PERTURBATIONS OF SOLUTIONS OF STIELTJES INTEGRAL EQUATIONS

BY

DAVID LOWELL LOVELADY

Abstract. Using multiplicative integration in two ways, formulae for solutions to perturbed Stieltjes integral equations are found in terms of unperturbed solutions. These formulae are used to obtain bounds on the difference between the perturbed solution and the unperturbed solution. The formulae are also used to explicitly solve, in terms of product integrals, a linear equation subject to nonlinear interface conditions.

I. **Introduction.** Let X be a Banach space with norm N_1. We propose to study relationships between the solutions f and g of $f(t) = p + (R) \int_t^c dF[f]$ and $g(t) = p + (R) \int_t^c dG[g]$, where p is in X, c is a real number, and F and G are functions from $[c, \infty)$ into a space of Lipschitz operators on X. J. W. Neuberger [9] has studied the first equation with F continuous, and J. S. Mac Nerney [6] has extended this study to the case in which F is of bounded variation. For the case in which the equations are equivalent to linear differential equations in matrix rings, C. C. Ross, Jr. [10] has obtained some multiplicative relationships for the solutions. Gantmacher [2, Chapter 14, §5] has also studied this case. R. H. Martin, Jr. [8] (see also [7]) has obtained bounds for solutions. We shall extend the work of Ross [10] to obtain formulae for the solutions of perturbed equations, and then use our formulae to obtain comparison bounds for perturbed solutions.

II. **Preliminary results.** Let H be the set to which A belongs only in case A is a function from X to X, $A(0) = 0$, and there is a number b so that $N_1[A(p) - A(q)] \leq b N_1[p - q]$ whenever (p, q) is in $X \times X$. If A is in H, let $N_2[A]$ be the least number b so that $N_1[A(p) - A(q)] \leq b N_1[p - q]$ whenever (p, q) is in $X \times X$, and let $N_3[A]$ be the least number b so that $N_1[A(p)] \leq b N_1[p]$ whenever p is in X.

Let c be a real number, and let $S = [c, \infty)$. Let OA be the set to which V belongs only in case V is a function from $S \times S$ to H so that if (x, y, z) is in $S \times S \times S$ and z is between x and y, then

(i) $V(x, z) + V(z, y) = V(x, y)$, and
(ii) there is a number M so that

$$\sum_{k=1}^n N_2[V(t_{k-1}, t_k)] \leq M \quad \text{whenever } (t_k)_{k=0}^n \text{ subdivides } (x, y).$$

Let OM be the set to which W belongs only in case W is a function from $S \times S$ to H so that whenever (x, y, z) is in $S \times S \times S$ and z is between x and y, then

(i) $W(x, z)W(z, y) = W(x, y)$, where the multiplication is by composition, and

(ii) there is a number M so that

$$\sum_{k=1}^{n} N_2[W(t_{k-1}, t_k) - I] \leqq M \quad \text{whenever } (t_k)_{k=0}^{n} \text{ subdivides } (x, y),$$

where I in H is given by $I(p) = p$.

Let BVH be the set to which f belongs only in case f is a function from S to H, of bounded N_2-variation on each bounded interval of S. Let BVX be the set to which f belongs only in case f is a function from S to X, of bounded N_1-variation on each bounded interval of S. If F is a function from S to either H or X, let dF be given on $S \times S$ by $dF(x, y) = F(y) - F(x)$.

Let h be a function from $S \times S$ to H, f a function from S to H, and g a function from S to H. If (a, b) is in $S \times S$, and p is in X, by ${}_a\sum^b h[p]$, ${}_a\prod^b h[p]$, $(L) \int_a^b fh[p]$, $(R) \int_a^b hg[p]$, and $(L, R) \int_a^b fhg[p]$ we mean the limit, in the sense of successive refinements of subdivisions, of members of X of the forms $\sum_{k=1}^{n} h(t_{k-1}, t_k)[p]$, $\prod_{k=1}^{n} h(t_{k-1}, t_k)[p]$, $\sum_{k=1}^{n} f(t_{k-1})[h(t_{k-1}, t_k)[p]]$, $\sum_{k=1}^{n} h(t_{k-1}, t_k)[g(t_k)[p]]$, and $\sum_{k=1}^{n} f(t_{k-1})[h(t_{k-1}, t_k)[g(t_k)[p]]]$, respectively, where $(t_k)_{k=0}^{n}$ is a monotone sequence into S so that $a = t_0$ and $b = t_n$.

