separated), the existence of a proper-admissible topology for 2^X implies that X is locally compact. In §3 we define a topology for 2^X (called the compact-open topology) which for Hausdorff spaces X is always proper, and is admissible if and only if X is locally compact. In §4 we will show that we can weaken the restriction on X, and instead require that $X\times Y$ be a k-space, and this will imply that $F\colon Y\to 2^X$ is continuous in the compact-open topology if and only if G_F is closed. We will also describe some other properties of this topology.

2. **Admissible and proper topologies.** We begin by giving a characterization of an admissible topology.

LEMMA 2.1. *A topology on 2^X is admissible if and only if*

$$\Omega = \{(x, A) \in X\times 2^X \mid x \in A\}$$

is closed in $X\times 2^X$.

Proof. If the topology is admissible, then Ω is closed since it is the graph of the identity from 2^X to itself, which is obviously continuous.

Suppose now that Ω is closed. Let Y be any space and $F\colon Y\to 2^X$ be continuous. Then $G_F=(1, F)^{-1}(\Omega)$ where $(1, F)\colon X\times Y\to X\times 2^X$ is given by $(1, F)(x, y) =(x, F(y))$ and is continuous. Hence G_F is closed.

If t_1 and t_2 are two topologies for the same set, and $t_1\subset t_2$, we say t_1 is weaker than t_2, and t_2 is stronger than t_1. When more than one topology is being considered at one time, we will denote by $2^X(t)$, the space 2^X with the topology t.

LEMMA 2.2. (a) *A topology stronger than an admissible topology is also admissible.*

(b) *A topology weaker than a proper topology is also proper.*

(c) *Any admissible topology is stronger than any proper topology.*

Proof. Parts (a) and (b) are clear. To prove (c), let t_1 be a proper topology and t_2 be an admissible topology. Then by Lemma 2.1, $\{(x, A) \mid x \in A\}$ is closed in $X\times 2^X(t_2)$, which implies $1\colon 2^X(t_2)\to 2^X(t_1)$ is continuous. Thus $t_1\subset t_2$.

From Lemma 2.2(c) we obtain the uniqueness of a topology which is both proper and admissible.

THEOREM 2.3. *For any topological space X, 2^X can have at most one topology which is both proper and admissible. Such a topology is necessarily the strongest proper topology on 2^X and the weakest admissible topology on 2^X. (By "strongest" or "weakest" we mean "containing" or "contained in" all other such topologies.)*

We are thus naturally led to consider the question of when 2^X can have a strongest proper topology or a weakest admissible topology. Such topologies must necessarily be unique. The first question is easily settled.

PROPOSITION 2.4. *For any X, 2^X has a strongest proper topology.*

Proof. Let $\{t_\alpha\}$ be the family of all proper topologies. Let t be the topology which has the set $\bigcup_\alpha t_\alpha$ as a subbasis. Since this is stronger than any proper topology, we need only show that it is proper.

Let Y be any space, and $F\colon Y \to 2^X$ be such that G_F is closed. Since any subbasic $V \in t$ is an open set for some proper topology, $F^{-1}(V)$ must be open and hence F is continuous.

On the question of a weakest admissible topology, we first state the following proposition whose proof is clear.

PROPOSITION 2.5. *If 2^X has a weakest admissible topology, it must be the intersection of all admissible topologies.*

We now prove the main result of this section.

THEOREM 2.6. *If X is a regular space and 2^X has a weakest admissible topology then X is locally compact.*

Proof. Let t be the weakest admissible topology on 2^X. Let $x \in X$. We will show that x has a relatively compact neighborhood.

Since t is admissible, there is an open set $V \subset X$ and a $\mathscr{U} \in t$ such that $x \in V$, $\varnothing \in \mathscr{U}$ and $V \times \mathscr{U} \subset \tilde{\Omega}$, the complement in $X \times 2^X$ of the set Ω defined in Lemma 2.1. We will show that $\overline{V}$, the closure of V, is compact.

