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COMPLETELY 0-SIMPLE SEMIRINGS

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Abstract. A completely (0-) simple semiring is a semiring R which is (0-) simple and is the union of its (0-) minimal left ideals and the union of its (0-) minimal right ideals. Structure results are obtained for such semirings. First the multiplicative semigroup of R is completely (0-) simple; for any $\mathscr{H}$-class H ($\neq 0$), H ($\cup \{0\}$) is a subsemiring. If furthermore R has a zero but is not a division ring, and if $(H \cup \{0\}, +)$ has a completely simple kernel for some H as above (for instance, if R is compact or if the $\mathscr{H}$-classes are finite), then (i) $(R, +)$ is idempotent; (ii) R has no zero divisors, additively or multiplicatively. Additional results are given, concerning the additive $\mathscr{J}$-classes of R and also (0-) minimal ideals of semirings in general.

A (0-) simple semiring R which has (0-) minimal left ideals and (0-) minimal right ideals need not be the set-theoretical union of its (0-) minimal left ideals, nor the union of its (0-) minimal right ideals; when R is equal to both unions, we say that it is completely (0-) simple. Our purpose is to describe the structure of these semirings. We prove that the multiplicative semigroup $(R, \cdot)$ of a completely 0-simple semiring R is a completely 0-simple semigroup. In case the $\mathscr{H}$-classes of R (which in this case coincide with that of $(R, \cdot)$) are finite, it turns out that, if R is not a division ring, then $(R, +)$ is an idempotent semigroup and furthermore R has no zero divisors. In this case, we also give an explicit description of the addition of R in each $\mathscr{J}$-class of $(R, +)$ and study how it relates to the multiplication and the addition in other $\mathscr{J}$-classes. A complete explicit description of the addition of R is not attempted, since any idempotent semigroup is the additive semigroup of some completely simple semiring.

The first section contains a number of basic results on 0-minimal ideals in semirings, culminating with the result that, if a 0-minimal two-sided ideal M of a semiring R with zero contains a 0-minimal left ideal of R, then it is generated by the union of its own 0-minimal left ideals (which coincide with the 0-minimal left ideals of R contained in M) (Theorem 1.9). Some of these results are used in §2 to give various characterizations of completely 0-simple semirings; namely, the following conditions are equivalent for any semiring R:

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(a) R is 0-simple and is the union of its 0-minimal left ideals and the union of its 0-minimal right ideals;

(b) R is 0-bisimple, has a 0-minimal left ideal and a 0-minimal right ideal, and $\bar{\Lambda}$ and $\bar{P}$ commute element by element (here $\bar{\Lambda}$ and $\bar{P}$ are the sets of translations of R and "0-bisimple" is defined by means of the semiring's Green's relation $\mathscr{D}$);

(c) R is 0-bisimple, $\bar{\Lambda}$ and $\bar{P}$ commute element by element and R has a primitive idempotent;

(d) $(R, \cdot)$ is completely 0-simple (Theorems 2.1, 2.6).

Furthermore, for any $\mathscr{H}$-class H of R such that $H^2 \neq 0$, $H \cup \{0\}$ is a subsemiring of R and, by itself, a 0-division semiring. All these results have a rather superficial yet helpful similarity with well-known results of semigroup theory. Finally, §3 deals with the actual structure of R in a number of particular cases; this includes the structure of finite 0-division semirings (Corollary 3.9). This is used in §4 to prove the results mentioned before in the case when any $\mathscr{H}$-class $H \neq 0$ of R is finite; actually we obtain these results under the somewhat weaker assumption that $(H \cup \{0\}, +)$ has a completely simple kernel (Theorems 4.1, 4.5, 4.6).

All these results have immediate corollaries concerning simple semirings.

The reader is referred to [2] for the basic notions on semigroups; we keep most of the notation in [2], but denote the triples in the Rees-Sushkevitsch theorem by (α, x, λ) instead of the usual $(x; i, \lambda)$. We have used the semigroup terminology, rather than the ring terminology, for such concepts as minimal ideals or division semirings, since it permits to deal with semirings without zero (a situation which has no analogue in ring theory). A number of basic notions and results on semirings are taken from [5] and briefly recalled below. By semiring, we understand a nonempty set R together with two associative operations $+$ and $\cdot$ on R, such that $x(y+z)=xy+xz$, $(y+z)x=yx+zx$ hold identically; it is not assumed that either operation is commutative or has an identity element.

In what follows, R is a semiring; $(R, +)$ and $(R, \cdot)$ denote the additive and multiplicative semigroups of R; R_1 denotes the semigroup $(R, \cdot)^1$. The smallest additive subsemigroup of R containing a subset A of R is denoted by $\langle A \rangle$; distributivity implies that $\langle A \rangle \langle B \rangle \subseteq \langle AB \rangle$ identically. A *zero* of R is an element 0 of R which is an identity element of $(R, +)$ and a zero element of $(R, \cdot)$; if R does not have a zero, R° denotes the semiring obtained by adjunction to R of a formal zero in the obvious manner.

A *left* (*right*, *two-sided*) *ideal* of R is a subset of R which is a subsemigroup of $(R, +)$ and a left (right, two-sided) ideal of $(R, \cdot)$. If A is a left ideal of $(R, \cdot)$, the smallest left ideal of R containing A is just $\langle A \rangle$; in particular the *join* of a family $(L_i)_{i \in I}$ of left ideals of R is $\bigvee_{i \in I} L_i = \langle \bigcup_{i \in I} L_i \rangle$; the *principal* left ideal of R generated by $a \in R$ is $(a] = \langle R_1 a \rangle$. Similar results hold for right or two-sided ideals, with $[a) = \langle a R_1 \rangle$, $(a) = \langle R_1 a R_1 \rangle$ for the principal ideals.

The *Green's relations* of the semiring R are defined by $a \mathscr{L} b$ if and only if $(a] = (b]$; $a \mathscr{R} b$ if and only if $[a) = [b)$; $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and $\mathscr{D}$ is the smallest equivalence

relation containing $\mathscr{L}$ and $\mathscr{R}$. These equivalence relations are larger than the corresponding Green's relations of $(R, \cdot)$; however, a multiplicative idempotent of R has same $\mathscr{L}$-class in R and $(R, \cdot)$, and similarly for $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$; if an $\mathscr{H}$-class H of R contains elements a, b such that $ab \in H$, then it is a (maximal) subgroup of $(R, \cdot)$ (see [5]). The letter $\mathscr{J}$ is reserved for the corresponding Green's relation of $(R, +)$.