Our first three lemmas have proofs almost identical to the proofs of Lemmas 4.1, 4.3, and 4.4 of [5], and we shall not exhibit these proofs here.

LEMMA 1. *Let each of f and g be in BVH, and suppose each value of f is linear. Let (a, b) be in $S \times S$, and let p be in X. Then if either of*

$$(L) \int_a^b f \, dg[p] \quad \text{and} \quad (R) \int_a^b (df)g[p]$$

exists, so also does the other, and

$$(L) \int_a^b f \, dg[p] = f(b)[g(b)[p]] - f(a)[g(a)[p]] - (R) \int_a^b (df)g[p].$$

LEMMA 2. *If f, g, a, b, and p are as in Lemma 1, then $(L) \int_a^b f \, dg[p]$ exists.*

LEMMA 3. *Let V be in OA, and let each of f, g, F, and G be in BVH, with each value of f linear. Suppose that whenever (a, b) is in $S \times S$, and p is in X, then*

$$dF(a, b)[p] = (L) \int_a^b fV[p] \quad \text{and} \quad dG(a, b)[p] = (R) \int_a^b Vg[p].$$

Then whenever (a, b) is in $S \times S$, and p is in X,

$$(R) \int_a^b (dF)g[p] = (L, R) \int_a^b fVg[p] = (L) \int_a^b f \, dG[p].$$

REMARK. The existence of $(R) \int_a^b Vg[p]$ was shown by Mac Nerney in [6].

LEMMA 4. *Let $(A_k)_{k=1}^n$ and $(B_k)_{k=1}^n$ be sequences into H. Let $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$, and $(c_k)_{k=1}^n$ be real-valued sequences so that, whenever $1 \leq k \leq n$, $N_3[A_k] \leq a_k$, $N_2[B_k] \leq b_k$, and $N_3[A_k - B_k] \leq c_k$. Then*

$$N_3\left[\prod_{k=1}^n A_k - \prod_{k=1}^n B_k\right] \leq \sum_{k=1}^n \left[\prod_{j=1}^{k-1} b_j\right] c_k \left[\prod_{j=k+1}^n a_j\right].$$

Proof.

$$N_3\left[\prod_{k=1}^n A_k - \prod_{k=1}^n B_k\right] = N_3\left[(A_1 - B_1)\prod_{k=2}^n A_k + B_1 \prod_{k=2}^n A_k - B_1 \prod_{k=2}^n B_k\right]$$

$$\leq c_1 \prod_{k=2}^n a_k + b_1 N_3\left[\prod_{k=2}^n A_k - \prod_{k=2}^n B_k\right].$$

The remainder of the proof is now clear.

LEMMA 5. *Let $(A_k)_{k=1}^n$ and $(B_k)_{k=1}^n$ be sequences into H. Let $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$, and $(c_k)_{k=1}^n$ be real-valued sequences so that, whenever $1 \leq k \leq n$, $N_2[A_k] \leq a_k$, $N_3[B_k] \leq b_k$, and $N_3[B_k - I] \leq c_k$. Then*

$$N_3\left[\prod_{k=1}^n A_k B_k - \prod_{k=1}^n A_k\right] \leq \left(\prod_{k=1}^n a_k\right)\sum_{k=1}^n \left[c_k \left(\prod_{j=k+1}^n b_j\right)\right].$$

REMARK. Lemma 5 is proved in much the same fashion as Lemma 4, so we shall not prove it here.