Let $\mathscr{A}$ be an open cover of $\overline{V}$, and let $\mathscr{A}^* = \mathscr{A} \cup \{X - \overline{V}\}$, an open cover of X. Let $\{F_\alpha\}$ be the collection of all closed subsets of X each of which is contained in some member of $\mathscr{A}^*$, and let the sets $\{A \in 2^X \mid A \subset X - F_\alpha\}$ be a subbasis for a topology t' on 2^X. We will show now that t' is admissible.

Suppose $y \in X$, $B \in 2^X$ and $y \notin B$. There is an $A^* \in \mathscr{A}^*$ such that $y \in A^*$. Since X is regular, there is an open set $W \subset X$ such that

$$y \in W \subset \overline{W} \subset \tilde{B} \cap A^*.$$

$\overline{W} \subset A^*$ implies $\overline{W} \in \{F_\alpha\}$, and $\overline{W} \subset \tilde{B}$ implies that $B \subset X - \overline{W}$. Thus

$$Z = W \times \{A \mid A \subset X - \overline{W}\}$$

is an open set in $X \times 2^X(t')$, $(y, B) \in Z$ and $Z \subset \tilde{\Omega}$, showing that Ω is closed in $X \times 2^X(t')$, and hence that t' is admissible.

Since t is the weakest admissible topology, we must have $t \subset t'$ and thus $\mathscr{U} \in t'$. From this it follows that there exist $F_1, \ldots, F_n$ in $\{F_\alpha\}$ such that

$$\varnothing \in \bigcap_{i=1}^{n} \{A \mid A \subset X - F_i\} = \left\{A \mid A \subset X - \bigcup_{i=1}^{n} F_i\right\} \subset \mathscr{U}.$$

We show that $V \subset \bigcup_{i=1}^{n} F_i$. Suppose, to the contrary, that there is a $y \in V - \bigcup F_i$. We have then that $\{y\} \in \{A \mid A \subset X - \bigcup F_i\} \subset \mathscr{U}$ and therefore $(y, \{y\}) \in V \times \mathscr{U} \subset \tilde{\Omega}$. However, this is impossible since $y \in \{y\}$, and thus we have $V \subset \bigcup F_i$. It follows that also $\overline{V} \subset \bigcup F_i$.

To complete the proof, for $i = 1, \ldots, n$ let $V_i \in \mathscr{A}^*$ be such that $F_i \subset V_i$. Therefore $\overline{V} \subset \bigcup_{i=1}^{n} V_i$, and we have produced a finite subcover of $\mathscr{A}^*$. We also have a finite subcover of $\mathscr{A}$ since we can omit $X - \overline{V}$ if it occurs among the V_i.

COROLLARY 2.7. *Let X be a regular space. If the intersection of all admissible topologies for 2^X is admissible, then X is locally compact.*

COROLLARY 2.8. *Let X be a regular space. If 2^X has a topology which is both proper and admissible, then X is locally compact.*

In the next section we will show that converses of Theorem 2.6 and the corollaries are true. That is, we will show that if X is a locally compact Hausdorff space, then it has a proper-admissible topology, and hence a weakest admissible topology. (It is clearly regular.)

The problem remains as to whether the hypothesis on X can be weakened, say, to Hausdorff. Theorem 2.6 says that if X is a regular space which is not locally compact (e.g. $X=Q$, the space of rationals in the real line), then 2^X has no weakest admissible topology (and thus no proper-admissible topology). An open question is whether a nonregular Hausdorff space can have a weakest admissible topology.