An *inner left* (*right*) *translation* of R is a finite pointwise sum of mappings of the form $x \mapsto ax$ (xa), $x \mapsto x$; $\bar{\Lambda}$ $(\bar{P})$ denotes the set of all such mappings. For any $a \in R$, $\bar{\Lambda}a = (a]$; hence $a \mathscr{L} b$ if and only if $a = \lambda b$, $b = \lambda' a$ for some $\lambda, \lambda' \in \bar{\Lambda}$; the dual result holds for $\mathscr{R}$. Any inner left translation λ of R is a left translation of $(R, \cdot)$ and is linked to some inner right translation ρ of R (i.e. $x(\lambda y) = (x\rho)y$ for all $x, y \in R$) (see [5]).

If $\lambda(x\rho) = (\lambda x)\rho$ for all $x \in R$, $\lambda \in \bar{\Lambda}$, $\rho \in \bar{P}$, we say that $\bar{\Lambda}$ and $\bar{P}$ commute *element by element*. This happens if and only if the restriction of each inner right translation to every principal left ideal $(a]$ of R is an additive homomorphism (of $(a]$ into R); a sufficient condition is that $R = R^2$. If $\bar{\Lambda}$ and $\bar{P}$ commute element by element, then $\mathscr{L}$ and $\mathscr{R}$ commute, so that $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. All these results are proved in [5]; proofs are also available in the first author's doctoral dissertation.

Before starting the exposition of our new results, we are glad to acknowledge our indebtedness to our referee, and to the members of the Tulane semigroup seminar, A. H. Clifford, L. Fuchs, W. R. Nico and others, for many valuable suggestions. We also owe to our referee a notable improvement in the proof of Theorem 2.1; the part (i) $\Rightarrow$ (iv) being entirely his.

1. 0-simple semirings and 0-minimal ideals.

1. A semiring R is *simple* in case it has no proper two-sided ideal, and 0-*simple* in case it has a zero, has no two-sided ideal besides R and 0, and $R^2 \neq 0$. The assumption that $R^2 \neq 0$ eliminates the following cases:

PROPOSITION 1.1. *Let R have a zero and no two-sided ideal except R and* 0. *If $R^2 = 0$, then either $R = 0$ or else $(R, +)$ is either a finite group of prime order or a two-element semigroup consisting of an identity and a zero.*

Proof. If $R^2 = 0$, $R \neq 0$, then, for each $a \in R - \{0\}$, $\{0, a\}$ is a two-sided ideal of $(R, \cdot)$, so that $\langle 0, a \rangle = R$; therefore, for any $b \in R - \{0\}$, $b = na$ for some positive integer n. If $(R, +)$ has an idempotent $f \neq 0$, then $R = \{0, f\}$ and $(R, +)$ consists of an identity and a zero. Assume now that $(R, +)$ has no idempotent except 0. Then, for any $c \in R - \{0\}$, $\langle c \rangle$ is finite (or else $\langle 0, 2c \rangle \subset \langle 0, c \rangle$), so that $pc = 0$ for some integer $p > 0$, and $(R, +)$ is a group. Since a proper subgroup of $(R, +)$ would be a proper two-sided ideal of R, and $R = \langle c \rangle$ has a commutative addition, $(R, +)$ is a finite group of prime order.

For any semiring R, the mapping $A \mapsto A \cup \{0\}$ sends the set of all two-sided ideals of R onto the set of all two-sided nonzero ideals of R°, and is one-to-one

and order-preserving. Therefore R is simple if and only if R° is 0-simple. For this reason we concentrate on 0-simple semirings.

PROPOSITION 1.2. *Let R be a 0-simple semiring. Then either $(R, +)$ is a group or, for all $x, y \in R$, $x+y=0$ implies $x=y=0$.*

Proof. The set $U=\{x \in R; x+y=0 \text{ for some } y \in R\}$ is a two-sided ideal of R. Hence either $U=R$ (and then $(R, +)$ is a group) or $U=0$ (and then $x+y=0$ implies $x=0$ and then $y=0$).

PROPOSITION 1.3. *Let R have a zero. The following are equivalent:*

(i) *R is 0-simple;*

(ii) *$\langle RxR\rangle = R$ for all $x \in R-\{0\}$;*

(iii) *$R^2 \neq 0$ and $\langle(\bar{\Lambda}x)\bar{P}\rangle = R$ for all $x \in R-\{0\}$.*

Proof. First let R be 0-simple. The set $N=\{x \in R; Rx=0\}$ is a two-sided ideal of R; since $N=R$ would imply $R^2=0$, therefore $N=0$ and $Rx \neq 0$ for all $x \in R-\{0\}$. Now, for any $x \in R-\{0\}$, $\langle RxR\rangle$ is a two-sided ideal of R. If $\langle RxR\rangle=0$, then $RxR=0$, $Rx \neq 0$ is a two-sided ideal of R, whence $Rx=R$ and $RxR=R^2 \neq 0$, a contradiction. Therefore $\langle RxR\rangle = R$, which proves that (i) implies (ii). Clearly (ii) implies (iii). If finally (iii) holds and $A \neq 0$ is a two-sided ideal of R, then for any $a \in A-\{0\}$, $R=\langle(\bar{\Lambda}a)\bar{P}\rangle \subseteq A$, whence $A=R$; since $R^2 \neq 0$, this proves that R is 0-simple.

2. A left ideal of R is called (0-) *minimal* if there is no left ideal of R properly contained in it (except 0); (0-) minimal right or two-sided ideals are defined similarly.

LEMMA 1.4. *Let M be a 0-minimal two-sided ideal of R such that $M^2 \neq 0$ and $L \neq 0$ be a left ideal of R contained in M. Then $L^2 \neq 0$.*

Proof. $\langle LR\rangle$ is a two-sided ideal of R contained in M. If $\langle LR\rangle=0$, then $LR=0$, L is a two-sided ideal of R contained in M, whence $L=M$ and $L^2=M^2 \neq 0$. If $\langle LR\rangle \neq 0$, then $M^2=\langle LR\rangle\langle LR\rangle \subseteq \langle LRLR\rangle \subseteq \langle LLR\rangle$ and again $L^2 \neq 0$.

PROPOSITION 1.5. *Let M be a 0-minimal two-sided ideal of R. If $M^2 \neq 0$, then M is a 0-simple subsemiring of R.*

Proof. Observe that M is a subsemiring of R and assume that $M^2 \neq 0$. The set $N=\{x \in M; Mx=0\}$ is a two-sided ideal of R contained in M. Since $N=M$ would imply $M^2=0$ we have $N=0$, whence $Mx \neq 0$ for all $x \in M-\{0\}$. If now $x \in M-\{0\}$, then $Mx \neq 0$ is a left ideal of R contained in M whence $MxMx \neq 0$ by 1.4 and $MxM \neq 0$. Finally $\langle MxM\rangle \neq 0$ is a two-sided ideal of R contained in M, which implies that $\langle MxM\rangle = M$. By 1.3, M is therefore 0-simple.