In [6, Theorem 1.1], Mac Nerney showed that if V is in OA, W is in OM, (a, b) is in $S \times S$, and p is in X, then each of $_a\prod^b [I+V][p]$ and $_a\sum^b [W-I][p]$ exists. Let E be the function on OA, each value of which is a function from $S \times S$ to H, such that if V is in OA, $E[V]$ is given by $E[V](a, b)[p] = _a\prod^b [I+V][p]$. Then [6, Theorem 1.1] the range of E is OM, E is a bijection between OA and OM, and if V is in OA, $W = E[V]$, then $V(a, b)[p] = _a\sum^b [W-I][p]$ whenever (a, b) is in $S \times S$ and p is in X. Mac Nerney also showed [6, Corollary 2.1] that if V is in OA, $W = E[V]$, and h is given by $h(t) = W(t, c)$, then h is in BVH, and is the only member of BVH such that, if p is in X,

$$h(t)[p] = p + (R)\int_t^c Vh[p]$$

for each t in S.

In [4], Herod showed that if each value of V, in OA, is linear, and $W = E[V]$, then each value of W has inverse in H only in case each of $I + V(t, t^+)$, $I + V(t, t^-)$, $I + V(t^+, t)$, and $I + V(t^-, t)$ has inverse in H whenever t is in S. From this it is clear that $W(a, b)$ has inverse in H whenever $a \geq b$ only in case each of $I + V(t, t^-)$ and $I + V(t^+, t)$ has inverse in H whenever t is in S. Let OAI be that subset of OA to which V belongs only in case each value of V is linear, and each of $I + V(t, t^-)$ and $I + V(t^+, t)$ has inverse in H whenever t is in S. Another consequence of [4]

is that if V is in OAI, then $W(\ , c)^{-1}$ is in BVH. We shall need this last fact to ensure the existence of certain integrals.

III. A variation of parameters approach. If V_1 and V_2 are in OA, we can, in some circumstances, find a member g of BVH so that

$$E[V_1 + V_2](t, c) = E[V_1](t, c)[g(t)]$$

for each t in S.

THEOREM 1. *Let V_1 be in OAI, V_2 in OA, and $W_1 = E[V_1]$. Let U be given by $U(a, b)[p] = (L, R) \int_a^b W_1(\ , c)^{-1} V_2 W_1(\ , c)[p]$. Then*

$$E[V_1 + V_2](t, c)[p] = W_1(t, c)[E[U](t, c)[p]],$$

for each t in S and p in X.

THEOREM 2. *Let V_1 be in OAI, V_2 in OA, $W_1 = E[V_1]$, and $W_2 = E[V_2]$. Let U be given by $U(a, b)[p] = (L, R) \int_a^b W_1(\ , c) V_2 W_1(\ , c)^{-1}[p]$. Then*

$$W_1(t, c)[W_2(t, c)[p]] = E[V_1 + U](t, c)[p]$$

whenever t is in S and p is in X.

REMARK. Since Theorem 2 follows from Theorem 1 simply by interchanging the roles of V_2 and U, we shall prove only Theorem 1. It should be noted that Theorem 2 includes Theorem 1 of [10].

Proof of Theorem 1. Let g be given by $g(t) = E[U](t, c)$, and let h be given by $h(t) = W_1(t, c)[g(t)]$. Now if t is in S and p is in X,

$$(R) \int_t^c V_2 h[p] = (L, R) \int_t^c W_1(\ , c) W_1(\ , c)^{-1} V_2 W_1(\ , c) g[p]$$

$$= (L, R) \int_t^c W_1(\ , c) U g[p]$$

$$= -(L) \int_t^c W_1(\ , c)\, dg[p]$$

$$= (R) \int_t^c (d[W_1(\ , c)]) g[p] - p + W_1(t, c)[g(t)[p]]$$

$$= -(R) \int_t^c V_1 W_1(\ , c) g[p] - p + h(t)[p]$$

$$= -(R) \int_t^c V_1 h[p] - p + h(t)[p].$$

This completes the proof.

The remainder of §III is devoted to exploring the consequences of Theorem 2 in the case in which each of V_1 and V_2 is in OAI. Our next result follows immediately.

COROLLARY 1. *Let each of V_1 and V_2 be in OAI, with $W_1 = E[V_1]$. Let U be given by*

$$U(a, b)[p] = (L, R) \int_a^b W_1(\ , c) V_2 W_1(\ , c)^{-1}[p].$$

Then $V_1 + U$ is in OAI.