Before proceeding to the next section, we will study an interesting consequence of the proof of Theorem 2.6. If X is regular, and $\mathscr{A}$ is a cover of X by open sets, then $\mathscr{A}$ gives rise to an admissible topology, $t_{\mathscr{A}}$ on 2^X as follows. Let $\{F_\alpha\}$ denote the set of closed subsets of X each of which is contained in some member of $\mathscr{A}$. Let $t_{\mathscr{A}}$ be the topology on 2^X whose subbasis consists of all sets of the form

$$\{A \in 2^X \mid A \subset X-F_\alpha\}.$$

PROPOSITION 2.9. (a) *Let X be a regular space. For each open cover $\mathscr{A}$ of X, $t_{\mathscr{A}}$ is an admissible topology on 2^X.*

(b) *If $\mathscr{A}$ and $\mathscr{B}$ are open covers, and $\mathscr{A}$ is a refinement of $\mathscr{B}$, then $t_{\mathscr{A}} \subset t_{\mathscr{B}}$.*

(c) *If $\mathscr{A}$ is an open cover which is closed under finite unions, then the sets $\{A \in 2^X \mid A \subset X-F_\alpha\}$ form a basis for $t_{\mathscr{A}}$.*

(d) *Let X be a normal space. If $\mathscr{B}$ is an open cover, and $\mathscr{A}$ is an open cover consisting of all elements of $\mathscr{B}$ and finite unions of elements of $\mathscr{B}$, then $t_{\mathscr{A}} = t_{\mathscr{B}}$ and hence a basis for $t_{\mathscr{B}}$ is formed by the sets given in* (c).

Proof. The proof of (a) is contained in the proof of Theorem 2.6. Part (b) follows from the fact that if $\mathscr{A}$ is a refinement of $\mathscr{B}$ then we have $\{F_\alpha\} \subset \{F_\beta\}$ where these are the respective collections used in the construction of $t_{\mathscr{A}}$ and $t_{\mathscr{B}}$. Part (c) follows from the fact that

$$\bigcap_{i=1}^{n} \{A \mid A \subset X-F_i\} = \left\{A \mid A \subset X-\bigcup_{i=1}^{n} F_i\right\}.$$

For part (d), since $\mathscr{B}$ is a refinement of $\mathscr{A}$, we have $t_{\mathscr{B}} \subset t_{\mathscr{A}}$. That $t_{\mathscr{A}} \subset t_{\mathscr{B}}$ follow from the fact that in a normal space if F is closed and $F \subset A_1 \cup \cdots \cup A_n$, where the A_i are open, then there exist closed sets F_i such that $F_i \subset A_i$ and $F = F_1 \cup \cdots \cup F_n$.

We remark here that if X is any Hausdorff space, and if we take the open cover $\mathscr{A} = \{X\}$, then the collection $\{F_\alpha\}$ consists of all closed subsets of X and $t_{\mathscr{A}}$ is the well-known *upper semifinite* topology. See Michael [7, p. 179] and Ponomarev [9] (where it is called the κ topology). Notice that for any open cover $\mathscr{B}$, $t_{\mathscr{B}} \subset t_{\mathscr{A}}$. Also, since for a regular space the weakest admissible topology t is weaker than $t_{\mathscr{A}}$, this topology t will not be Hausdorff.

In the next section we shall consider a topology for 2^X which for locally compact Hausdorff X can be described as that derived from the cover of X by relatively compact open sets. It therefore must be admissible. In fact, we will show it is also proper, and thus is the weakest admissible topology.

3. **Compact-open topology: 2^X as a function space.** Let X be a topological space, and let $\{C_\alpha\}$ be the set of all *compact* subsets of X. We define the *compact-open topology* on 2^X to be the topology with the basis consisting of sets of the form $\{A \in 2^X \mid A \subset X - C_\alpha\}$.

THEOREM 3.1. *Let X be a Hausdorff space. Then the compact-open topology on 2^X is admissible if and only if X is locally compact.*

Proof. Suppose X is locally compact and let $\mathscr{C}$ be the open cover of X consisting of all relatively compact open sets. It follows that the compact-open topology is the topology $t_{\mathscr{C}}$, and since X is regular, Proposition 2.9(a) shows that $t_{\mathscr{C}}$ is admissible.