PROPOSITION 1.6. *Let M be a 0-minimal two-sided ideal of R such that $M^2 \neq 0$. Then the 0-minimal left ideals of R contained in M coincide with the 0-minimal left ideals of M.*

Proof. Let L be a 0-minimal left ideal of R contained in M. Then $L^2 \neq 0$ by 1.4. Let $A \neq 0$ be a left ideal of M contained in L. Since $\langle RA \rangle \subseteq L$, either $\langle RA \rangle = 0$ or $\langle RA \rangle = L$. In the first case A is a left ideal of R, so that $A = L$. In the second case, $L^3 = \langle RA \rangle \langle RA \rangle \subseteq \langle RMRA \rangle \subseteq \langle MA \rangle$, so that $\langle MA \rangle \neq 0$; since $\langle MA \rangle \subseteq L$ is a left ideal of R, $L = \langle MA \rangle \subseteq A$ and again $A = L$. Therefore L is a 0-minimal left ideal of M.

Conversely, let L be a 0-minimal left ideal of M. For any $x \in L - \{0\}$, $MxM \neq 0$ by 1.3, 1.5, whence $Mx \neq 0$ and $ML \neq 0$. Now $\langle ML \rangle \neq 0$ is a left ideal of M contained in L, so that $L = \langle ML \rangle$. Since any left ideal of R contained in M is also a left ideal of M, it follows that L is a 0-minimal left ideal of R. This completes the proof.

LEMMA 1.7. *Let L be a 0-minimal left ideal of R. For any $x \in R$, either $Lx = 0$ or Lx is a 0-minimal left ideal of R. The same holds for $L\rho$ (where $\rho \in \bar{P}$) in case $\bar{\Lambda}$ and $\bar{P}$ commute element by element.*

Proof. The first part is Lemma 1 of [1]. For the second part, observe that L is a principal left ideal of R (generated by any of its nonzero elements); therefore the restriction of ρ to L is an additive homomorphism of L into R, since $\bar{\Lambda}$ and $\bar{P}$ commute element by element. Now $L\rho$ is a left ideal of R, since it is closed under addition and $y(L\rho) = (yL)\rho \subseteq L\rho$ for any $y \in R$. Assume that $L\rho \neq 0$ and let A be a left ideal of R contained in $L\rho$; let $B = \{x \in L;\ x\rho \in A\}$. Then B is closed under addition and, for any $b \in B$, $y \in R$, $(yb)\rho = y(b\rho) \in A$, so that $yb \in B$. Since $B \subseteq L$, therefore either $B = 0$ or $B = L$; since $A = B\rho$, then either $A = 0$ or $A = L\rho$, so that $L\rho$ is a 0-minimal left ideal of R.

3. Now we come to the main results of this section.

THEOREM 1.8. *Let R be a 0-simple semiring which has a 0-minimal left ideal L. Then R is the join of its 0-minimal left ideals.*

Proof. It follows from 1.7 (first part) that the union of all the 0-minimal left ideals of R is a two-sided ideal of $(R, \cdot)$. Therefore the join of the 0-minimal left ideals of R is a two-sided ideal of R; it cannot be 0, hence must be R.

THEOREM 1.9. *Let M be a 0-minimal two-sided ideal of R such that $M^2 \neq 0$, which contains a 0-minimal left ideal of R. Then M is the join of the 0-minimal left ideals of R contained in M (which are the 0-minimal left ideals of M).*

All the previous results have immediate corollaries concerning minimal ideals. In the case of 1.8, a converse can be added in this situation, to give

THEOREM 1.10. *Let R be a semiring which has a minimal left ideal. Then R is simple if and only if R is the join of its minimal left ideals.*

Proof. The condition is necessary by 1.8. Conversely, assume that $R = \bigvee_{i \in I} L_i$, where $(L_i)_{i \in I}$ is the family of all minimal left ideals of R. By 1.3 it is enough to prove that, for any $x, y \in R$, $y = \sum_{j=1}^{j=n} a_j x b_j$ for some $a_j, b_j \in R$. Since y is a sum

of elements of $\bigcup_{i\in I} L_i$, we may even assume that y is in some minimal left ideal L of R. But then $\langle RxL\rangle$ is a left ideal of R contained in L and is therefore equal to L; so that $y \in \langle RxL\rangle \subseteq \langle RxR\rangle$. This completes the proof.

2. Completely 0-simple semirings.

1. A simple semiring having a minimal left ideal is the join of its minimal left ideals, but need not be the union of its minimal left ideals. The following semiring is a counterexample:

$+$	a	b	c	d	e	f
a	a	b	a	e	e	f
b	a	b	f	b	e	f
c	a	b	c	d	e	f
d	a	b	c	d	e	f
e	a	b	a	e	e	f
f	a	b	f	b	e	f

$\cdot$	a	b	c	d	e	f
a	a	a	c	c	a	a
b	b	b	d	d	b	b
c	a	a	c	c	a	a
d	b	b	d	d	b	b
e	b	b	d	d	b	b
f	a	a	c	c	a	a

This semiring is simple and has minimal left ideals $\{a, b\}$, $\{c, d\}$ and minimal right ideals $\{a, c\}$, $\{b, d\}$. Moreover it is easily verified that any inner left translation of R is an inner left translation of $R_1=(R, \cdot)^1$, so that $\overline{\Lambda}$ and $\overline{P}$ commute element by element. The $\mathscr{D}$-classes of R are $\{a, b, c, d\}$, $\{e\}$, $\{f\}$.

From this follows that a 0-simple semiring having 0-minimal left ideals and 0-minimal right ideals need not be the union of its 0-minimal left (right) ideals, even if $\overline{\Lambda}$ and $\overline{P}$ commute element by element. For this, we need an additional condition.

2. A semiring R is called *bisimple* if it consists of one $\mathscr{D}$-class only; and 0-*bisimple* if it has a zero, $R^2\neq 0$ and the $\mathscr{D}$-classes of R are 0 and $R-\{0\}$. Since two elements of the same $\mathscr{D}$-class generate the same principal two-sided ideal, a (0-) bisimple semiring is also (0-) simple.