Now we produce a necessary and sufficient condition for commutativity. Our result includes Theorem 2 of [10].

THEOREM 3. *Let each of V_1 and V_2 be in OAI, with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Then these are equivalent:*

(i) $\qquad\qquad W_1(t, c) W_2(t, c) = W_2(t, c) W_1(t, c) \quad$ *for each t in S.*

(ii)
$$V_1(a, b)[p] - V_2(a, b)[p]$$
$$= (L, R) \int_a^b [W_2(\ , c) V_1 W_2(\ , c)^{-1}[p] - W_1(\ , c) V_2 W_1(\ , c)^{-1}[p]]$$

whenever p is in X and $a \geq b$.

Proof. Choose U_1 and U_2, according to Theorem 2, so that if t is in S,

$$W_1(t, c) W_2(t, c) = E[V_1 + U_1](t, c) \quad \text{and} \quad W_2(t, c) W_1(t, c) = E[V_2 + U_2](t, c).$$

Now (i) holds only in case $E[V_1 + U_1](t, c) = E[V_2 + U_2](t, c)$ for each t in S. Since each of $V_1 + U_1$ and $V_2 + U_2$ is in OAI, this is equivalent to $E[V_1 + U_1](a, b) = E[V_2 + U_2](a, b)$ whenever $a \geq b$, which in turn is equivalent to $V_1(a, b) - V_2(a, b) = U_2(a, b) - U_1(a, b)$ whenever $a \geq b$, which is (ii). The proof is complete.

An interesting question is: When does E preserve the exponential identity on OAI? We answer this question with a theorem which includes Theorem 3 of [10].

THEOREM 4. *Let each of V_1 and V_2 be in OAI, with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Then these are equivalent:*
  (i) $W_1(t, c) W_2(t, c) = E[V_1 + V_2](t, c)$ *for each t in S.*
  (ii) $V_2(a, b)[p] = (L, R) \int_a^b W_1(\ , c) V_2 W_1(\ , c)^{-1}[p]$ *for $a \geq b$ and p in X.*

REMARK. Theorem 4 follows immediately from Theorem 2 and Corollary 1, so we shall not prove it here.

An interesting special case in the multiplication of solutions comes in taking powers. We now handle that case.

THEOREM 5. *Let V be in OAI, and let $W = E[V]$. Then these are equivalent:*
  (i) $E[kV](t, c) = [W(t, c)]^k$ *for each t in S and each positive integer k.*
  (ii) $V(a, b)[p] = (L, R) \int_a^b W(\ , c) V W(\ , c)^{-1}[p]$ *for $a \geq b$ and p in X.*

  IV. An integrated multiplication approach. If V_1 and V_2 are in OA, we can, under certain circumstances, find a formula for $E[V_1 + V_2]$ in terms of products of $E[V_1]$ and $E[V_2]$.

THEOREM 6. *Let each of V_1 and V_2 be in OA, with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Suppose that whenever (a, b) is in $S \times S$, then ${}_a\sum^b N_3[V_1[I + V_2] - V_1] = 0$. Then whenever (a, b) is in $S \times S$ and p is in X, ${}_a\prod^b W_1 W_2[p]$ exists. Furthermore, if W_3 is given by $W_3(a, b)[p] = {}_a\prod^b W_1 W_2[p]$, then $W_3 = E[V_1 + V_2]$.*

REMARK. Helton [3, §5] has studied this kind of integrated multiplication in connection with linear equations.