Now suppose the compact-open topology is admissible and let $U \subset X$ be open and let $x \in U$. We will find a relatively compact open set $V \subset X$ such that

$$x \in V \subset \overline{V} \subset U.$$

Since Ω is closed in $X \times 2^X$, we can find an open set $V \subset X$ and a basic open set $\mathscr{W} = \{A \in 2^X \mid A \subset X - C\}$, C compact, so that

$$(x, X - U) \in V \times \mathscr{W} \subset \tilde{\Omega}.$$

It follows that $V \subset C$ and thus we have

$$x \in V \subset \overline{V} \subset C \subset U,$$

and the proof is complete.

We will also show that for any space X, the compact-open topology on 2^X is proper. Before doing this, however, we will discuss a perhaps more familiar space which is homeomorphic to 2^X with the compact-open topology.

We let $\mathscr{S}$ denote the space having two points, 0 and 1, for which the open sets are $\varnothing$, $\mathscr{S}$, and $\{0\}$. (This is known as the Sierpinski space, see [2, p. 63].) Consider the *set* of all continuous functions from X into $\mathscr{S}$, denoted $\mathscr{S}^X$. For each $f \in \mathscr{S}^X$, there corresponds a closed set $\Gamma(f) \in 2^X$ given by $\Gamma(f) = f^{-1}(1)$. Similarly, for each closed set $A \in 2^X$, we can define $\Phi(A) \in \mathscr{S}^X$ by

$$\begin{aligned} \Phi(A)(x) &= 1 \quad \text{if } x \in A, \\ &= 0 \quad \text{if } x \notin A. \end{aligned}$$

Clearly, $\Gamma(\Phi(A)) = A$ and $\Phi(\Gamma(f)) = f$. Thus we have a bijection of the set $\mathscr{S}^X$ onto the set 2^X. Now consider 2^X as a topological space with the compact-open topology defined earlier, and consider $\mathscr{S}^X$ with the compact-open topology as defined for spaces of continuous functions (see, e.g., [2, p. 257]).

LEMMA 3.2. *If $\mathscr{S}^X$ and 2^X have the respective compact-open topologies, the bijection $\Gamma: \mathscr{S}^X \to 2^X$ is a homeomorphism.*

Proof. We need only verify that $\Gamma(\{f \mid f(C) \subset \{0\}\}) = \{A \mid A \subset X - C\}$ for any compact $C \subset X$. But

$$\Gamma(\{f \mid f(C) \subset \{0\}\}) = \{A \mid \Phi(A)(C) \subset \{0\}\} = \{A \mid A \cap C = \varnothing\},$$

which concludes the proof.

Thus the compact-open topology on 2^X is in fact the compact-open topology on a space of continuous functions. We will use this connection to show that the compact-open topology on 2^X is proper.

Suppose Y is any space. For every map $\alpha: X \times Y \to \mathscr{S}$, where α is continuous on X for each fixed $y \in Y$, there is an associated map $\alpha: Y \to \mathscr{S}^X$ given by $[\hat{\alpha}(y)](x) = \alpha(x, y)$. Conversely, given $\hat{\alpha}: Y \to \mathscr{S}^X$, we can define $\alpha: X \times Y \to \mathscr{S}$. Recall that a topology on the function space $\mathscr{S}^X$ is said to be proper if for any space Y and function $\alpha: X \times Y \to \mathscr{S}$, the continuity of α (jointly in x and y) implies the continuity of $\hat{\alpha}$. Similarly, a topology on $\mathscr{S}^X$ is said to be admissible if for any space Y and function $\hat{\alpha}: Y \to \mathscr{S}^X$, the continuity of $\hat{\alpha}$ implies that of α. Equivalently, $\mathscr{S}^X$ has an admissible topology if and only if the evaluation map $\omega: \mathscr{S}^X \times X \to \mathscr{S}$, given by $\omega(f, x) = f(x)$, is continuous. See [2, p. 274].