The main result of this section is the following:

THEOREM 2.1. *The following properties are equivalent for any semiring R with zero*:

(i) *R is* 0-*simple and is the union of its* 0-*minimal left ideals and the union of its* 0-*minimal right ideals*;

(ii) *R is* 0-*bisimple, has a* 0-*minimal left ideal and a* 0-*minimal right ideal, and $\overline{\Lambda}$ and $\overline{P}$ commute element by element*;

(iii) *R has a* 0-*minimal left ideal and a* 0-*minimal right ideal, $\overline{\Lambda}$ and $\overline{P}$ commute element by element, $R^2\neq 0$ and, for any $x, y \in R$, $x\neq 0$, there exist $\lambda \in \overline{\Lambda}$, $\rho \in \overline{P}$ such that $y=\lambda x\rho$*;

(iv) *$(R, \cdot)$ is a completely* 0-*simple semigroup.*

Proof. We start by showing that (i) implies (iv). First let L be a 0-minimal left ideal of R. For each $x \in L-\{0\}$, Rx is a left ideal of R contained in L and nonzero

by 1.3, so that $L=Rx$. It follows that L is a 0-minimal left ideal of $(R, \cdot)$. Dually every 0-minimal right ideal of R is also a 0-minimal right ideal of $(R, \cdot)$.

If now $x, y \in R-\{0\}$, then $\langle RxRy\rangle = \langle RxR\rangle y = Ry \neq 0$ by 1.3, so that $RxRy \neq 0$. By (i), y belongs to some 0-minimal ideal L of R; then $Ry=L$ and, since $RxRy$ is a left ideal of $(R, \cdot)$ contained in Ry, then $RxRy=Ry$. Thus $RxR^2 = \bigcup RxRy = \bigcup Ry = R$, so that $RxR=R$ for all $x \in R-\{0\}$, which shows that $(R, \cdot)$ is 0-simple. Therefore $(R, \cdot)$ is completely 0-simple.

Next we prove that (iv) implies (ii). If $(R, \cdot)$ is completely 0-simple, then any $\mathscr{L}$-class of $(R, \cdot)$ contains an idempotent, hence is an $\mathscr{L}$-class of R; the same holds for $\mathscr{R}$, dually. It follows that the Green's relations $\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}$ of R coincide with that of $(R, \cdot)$. In particular R is 0-bisimple. Moreover, any 0-minimal left ideal of $(R, \cdot)$ has the form Rx for some $x \in R$ and is therefore a left ideal of R, obviously 0-minimal; dually, R has 0-minimal right ideals. Finally $\overline{\Lambda}$ and $\overline{P}$ commute element by element, since $R^2=R$. Thus (ii) holds.

If (ii) holds, then $R^2 \neq 0$. Furthermore if $x, y \in R-\{0\}$ then $x \mathscr{D} y$; since $\mathscr{L}$ and $\mathscr{R}$ commute by (ii), then $x \mathscr{L} z \mathscr{R} y$ for some $z \in R$; then $z=\lambda x$, $y=z\rho$ for some λ, ρ, and $y=\lambda x \rho$. It follows that (ii) implies (iii).

Finally, let (iii) hold. Let y be an arbitrary element of R; let L be a 0-minimal left ideal of R and $x \in L-\{0\}$. If $y=0$, then $y \in L$. If $y \neq 0$, write $y=\lambda x \rho$, where $\lambda \in \overline{\Lambda}$, $\rho \in \overline{P}$. Then $y \in L\rho$; in particular $L\rho \neq 0$, and $L\rho$ is a 0-minimal left ideal of R, by 1.7. Therefore R is the union of its 0-minimal left ideals. Dually R is the union of its 0-minimal right ideals. Finally R is 0-simple, by 1.3. Therefore (i) holds. This completes the proof.

The corollary of Theorem 2.1 concerning simple semirings becomes, in view of Theorem 1.10:

THEOREM 2.2. *The following conditions are equivalent for any semiring R:*

(i) *R is the union of its minimal left ideals and the union of its minimal right ideals;*

(ii) *R is bisimple, has a minimal left ideal and a minimal right ideal, and $\overline{\Lambda}$ and $\overline{P}$ commute element by element;*

(iii) *R has a minimal left ideal and a minimal right ideal, $\overline{\Lambda}$ and $\overline{P}$ commute element by element and, for any $x, y \in R$, there exist $\lambda \in \overline{\Lambda}$, $\rho \in \overline{P}$ such that $y=\lambda x \rho$;*

(iv) *$(R, \cdot)$ is a completely simple semigroup.*

3. A semiring which satisfies any of the equivalent conditions of Theorem 2.1 (2.2) will be called *completely 0-simple* (*completely simple*). We can immediately add some additional properties of these semirings.

PROPOSITION 2.3. *Let R be completely 0-simple. Then*

(i) *the 0-minimal left (right) ideals of R and $(R, \cdot)$ coincide;*

(ii) *the Green's relations $\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}$ of R and $(R, \cdot)$ coincide;*

(iii) *if $H \neq 0$ is an $\mathscr{H}$-class of R, then $H \cup \{0\}$ is a subsemiring of R; if $H^2 \neq 0$, then $(H, \cdot)$ is a group.*

Proof. (i) and (ii) have been shown incidentally in the proof of Theorem 2.1. To prove (iii), observe that if L is a 0-minimal left ideal of R, then $L-\{0\}$ is an $\mathscr{L}$-class of R; since R is completely 0-simple, every nonzero $\mathscr{L}$-class of R is obtained in that fashion. The $\mathscr{R}$-classes are obtained dually. Thus for any $\mathscr{H}$-class $H\neq 0$ of R, $H\cup\{0\}$ is the intersection of a left ideal of R and a right ideal of R and is therefore a subsemiring of R. If $H^2\neq 0$, then $(H, \cdot)$ is a group by (ii).

We call *division semiring* a semiring R such that $(R, \cdot)$ is a group; and 0-*division semiring* a semiring R with zero but without zero divisors, such that $(R-\{0\}, \cdot)$ is a group (0-division semirings are usually called "division semirings"). Then 2.3(iii) can be restated thus: if H is an $\mathscr{H}$-class of a completely (0-) simple semiring (and if $H^2\neq 0$), then H ($\cup\{0\}$) is a (0-) division semiring.

PROPOSITION 2.4. *Let H, H' be two nonzero $\mathscr{H}$-classes of a completely* 0-*simple semiring R. Then $H\cup\{0\}$ and $H'\cup\{0\}$ are isomorphic additive semigroups; if furthermore $H^2\neq 0$, $H'^2\neq 0$, then $H\cup\{0\}$ and $H'\cup\{0\}$ are isomorphic* 0-*division semirings.*

Proof. Since H, H' are $\mathscr{H}$-classes of $(R, \cdot)$ and $(R, \cdot)$ is 0-bisimple, there exist $u, v\in R$ such that $x\mapsto uxv$ maps H one-to-one onto H'. Clearly this mapping preserves the addition and also sends $H\cup\{0\}$ one-to-one onto $H'\cup\{0\}$, thus providing the desired isomorphism. If furthermore $H^2\neq 0$, $H'^2\neq 0$, let e (e') be the identity element of $(H, \cdot)$ ($(H', \cdot)$); take $a\in R_e\cap L_{e'}$; by Theorem 2.20 of [2], a has an inverse $a'\in L_e\cap R_{e'}$ and $x\mapsto a'xa$ maps H one-to-one onto H' and preserves the multiplication; in this case the proof is concluded as above.