Proof. Let (a, b) be in $S \times S$ and p in X. It is clear that ${}_a\prod^b W_1 W_2[p]$, if it exists, is given by ${}_a\prod^b [I + V_1][I + V_2][p]$. Let $(t_k)_{k=0}^n$ be a partition of (a, b). Now by Lemma 4,

$$N_1\left[\prod_{k=1}^n [I + V_1(t_{k-1}, t_k)][I + V_2(t_{k-1}, t_k)][p]\right.$$
$$\left. - \prod_{k=1}^n [I + V_1(t_{k-1}, t_k) + V_2(t_{k-1}, t_k)][p]\right]$$
$$\leqq N_1[p] \prod_{k=1}^n [1 + N_2[V_1(t_{k-1}, t_k)]][1 + N_2[V_2(t_{k-1}, t_k)]]$$
$$\cdot \sum_{k=1}^n N_3[V_1(t_{k-1}, t_k)[I + V_2(t_{k-1}, t_k)] - V_1(t_{k-1}, t_k)].$$

The proof is now clear.

We can use this theorem, and the following lemma, to get two companion theorems to Theorem 5.

LEMMA 6. *If A is in H, and n is a positive integer, then*

$$[I + A]^n = I + A + \sum_{k=1}^{n-1} A[I + A]^k.$$

Consequently,

$$[I + A]^n - [I + nA] = \sum_{k=1}^{n-1} [A[I + A]^k - A].$$

THEOREM 7. *Let V be in OA, $W = E[V]$, and let n be a positive integer. Suppose that whenever (a, b) is in $S \times S$, and k is an integer, $1 \leqq k \leqq n$, then*

$${}_a\sum^b N_3[V[I + V]^k - V] = 0.$$

Then if (a, b) is in $S \times S$ and p is in X, ${}_a\prod^b W^n[p]$ exists. Furthermore, if M is given by $M(a, b)[p] = {}_a\prod^b W^n[p]$, then $M = E[nV]$.

THEOREM 8. *Let V be in OA, $W = E[V]$, and suppose each value of V is linear. Suppose that if (a, b) is in $S \times S$, then ${}_a\sum^b N_3[V^2] = 0$. Then if k is a positive integer, (a, b) is in $S \times S$, and p is in X, then*

$${}_a\prod^b W^k[p] = {}_a\prod^b [I + kV][p].$$

REMARKS. The hypotheses of Theorem 7 will be satisfied if V is N_3-continuous. Herod, in [4], has shown that if each value of V is linear, and $_a\sum^b N_3[V^2]=0$ for each (a, b) in $S \times S$, then V is in OAI. To prove Theorem 7, imitate the proof of Theorem 6, and use Lemma 5. Theorem 8 follows quickly from Theorem 7.

V. **The comparison of perturbed solutions.** In differential equations, much use is made of the variation-of-parameters formula in the study of perturbed solutions. Our first result extends that idea.

LEMMA 7. *Let V_1 be in OAI, V_2 in OA, and $W_1=E[V_1]$. Let h be that member of BVH such that if (t, p) is in $S \times X$,*

$$h(t)[p] = p+(R) \int_t^c V_1 h[p]+(R) \int_t^c V_2 h[p].$$

Then if (t, p) is in $S \times X$,

$$h(t)[p] = W_1(t, c)[p]+(L, R) \int_t^c W_1(t, \ )V_2 h[p].$$

Proof. Let g be that member of BVH such that if (t, p) is in $S \times X$, then $g(t)[p] = p+(L, R) \int_t^c W_1(\ , c)^{-1}V_2 W_1(\ , c)g[p]$. Now by Theorem 1 we have $h(t) = W_1(t, c)g(t)$ for each t in S. Since each value of W_1 is linear, this says

$$h(t)[p] = W_1(t, c)[p] + W_1(t, c) \cdot (L, R) \int_t^c W_1(\ , c)^{-1}V_2 h[p]$$

$$= W_1(t, c)[p]+(L, R) \int_t^c W_1(t, \ )V_2 h[p],$$

whenever (t, p) is in $S \times X$. This completes the proof.

Since X is a Banach space, the set of real numbers can be considered as a subset of H. Let OAR (respectively OMR) be the subset of OA (respectively OM) which consists of real-valued functions. Martin, in [8] (see also [7]), has shown that if V is in OA, and (a, b) is in $S \times S$, then $_a\sum^b (N_3[I+V]-1)$ exists. Furthermore, if γ on $S \times S$ is given by $\gamma(a, b)=_a\sum^b (N_3[I+V]-1)$, then γ is in OAR. Also, if $W=E[V]$ and $\lambda=E[\gamma]$, then λ is the least member of OMR such that $N_3[W(a, b)] \leq \lambda(a, b)$ for each (a, b) in $S \times S$, i.e., $\lambda(a, b)=_a\prod^b N_3[W]$ [7, Theorems 3.2 and 3.3] (see also [6, Theorems 1 and 2]).