Since the sets $\mathscr{S}^X$ and 2^X are in a one-to-one correspondence, a topology on one can be viewed as a topology on the other, making Γ a homeomorphism. We now show that the various definitions of admissible and proper are compatible.

LEMMA 3.3. *A topology on the set of subsets 2^X is admissible [proper] if and only if as a topology on the function space $\mathscr{S}^X$ it is admissible [proper].*

Proof. The proof follows from the observation that $\alpha: X \times Y \to \mathscr{S}$ is continuous if and only if

$$\begin{aligned} \alpha^{-1}(1) &= \{(x, y) \in X \times Y \mid [\hat{\alpha}(y)](x) = 1\} \\ &= \{(x, y) \mid x \in \Gamma(\hat{\alpha}(y))\} \\ &= G_{\Gamma \circ \hat{\alpha}} \end{aligned}$$

is closed.

We remark here that the continuity of the evaluation map ω is equivalent to the closure of the set Ω defined in §2.

COROLLARY 3.4. *The compact-open topology on 2^X is always proper.*

Proof. This follows by Lemmas 3.2 and 3.3 and the fact that the compact-open topology on $\mathscr{S}^X$ is always proper (see [2, Theorem 3.1, p. 261]).

COROLLARY 3.5. *Let X be a Hausdorff space. Then the compact-open topology on $\mathscr{S}^X$ is admissible if and only if X is locally compact.*

The next two results provide converses to Theorem 2.6 and its corollaries.

THEOREM 3.6. *If X is a locally compact Hausdorff space, then the compact-open topology on 2^X is both proper and admissible, and hence is the weakest admissible topology on 2^X.*

COROLLARY 3.7. *If X is a locally compact Hausdorff space, then the intersection of all admissible topologies for 2^X is admissible.*

In the next section, we will exploit further the function space structure of 2^X.

4. **Compact-open topology: Further properties.** Each space of subsets and function space considered in this section will be assumed to have the respective compact-open topology.

We now change the original problem somewhat. Given spaces X *and* Y we ask under what conditions can we assert that, for any function F: $Y \to 2^X$, the continuity of F (in the compact-open topology) is equivalent to its graph G_F being closed.

To give an answer to that question, we must recall the definition of a k-space. A Hausdorff space Z is said to be a k-space if $U \subset X$ is open in X if and only if for every compact subset C of Z, $U \cap C$ is open in C. It is known (see [2, p. 248]) that all locally compact and all first countable Hausdorff spaces (and hence all metric spaces) are k-spaces. Also if both X and Y are first countable Hausdorff spaces, or one is a locally compact Hausdorff space and the other is a k-space then $X \times Y$ is a k-space (see [2, p. 263]). This is important because of the following result.

THEOREM 4.1. *If X is locally compact Hausdorff or if $X \times Y$ is a k-space, then the continuity of F: $Y \to 2^X$ is equivalent to its graph G_F being closed.*

Proof. For the case where X is locally compact Hausdorff, this is Theorem 3.6. Since the compact-open topology is always proper, we always have that closed graph implies continuity. Thus we need only prove that if $X \times Y$ is a k-space, then continuity of F implies G_F is closed. This follows from [2, Corollary 3.2, p. 261] and the observation made in the proof of Lemma 3.3.

Theorem 4.1 tells us that, with the right conditions on X or $X \times Y$, any continuous function F: $Y \to 2^X$ gives rise to a closed subset of $X \times Y$. Conversely, any closed subset $A \subset X \times Y$ gives rise to a continuous function F: $Y \to 2^X$, defined by $F(y) = \{x \mid (x, y) \in A\}$, having the property that $G_F = A$. Thus we have a bijection G between the set of continuous functions from Y into 2^X, denoted $(2^X)^Y$, and the set of closed subsets of $X \times Y$, $2^{X \times Y}$, defined by $G(F) = G_F$.