COROLLARY 2.5. *Let H, H' be $\mathscr{H}$-classes of a completely simple semiring. Then H and H' are isomorphic division semirings.*

4. Completely 0-simple semigroups can be characterized by the existence of primitive idempotents; a similar characterization exists for completely 0-simple semirings. First, define a *primitive idempotent* of a semiring R as a primitive idempotent of $(R, \cdot)$. Observe that the semiring given as an example at the beginning of this section is simple and has primitive idempotents, yet is not completely simple, even though $\overline{\Lambda}$ and $\overline{P}$ commute element by element; however, this semiring is not bisimple.

THEOREM 2.6. *Let R be a* 0-*bisimple semiring such that $\overline{\Lambda}$ and $\overline{P}$ commute element by element. Then R is completely* 0-*simple if and only if it contains a primitive idempotent.*

Proof. If R is completely 0-simple, then R has nonzero multiplicative idempotents, and they are all primitive. For the converse, let e be a primitive idempotent of R. First we prove that Re is a 0-minimal left ideal of R. It is clearly a nonzero left ideal of R. Let $A\neq 0$ be a left ideal of R contained in Re; take $a\in A-\{0\}$. First

$e=\lambda a\rho$ for some $\lambda \in \bar{\Lambda}$, $\rho \in \bar{P}$ (see the proof that (ii) implies (iii) in Theorem 2.1). Let $\lambda' \in \bar{\Lambda}$, $\rho' \in \bar{P}$ be linked to ρ and λ respectively. Then

$$e = e(\lambda a\rho)e = (e\rho')(a\rho)e = (e\rho')a(\lambda' e).$$

Set $e\rho'=x$, $\lambda' e=y$, so that $e=xay$, $ex=x$, $ye=y$; note that $ae=a$ since $a \in Re$, whence $e=xaey$. Set $f=eyxa$; then $f^2=eyxaeyxa=eyxa=f$ and $ef=fe=f$; furthermore $xafy=xaeyxay=e^2\neq 0$, so that $f\neq 0$. Since e is primitive, $f=e$. Therefore $e=eyxa \in A$ and $A=Re$. This shows that Re is a 0-minimal left ideal of R. Dually eR is a 0-minimal right ideal of R, so that R is completely 0-simple.

3. Some particular cases.

1. In this section we give some information on various kinds of completely 0-simple semirings.

We start with the case when $(R, \cdot)$ is a Rees matrix semigroup $\mathscr{M}^\circ(E; \Gamma, \Delta; p)$ over a trivial group E. In this case we write the nonzero elements of R as ordered pairs (α, λ) (where $\alpha \in \Gamma$, $\lambda \in \Delta$), so that $R-\{0\}=\Gamma\times\Delta$ (as a set) and $(\alpha, \lambda)(\beta, \mu) =(\alpha, \mu)$ whenever $p_{\lambda,\beta}\neq 0$.

PROPOSITION 3.1. *Let R be a semiring such that $(R, \cdot)=\mathscr{M}^\circ(E; \Gamma, \Delta; p)$, where E is a trivial group, and that $(R, +)$ is not a group. Then*

(i) *for any $\alpha, \beta \in \Gamma$ there exists $\xi \in \Delta$ such that $p_{\xi,\alpha}\neq 0$, $p_{\xi,\beta}\neq 0$; for any $\lambda, \mu \in \Delta$ there exists $\gamma \in \Gamma$ such that $p_{\gamma,\lambda}\neq 0$, $p_{\gamma,\mu}\neq 0$;*

(ii) *$R-\{0\}$ is a subsemigroup of $(R, +)$ and there exist semigroup structures on Γ, Δ such that $(R-\{0\}, +)=(\Gamma, +)\times(\Delta, +)$.*

Proof. Assume that $\alpha, \beta \in \Gamma$ are such that, for each $\xi \in \Delta$, either $p_{\xi,\alpha}=0$ or $p_{\xi,\beta}=0$. Let $\lambda, \mu \in \Delta$ be such that $p_{\lambda,\alpha}\neq 0$, $p_{\mu,\beta}\neq 0$; then $p_{\mu,\alpha}=p_{\lambda,\beta}=0$. For any $\gamma \in \Gamma$, set $a=(\alpha, \lambda)$, $b=(\beta, \mu)$, $c=(\gamma, \lambda)$, $d=(\gamma, \mu)$ and $a+b=(A, B)$ (note that $a+b\neq 0$ by 1.2). Then $ca\neq 0$, $db\neq 0$, $cb=da=0$ and, by distributivity, $c(a+b) =ca\neq 0$, $d(a+b)=db\neq 0$. Therefore

$$(\gamma, \lambda) = ca = c(a+b) = (\gamma, B) = d(a+b) = db = (\gamma, \mu),$$

whence $\lambda=\mu$, which is impossible. This proves the first part of (i); the second part follows by duality. (N.B. It will follow from Theorem 4.5 below that in fact $p_{\lambda,\alpha}\neq 0$ for all λ, α.)

Next, $R-\{0\}$ is a subsemigroup of $(R, +)$, by 1.2; for any $(\alpha, \lambda), (\beta, \mu) \in R-\{0\}$, we may then set

$$(\alpha, \lambda)+(\beta, \mu) = (A(\alpha, \beta; \lambda, \mu), B(\alpha, \beta; \lambda, \mu)).$$

By (i) there exists $\gamma \in \Gamma$ such that $p_{\lambda,\gamma}\neq 0$, $p_{\mu,\gamma}\neq 0$. For any $\xi \in \Delta$, $(\alpha, \lambda)(\gamma, \xi)\neq 0$, so $(\alpha, \lambda)(\gamma, \xi)+(\beta, \mu)(\gamma, \xi)\neq 0$; hence

$$\begin{aligned}(A(\alpha, \beta; \xi, \xi), B(\alpha, \beta; \xi, \xi)) &= (\alpha, \lambda)(\gamma, \xi)+(\beta, \mu)(\gamma, \xi)\\ &= ((\alpha, \lambda)+(\beta, \mu))(\gamma, \xi) = (A(\alpha, \beta; \lambda, \mu), \xi),\end{aligned}$$

and $A(\alpha, \beta; \xi, \xi) = A(\alpha, \beta; \lambda, \mu)$. Since ξ is arbitrary, it follows that $A(\alpha, \beta; \lambda, \mu)$ depends only on α, β; hence we can turn Γ into a groupoid, with $\alpha+\beta = A(\alpha, \beta; \lambda, \mu)$. Dually, we can turn Δ into a groupoid, with $\lambda+\mu = B(\alpha, \beta; \lambda, \mu)$. Then it is clear that $(R-\{0\}, +) = (\Gamma, +)\times(\Delta, +)$, whence $(\Gamma, +)$ and $(\Delta, +)$ are semigroups. This completes the proof. (N.B. It will follow from Theorem 4.1 below that they can be arbitrary idempotent semigroups. If $(R, +)$ is a group, then R is a two element field.)