THEOREM 4.2. *Let X be a locally compact Hausdorff space, or let $X \times Y$ be a k-space. Then the function G: $(2^X)^Y \to 2^{X \times Y}$, which assigns to each continuous F its graph G_F, is defined and is a homeomorphism.*

Proof. This follows from Lemma 3.2 and [2, Theorem 5.3, p. 265].

COROLLARY 4.3. *If $X\times Y$ is a k-space, then $(2^X)^Y$ is homeomorphic to $(2^Y)^X$. In fact the homeomorphism is given by mapping the function $F\colon Y\to 2^X$ to the function $F^*\colon X\to 2^Y$ given by $F^*(x)=\{y \mid x\in F(y)\}$.*

COROLLARY 4.4. *Let $F\colon Y\to 2^X$ and let $F^*(x)=\{y \mid x\in F(y)\}$ for all $x\in X$. If X is locally compact Hausdorff, then F is continuous implies $F^*\colon X\to 2^Y$ is defined and is continuous. If $X\times Y$ is a k-space, then F is continuous if and only if F^* is defined and is continuous. (Note that by F^* is defined we mean $F^*(x)$ is closed for all $x\in X$.)*

Thus, under the proper conditions, the study of continuous closed-set valued functions from Y to X is equivalent to the study of closed subsets of $X\times Y$. Conversely, consideration of closed subsets of $X\times Y$ leads to the study of continuous closed-set valued functions from one factor into the other.

The remaining results of this section describe some further continuity properties of the compact-open topology on 2^X. The following is obvious.

PROPOSITION 4.5. *Let X be a Hausdorff space. Then*

(a) *the map $i\colon X\to 2^X$ given by $i(x)=\{x\}$ is continuous, and*

(b) *if $f\colon Y\to X$ is continuous, the map $F\colon Y\to 2^X$ given by $F(y)=\{f(y)\}$ is continuous.*

THEOREM 4.6. *Let X and Y be any spaces. Then the map $H\colon Y^X\times 2^Y\to 2^X$ given by $H(f, B)=f^{-1}(B)$ is continuous in f for fixed B and in B for fixed f. If Y is a locally compact Hausdorff space, then H is continuous jointly in f and B.*

Proof. The theorem is true for the map $T\colon Y^X\times \mathscr{S}^Y\to \mathscr{S}^X$ given by $T(f, g)=g\circ f$ [2, p. 259]. Noting that if $g\in \mathscr{S}^Y$ is such that $B=g^{-1}(1)$, then $(g\circ f)^{-1}(1)=f^{-1}(g^{-1}(1))=f^{-1}(B)$, the proof follows from Lemma 3.2.

We will denote by $H_B\colon Y^X\to 2^X$ the map $H_B(f)=H(f, B)$ and by $H_f\colon 2^Y\to 2^X$ the map $H_f(B)=H(f, B)$.

COROLLARY 4.7. *Let X be a Hausdorff space. Let $f\colon Y\to X$ be continuous. Then the map $f^{-1}\colon X\to 2^Y$ is continuous.*

Proof. The map $H_f\circ i\colon X\to 2^X\to 2^Y$ is continuous, and $H_f\circ i(x)=H_f(\{x\})=f^{-1}(x)$.

We remark here that if $f\colon Y\to X$ and if $F\colon Y\to 2^X$ is given by $F(y)=\{f(y)\}$, then $F^*\colon X\to 2^Y$ is given by $F^*(x)=\{y \mid x\in\{f(y)\}\}=f^{-1}(x)$, provided F^* is defined, i.e., provided $f^{-1}(x)$ is closed for all $x\in X$. Thus by Corollary 4.4 we get that if $X\times Y$ is a k-space then f^{-1} is defined and is continuous if and only if F is continuous. Notice that F^* acts in some sense as an inverse of F.