THEOREM 3.2. *Let R be a semiring such that $(R, \cdot) = \mathscr{M}^\circ(E; \Gamma, \Delta; p)$, where E is a trivial group, and that $(R, +)$ is a semilattice. Then R has no zero divisors and there exist semilattice structures on Γ, Δ such that $(R-\{0\}, +) = (\Gamma, +)\times(\Delta, +)$.*

Proof. Since $R \neq 0$, $(R, +)$ is not a group and its structure follows from 3.1. Assume, then, that $p_{\lambda,\alpha} = 0$ for some λ, α. Let μ, β be such that $p_{\mu,\alpha} \neq 0$, $p_{\lambda,\beta} \neq 0$, and let γ, $\delta \in \Gamma$ be arbitrary. Multiplying $(\gamma, \lambda)+(\delta, \mu) = (\gamma+\delta, \lambda+\mu)$ on the right by (α, λ) yields

$$(\delta, \lambda) = ((\gamma, \lambda)+(\delta, \mu))(\alpha, \lambda) = (\gamma+\delta, \lambda).$$

This shows that $\gamma+\delta = \delta$ identically. Since $(\Gamma, +)$ is commutative, then Γ has only one element. But then $\alpha = \beta$, and $p_{\lambda,\alpha} \neq 0$, a contradiction. This completes the proof.

COROLLARY 3.3. *Let R be a semiring such that $(R, \cdot) = \mathscr{M}(E; \Gamma, \Delta; p)$, where E is a trivial group, and that $(R, +)$ is a semilattice. There exist semilattice structures on Γ, Δ such that $(R, +) = (\Gamma, +)\times(\Delta, +)$.*

2. The next case is when $(R, +)$ is a group.

THEOREM 3.4. *Let R be a completely 0-simple semiring such that $(R, +)$ is a group. Then R is a division ring.*

Proof. Since $R^2 = R$, $(R, +)$ must be an abelian group; indeed for any $a = xy$, $b = uv \in R$,

$$xv+xy+uv+uy = (x+u)(v+y) = xv+uv+xy+uy,$$

whence $a+b = b+a$. Therefore R is a ring. Then the theorem follows from the known result that a ring whose multiplicative semigroup is completely 0-simple must be a division ring (see, for instance, [4]. There is also a short proof which uses Litoff's theorem (IV.15.3 of [3]); it goes as follows. Let R be a ring such that $(R, \cdot)$ is completely 0-simple. Then R is a simple ring and, for any idempotent $e \neq 0$, Re is a minimal left ideal of R; dually R has a minimal right ideal. By Litoff's theorem, R is locally matrix over a division ring. If then e, f are nonzero idempotents of R, then e, f lie in a subring S of R which is isomorphic to a full matrix ring over a division ring and has therefore an identity g; g must be primitive, and $ge = eg = e$, $gf = fg = f$ implies $e = g = f$. Therefore $(R, \cdot)$ has only one nonzero idempotent and must be a group with zero).

3. The next case is when R is completely simple and $(R, +)$ is a rectangular band. A rectangular band is a completely simple idempotent semigroup and is therefore isomorphic to a semigroup $I \times J$, where I, J are nonempty sets and $I \times J = I \times J$ (as a set) with addition $(i, j)+(k, m)=(i, m)$.

PROPOSITION 3.5. *Let R be a semiring such that $(R, +)=I \times J$. Then there exist semigroup structures on I, J such that $(R, \cdot)=(I, \cdot)\times(J, \cdot)$.*

Proof. It parallels the proof of 3.1(ii). For any (i, j), $(u, v) \in R$, set

$$(u, v)(i, j) = (A(u, i; v, j), B(u, i; v, j)).$$

Then, for all $t, u, i, k \in I$, $v, w, j, m \in J$,

$$\begin{aligned}(A(t, i; w, m), B(t, i; w, m)) &= ((t, v)+(u, w))((i, j)+(k, m)) \\ &= (t, v)(i, j)+(t, v)(k, m)+(u, w)(i, j)+(u, w)(k, m) \\ &= (A(t, i; v, j), B(u, k; w, m)),\end{aligned}$$

so that $A(t, i; w, m)=A(t, i; v, j)$, $B(t, i; w, m)=B(u, k; w, m)$ identically. In other words, $A(u, i; v, j)$ depends only on u, i and we may turn I into a groupoid by defining $ui=A(u, i; v, j)$. Similarly J can be turned into a groupoid by putting $vj=B(u, i; v, j)$. Then $(R, \cdot)=(I, \cdot)\times(J, \cdot)$, whence $(I, \cdot)$ and $(J, \cdot)$ are semigroups. This proves the result.

If I and J are arbitrary semigroups, we can, conversely, define operations on $I \times J$ by

(1) $(i, j)+(k, m)=(i, m)$

(2) $(i, j)(k, m)=(ik, jm)$;

it is clear that this turns $I \times J$ into a semiring. This semiring will be denoted by $I * J$.

PROPOSITION 3.6. *Let I, J, K, L be semigroups and φ be an isomorphism of $I * J$ onto $K * L$. Then there exist isomorphisms $\zeta: I \cong K$, $\eta: J \cong L$ such that $\varphi(i, j)=(\zeta(i), \eta(j))$ identically.*

Proof. First φ is an isomorphism of $I \times J$ onto $K \times L$, so that by Corollary 3.12 of [2] there exist one-to-one mappings ζ of I onto K, η of J onto L such that $\varphi(i, j)=(\zeta(i), \eta(j))$ identically. Since φ is also a multiplicative isomorphism, then ζ, η are semigroup isomorphisms.

PROPOSITION 3.7. *Let R be a completely simple semiring such that $(R, +)$ is a rectangular band. Then there exist completely simple semigroups U, V such that $R \cong U * V$.*

Proof. By 3.5 there exist semigroups U, V such that $R \cong U * V$. Since $U \times V \cong (R, \cdot)$, U, V must be completely simple.

4. Finally 3.5 can be used to describe finite division semirings. More generally:

THEOREM 3.8. *Let R be a semiring such that $(R, +)$ has a completely simple kernel. Then R is a division semiring if and only if there exist groups F, G such that $R \cong F * G$.*

Proof. Assume that R is a division semiring, and let $S=(R, +)$ and K be the kernel of S. For any $a \in R$, $x \mapsto ax$ is an automorphism of S; therefore $aK=K$. But any element of S has the form ak for some $a \in R$, $k \in K$; it follows that $S=K$ and S is completely simple. Moreover, let f be an idempotent of S; any element of S can be written in the form af for some $a \in R$ and is therefore an idempotent of S. We have proved that S is a rectangular band. By 3.5 there exist semigroups F, G such that $R \cong F * G$; since $F \times G \cong (R, \cdot)$, F and G are in fact groups. The converse is trivial.