THEOREM 4.8. *Let X be a compact Hausdorff space and Y a k-space. Then $f\colon Y\to X$ is continuous if and only if $f^{-1}\colon X\to 2^Y$ is defined and is continuous.*

Proof. By Corollary 4.7 and the preceding discussion, all we need prove is that if $F(y)=\{f(y)\}$ is continuous then f is continuous.

Let $U \subset X$ be open. Since X is compact Hausdorff, $\mathscr{V} = \{A \in 2^X \mid A \subset U\}$ is open in 2^X, hence the continuity of F implies $F^{-1}(\mathscr{V})$ is open in Y. But

$$F^{-1}(\mathscr{V}) = \{y \in Y \mid f(y) \in U\} = f^{-1}(U).$$

PROPOSITION 4.9. *The map*

$$\Sigma: 2^X \times 2^X \to 2^X,$$

given by $\Sigma(A, B) = A \cup B$, *is continuous.*

Proof. Notice that

$$\{(A, B) \in 2^X \times 2^X \mid A \cup B \subset U\} = \{A \in 2^X \mid A \subset U\} \times \{B \in 2^X \mid B \subset U\}.$$

COROLLARY 4.10. *If* $F_i: Y \to 2^X$ *is continuous for* $i = 1, 2$, *then* $F_1 \cup F_2: Y \to 2^X$, *given by* $(F_1 \cup F_2)(y) = F_1(y) \cup F_2(y)$, *is continuous.*

Proof. Let $F_1 \times F_2: Y \to 2^X \times 2^X$ be the product map, then $F_1 \cap F_2 = \Sigma \circ (F_1 \times F_2)$.

LEMMA 4.11. *Let* $F_\alpha: Y \to 2^X$ *and let* $\bigcap F_\alpha: Y \to 2^X$ *be given by* $(\bigcap F_\alpha)(y) = \bigcap F_\alpha(y)$. *Then* $G_{\cap F_\alpha} = \bigcap G_{F_\alpha}$.

Proof.

$$\begin{aligned} G_{\cap F_\alpha} &= \{(x, y) \mid x \in \bigcap F_\alpha(y)\} \\ &= \{(x, y) \mid x \in F_\alpha(y) \text{ for all } \alpha\} \\ &= \bigcap \{(x, y) \mid x \in F_\alpha(y)\} \\ &= \bigcap G_{F_\alpha}. \end{aligned}$$

PROPOSITION 4.12. *Let* X *be a locally compact Hausdorff space or let* $X \times Y$ *be a* k*-space. Then if* $F_\alpha: Y \to 2^X$ *is continuous for all* α, *then* $\bigcap F_\alpha$ *is continuous.*

Proof. By Theorem 4.1, G_{F_α} is closed for all α. From Lemma 4.11, we conclude that $G_{\cap F_\alpha}$ is closed, hence that $\bigcap F_\alpha$ is continuous.

The last few results suggest some applications of the continuity properties of the compact-open topology.

PROPOSITION 4.13. *Let* X *be a locally compact topological group. Then the map* $C: 2^X \times X \to 2^X$, *given by* $C(A, x) = xA$, *is continuous.*

Proof. X is necessarily Hausdorff (see [8, p. 27]). We need only show G_C is closed. But

$$\begin{aligned} G_C &= \{(A, x, y) \mid y \in xA\} \\ &= \{(A, x, y) \mid x^{-1}y \in A\} \\ &= (1, m)^{-1}(\Omega), \end{aligned}$$

where $(1, m): 2^X \times X \times X \to 2^X \times X$ is given by $(1, m)(A, x, y) = (A, x^{-1}y)$ and is continuous. Thus G_C is closed.

COROLLARY 4.14. *Multiplication by an element of a locally compact topological group X is a homeomorphism of 2^X onto itself.*

In fact, Corollary 4.14 also follows from a more general result. Recall the definition $H_f(B)=f^{-1}(B)$.