COROLLARY 3.9. *Let R be a semiring with zero such that $(R, +)$ has a completely simple kernel. Then R is a 0-division semiring if and only if either R is a division ring or there exist groups F, G such that $R \cong (F * G)^\circ$.*

Proof. If $(R, +)$ is a group, then R is a division ring by Theorem 3.4. If $(R, +)$ is not a group, then by 1.2, $R-\{0\}$ is a subsemiring of R and obviously a division semiring; furthermore $(R-\{0\}, +)$ has same kernel as $(R, +)$. Hence it follows from 3.8 that $R \cong (R-\{0\})^\circ \cong (F * G)^\circ$ for some groups F, G. The converse is trivial.

Note that some finiteness condition is necessary in 3.8, 3.9. For instance, the positive real numbers form a division semiring under the ordinary operations, but their addition is not idempotent.

4. The structure of completely 0-simple semirings.

1. If R is a completely 0-simple semiring, the structure of $(R, \cdot)$ is given by Theorem 2.1(iv) and the Rees-Sushkevitsch theorem. On the other hand the only information we have on $(R, +)$ in the general situation is provided by 1.2.

Sharper results can be obtained if R satisfies the following condition:

Condition (F). When R has no zero (has a zero), there is an $\mathscr{H}$-class H of R $(H \neq 0)$ such that $(H, +)$ $((H \cup \{0\}, +))$ has a completely simple kernel. (N.B. If this holds for one $\mathscr{H}$-class, then by 2.4 it will hold for every other (nonzero) $\mathscr{H}$-class.)

For instance, (F) holds when the $\mathscr{H}$-classes of R are finite and in particular when R itself is finite. Another important case is when R is compact (=has a compact Hausdorff topology under which both operations are continuous); in the case with zero, any $\mathscr{H}$-class H such that $H^2 \neq 0$ is a maximal subgroup of $(R, \cdot)$ and is compact, so that $(H \cup \{0\}, +)$ is a compact semigroup.

This condition enables us to use the results of §3, starting with 3.9.

THEOREM 4.1. *Let R be a completely 0-simple semiring such that* (F) *holds. Then either R is a division ring or $(R, +)$ is an idempotent semigroup.*

Proof. Let $H \neq 0$ be an $\mathscr{H}$-class of R such that $(H \cup \{0\}, +)$ has a completely simple kernel; by 2.4 we may assume that $H^2 \neq 0$. Then by 2.3 and 3.9 $(H \cup \{0\}, +)$ is either an idempotent semigroup or a group. In the first case each element of R is an additive idempotent by 2.4. In the second case it follows again from 2.4 that

any element of R has an additive inverse, so that $(R, +)$ is a group. Then R is a division ring by Theorem 3.4.

COROLLARY 4.2. *Let R be a completely simple semiring such that* (F) *holds. Then $(R, +)$ is an idempotent semigroup.*

2. From now on, we assume that (F) holds and that R is not a division ring. If H is an $\mathscr{H}$-class of R such that $H^2 \neq 0$, then, by 2.3, 3.9, $H \cong F * G$, where F, G are groups. From now on we identify first H and $F * G$, then $(R, \cdot)$ and the Rees matrix semigroup $\mathscr{M}^\circ((H, \cdot); \Gamma, \Delta; p) = \mathscr{M}^\circ(F \oplus G; \Gamma, \Delta; p)$, and do it in such a way that $x \in H$ coincides with $(1, x, 1)$, where $1 \in \Gamma$, $1 \in \Delta$ are such that $p_{1,1} = e = (f, g)$ is the identity element of $F \oplus G$. We add a formal zero to F, G and put $p_{\lambda,\alpha} = (q_{\lambda,\alpha}, r_{\lambda,\alpha})$ if $p_{\lambda,\alpha} \neq 0$, $q_{\lambda,\alpha} = r_{\lambda,\alpha} = 0$ if $p_{\lambda,\alpha} = 0$; finally we identify all triples $(\alpha, 0, \lambda)$ with 0, so that the multiplication of R is given in all cases by

$$(\alpha, (i, j), \lambda)(\beta, (k, m), \mu) = (\alpha, (iq_{\lambda,\beta}k, jr_{\lambda,\beta}m), \mu); \tag{3}$$

then the addition in H is given by (1), i.e.

$$(1, (i, j), 1) + (1, (k, m), 1) = (1, (i, m), 1). \tag{4}$$

We call this description of R the representation by triples.

Since $(R, +)$ is idempotent, it is by Clifford's theorem (4.9 of [2]) a semilattice of rectangular bands, namely its $\mathscr{J}$-classes. We start by studying $\mathscr{J}$.

PROPOSITION 4.3. *Let R be a completely 0-simple semiring such that* (F) *holds, which is not a division ring. Then $\mathscr{J}$ is a congruence of R; $R/\mathscr{J}$ is a completely 0-simple semiring in which $\mathscr{H}$ is the equality, whose addition is a semilattice. In terms of the representation of R by triples, there exist equivalence relations $\equiv$ on Γ, Δ such that $(\alpha, x, \lambda)\ \mathscr{J}\ (\beta, y, \mu)$ if and only if either $x = y = 0$ or $\alpha \equiv \beta$, $\lambda \equiv \mu$ and $x, y \neq 0$.*

Proof. It follows from Clifford's theorem that $\mathscr{J}$ is a congruence of $(R, +)$ and that $(R, +)/\mathscr{J}$ is a semilattice. On the other hand, it is easily seen that, in any semiring, $\mathscr{J}$ is a congruence of $(R, \cdot)$; hence in our case $\mathscr{J}$ is a congruence of R. If $\mathscr{J}$ were the universal congruence, then $(R, +)$ would be completely simple and, having an identity, would be a group; then R would be a division ring by Theorem 3.4, contradicting the hypothesis. Hence $\mathscr{J}$ is not the universal congruence; in particular $(R, \cdot)/\mathscr{J}$ is completely 0-simple, and so is $R/\mathscr{J}$.

Furthermore, let $H \neq 0$ be an $\mathscr{H}$-class of R. Since $(H, +)$ is a rectangular band, any two elements of H are $\mathscr{J}$-equivalent in $(H, +)$, hence also in $(R, +)$. From this follows that, in $R/\mathscr{J}$, $\mathscr{H}$ is the equality. The description of $\mathscr{J}$ then follows from Preston's theorem (10.48 of [2]).

COROLLARY 4.4. *Let R be a completely simple semiring such that* (F) *holds. Then $\mathscr{J}$ is a congruence of R; $R/\mathscr{J}$ is a completely simple semiring in which $\mathscr{H}$ is the equality, whose addition is a semilattice. In terms of the representation by triples*

there exist equivalence relations $\equiv$ *on* Γ, Δ *such that* $(\alpha, x, \lambda)\ \mathscr{J}\ (\beta, y, \mu)$ *if and only if* $\alpha\equiv\beta$ *and* $\lambda\equiv\mu$.

Under the hypothesis of 4.3, Theorem 3.2 can be applied to $R/\mathscr{J}$ and yields:

THEOREM 4.5. *Let R be a completely* 0-*simple semiring such that* (F) *holds. Then R has no zero divisors. If R is not a division ring, then $R-\{0\}$ is a completely simple semiring satisfying* (F), *and* $R\cong(R-\{0\})^\circ$.

Proof. Since a division ring has no zero divisors, we may assume that R is not a division ring throughout. Then by 3.7 $R/\mathscr{J}$ satisfies the hypotheses of Theorem 3.2. Let π be the canonical homomorphism of R onto $R/\mathscr{J}$; observe that $\pi^{-1}0=0$. If $x, y\in R$, $x, y\neq 0$, then $\pi(x), \pi(y)\neq 0$, so that $\pi(x)+\pi(y)\neq 0$ by 1.2, $\pi(x)\pi(y)\neq 0$ by 3.2, and $x+y\neq 0$, $xy\neq 0$. The theorem follows.

3. By Theorem 4.5 we may concentrate exclusively on the case of a completely simple semiring R. In this case we shall not attempt a complete description of the addition of R; indeed *any* idempotent semigroup is the additive semigroup of some completely simple semiring (the multiplication may be defined by $xy=y$ identically). It is, however, possible to describe the addition inside each $\mathscr{J}$-class in terms of the given representation by triples. More specifically,

THEOREM 4.6. *Let R be a completely simple semiring such that* (F) *holds, and let* $\equiv$ *be the equivalence relations on* Γ, Δ *which determine* $\mathscr{J}$ *in terms of the representation by triples. For each class* $\Gamma_1\subseteq\Gamma$, $(\Delta_1\subseteq\Delta)$ *modulo* $\equiv$, *there exist sets* Γ', Γ'' (Δ', Δ'') *such that one may assume* $\Gamma_1=\Gamma'\times\Gamma''$ $(\Delta_1=\Delta'\times\Delta'')$ *and that the following then hold*:

(i) *if* $\alpha=(\alpha', \alpha'')\equiv\beta=(\beta', \beta'')\in\Gamma$, $\lambda=(\lambda', \lambda'')\equiv\mu=(\mu', \mu'')\in\Delta$, *then* $q^{-1}_{\xi,(\alpha',\beta'')}q_{\xi,\alpha}=u_{\alpha,\beta}$ *and* $r^{-1}_{\xi,(\alpha',\beta'')}r_{\xi,\beta}=w_{\alpha,\beta}$ *depend only on* α, β, *while* $q_{\lambda,\gamma}q^{-1}_{(\lambda',\mu''),\gamma}=v_{\lambda,\mu}$ *and* $r_{\mu,\gamma}r^{-1}_{(\lambda',\mu''),\gamma}=t_{\lambda,\mu}$ *depend only on* λ, μ;

(ii) *if* $\alpha, \beta, \lambda, \mu$ *are as before, then*

$$((\alpha', \alpha''), (i, j), (\lambda', \lambda''))+((\beta', \beta''), (k, m), (\mu', \mu''))$$
$$= ((\alpha', \beta''), (u_{\alpha,\beta}iv_{\lambda,\mu}, w_{\alpha,\beta}mt_{\lambda,\mu}), (\lambda', \mu''))$$

for all $i, k\in F$, $j, m\in G$.

Note that this theorem does not only give the promised information on the addition; it produces further invariants of the semiring (the sets Γ', Γ'', Δ', Δ'' associated with each class modulo $\equiv$; and the functions u, v, w, t, which are invariants inasmuch as p is). From part (i) we also see that, when the multiplication is given, then $\mathscr{J}$ cannot be too large (by the fact that $u_{\alpha,\beta}$ does not depend on the choice of $\xi\in\Delta$, etc.).

The proof of the theorem itself is long but does not present any particular difficulty; we shall give only an outline. We start by extending (4) by distributivity, which gives

$$(\alpha, (i, j), \lambda)+(\alpha, (k, m), \lambda) = (\alpha, (i, m), \lambda), \tag{5}$$

where α, λ are now arbitrary.

Next, take a $\mathscr{J}$-class J of $(R, +)$, determined by Γ_1, Δ_1. Then J is a completely simple subsemiring of R, and $(J, +)$ is a rectangular band; by 3.7 there exist completely simple semigroups $U=\mathscr{M}(H'; \Gamma', \Delta'; p')$, $V=\mathscr{M}(H''; \Gamma'', \Delta''; p'')$ such that $J \cong U * V$. The isomorphism $(J, \cdot) \cong U \times V$ of completely simple semigroups can be described as in Corollary 3.12 of [2]; one uses (5) to show that $H' \cong F$, $H'' \cong G$, so that one may assume without loss of generality that $H'=F$, $H''=G$, and also $\Gamma_1=\Gamma' \times \Gamma''$, $\Delta_1=\Delta' \times \Delta''$. Then one obtains an explicit description of the addition in J, i.e. part (ii) of the theorem, with u, v, w, v given as in part (i) with the restriction that $\xi \in \Delta_1$, $\gamma \in \Gamma_1$.

Then it only remains to show that Γ', Γ'', u, w may be chosen so as not to depend on Δ_1, only on Γ_1, and similarly for the other data. This is done in two steps. First one notes that the given functions u, v, etc. associated with J satisfy $u_{\alpha,\alpha}=f$, etc.; the first step is to show that, if functions u', v', etc. determine the addition of J as in (ii) and satisfy the extra conditions $u'_{\alpha,\alpha}=f$ etc., then they must coincide with the given u, v, etc. Once this uniqueness result is established, it is easy to show, by distributivity, that if Γ', Γ'', u, w serve for J, then they also serve for any other $\mathscr{J}$-class with same Γ_1; and similarly for the other side. Then it follows that Γ', Γ'' depend only on Γ_1, similarly for the Δ's, and that (i) holds in full, and the proof is complete.

The uniqueness result concerning the functions u, v, w, t which serve in (ii) can be sharpened as follows. If one does not require that $u_{\alpha,\alpha}=f$ etc., then these functions are unique modulo the centers of F and G only; if F and G have trivial centers, then they are unique without any further restriction.

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