PROPOSITION 4.15. *Let $f\colon X\to Y$ and $g\colon Y\to Z$ be continuous, then $H_{g\circ f}=H_f\circ H_g$. In particular, if f is a homeomorphism of X onto Y then H_f is a homeomorphism of 2^Y onto 2^X.*

Now consider the space $2^{X\times X}$ as the space of all closed binary relations R on X. We define the maps

$$M\colon 2^{X\times X}\to 2^X$$

by $M(R)=\{x\in X \mid \{x\}\times X\subset R\}$, and

$$E\colon 2^{X\times X}\times X\to 2^X$$

by $E(R,x)=\{y\in X \mid (x,y)\in R,\ (y,x)\in R\}$. It is easy to show that $M(R)$ and $E(R,x)$ are always closed sets, thus M and E are defined. The map M takes R to its set of "maximum" elements, while the map E takes R and x to the "equivalence class" of x with respect to R.

THEOREM 4.16. *If X is locally compact, then both M and E are continuous.*

Proof. Let $\Omega_2=\{(R,x,y) \mid (x,y)\in R\}$, by hypothesis a closed set in $2^{X\times X}\times X\times X$. Then

$$\begin{aligned} G_M &= \{(R,x) \mid x\in M(R)\} \\ &= \{(R,x) \mid \{x\}\times X\subset R\} \\ &= \{(R,x) \mid (R,x,y)\in\Omega_2, \text{ for all } y\in X\} \\ &= (2^X\times X)-p(\tilde{\Omega}_2), \end{aligned}$$

where p is the projection of $2^{X\times X}\times X\times X$ onto the first two coordinates. Hence G_M is closed.

Also,

$$\begin{aligned} G_E &= \{(R,x,y) \mid y\in E(R,x)\} \\ &= \{(R,x,y) \mid (x,y)\in R,\ (y,x)\in R\} \\ &= \Omega_2\cap\Omega_2', \end{aligned}$$

where $\Omega_2'=\{(R,x,y) \mid (y,x)\in R\}$ is also closed. Thus G_E is closed.

5. **Discussion.** If we had viewed the space 2^X as a function space earlier, then Lemma 2.2, Theorem 2.3 and Proposition 2.4 could be seen as following from known results about function spaces [2, p. 275]. However, it seemed simpler to directly indicate their proofs, which follow the proofs in [2].

The proof of Theorem 2.6 follows Arens' proof of the fact that if I^X has a weakest (he calls it strongest) admissible topology where I is the unit interval and X is *completely regular*, then X is locally compact [1, Theorem 3]. Note that what we have essentially proved is that if the function space $\mathscr{S}^X$ has a weakest admissible topology where X is *regular*, then X is locally compact.

Notice that the compact-open topology coincides with the upper-semifinite topology (defined at the end of §2 or see [7, p. 179]) when the space X is compact Hausdorff.

We note here that if X is a locally compact Hausdorff space with a countable base, then by [2, Theorem 5.2, p. 65], 2^X will have a countable base. In addition, for any space X, 2^X with the compact-open topology is trivially compact since 2^X itself must belong to any open cover.

The compact-open topology for 2^X appears as half of the generating set for Fell's H-topology [4]. It was later isolated by Effros, who called it the global topology [3, p. 931]. Effros describes convergence in this topology when X is a locally compact Hausdorff space. It is interesting to note that this description is equivalent to Arens' description of convergence in Y^X for locally compact Hausdorff X [1, Theorem 4] when one consider 2^X as the space $\mathscr{S}^X$. See also [5].

Finally, an equivalent version of the compact-open topology (for spaces of open subsets) was considered by Kannai [6] in connection with some problems in Mathematical Economics. Theorem 2.6 and Theorem 3.1 would seem to indicate that Kannai's methods will not work in spaces which are not locally compact.

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