LEMMA 8. *Let each of V_1 and V_2 be in OA, $W_1=E[V_1]$, and $W_2=E[V_2]$. For $i=1, 2$, let γ_i be given by $\gamma_i(a, b)=_a\sum^b (N_3[I+V_i]-1)$, and let $\lambda_i=E[\gamma_i]$. Let γ_3 be given by $\gamma_3(a, b)=_a\sum^b (N_3[I+V_1+V_2]-1)$, and let $\lambda_3=E[\gamma_3]$. Suppose that whenever (a, b) is in $S \times S$, then $_a\sum^b N_3[V_1[I+V_2]-V_1]=0$ and $_a\sum^b |\gamma_1\gamma_2|=0$. Then whenever (a, b) is in $S \times S$,*

$$\gamma_3(a, b) \leq \gamma_1(a, b)+\gamma_2(a, b),$$

and

$$\lambda_3(a, b) \leq \lambda_1(a, b)\lambda_2(a, b).$$

Proof. Let $W_3 = E[V_1 + V_2]$. Now, if (a, b) is in $S \times S$,

$$\lambda_3(a, b) = {}_a\prod{}^b N_3[W_3] \leqq {}_a\prod{}^b N_3[W_1]N_3[W_2] = \lambda_1(a, b)\lambda_2(a, b),$$

$$\gamma_3(a, b) = {}_a\sum{}^b [\lambda_3 - 1] \leqq {}_a\sum{}^b [\lambda_1\lambda_2 - 1] = \gamma_1(a, b) + \gamma_2(a, b).$$

The proof is complete.

THEOREM 9. *Let V_1, V_2, γ_1, and γ_2 be as in Lemma 8. Let h be that member of BVH such that*

$$h(t)[p] = p + (R)\int_t^c V_1 h[p] + (R)\int_t^c V_2 h[p]$$

whenever (t, p) is in $S \times X$, and let F be that real-valued member of BVH such that

$$F(t) = 1 + (R)\int_t^c \gamma_1 F + (R)\int_t^c \gamma_2 F$$

for each t in S. Then $N_3[h(t)] \leqq F(t)$ for each t in S.

REMARK. Note that if $\lambda_1 = E[\gamma_1]$, and λ_1 is never zero, then, by Theorem 1, $F(t) = \lambda_1(t, c) \, {}_t\prod{}^c [1 + \lambda_1^{-1}\gamma_2]$ for each t in S. Theorem 9 follows immediately from Lemma 8 and the aforementioned work of Martin [8] (see also [7]).

Now we obtain a comparison bound for a nonlinear perturbation of a linear equation. Our theorem follows directly from Theorem 9 and Lemma 7.

THEOREM 10. *Let V_1 be in OAI, V_2 in OA, and $W_1 = E[V_1]$. Let λ_1 and γ_2 be given by $\lambda_1(a, b) = {}_a\prod{}^b N_3[W_1]$ and $\gamma_2(a, b) = {}_a\sum{}^b (N_3[V_2 + I] - 1)$. Let β be a member of OAR such that $N_3[V_2(a, b)] \leqq \beta(a, b)$ for each (a, b) in $S \times S$. Let h in BVH be given by $h(t) = E[V_1 + V_2](t, c)$. Let G be given by $G(t) = {}_t\prod{}^c [1 + \lambda_1^{-1}\gamma_2]$. Then if t is in S,*

$$N_3[h(t) - W_1(t, c)] \leqq \lambda_1(t, c) \cdot (R)\int_t^c \lambda_1^{-1}\beta G.$$

REMARK. One might speculate on replacing β, in the conclusion of Theorem 10, by γ_2. This, in general, is not possible since γ_2 may have negative values.

We can get a bound for the difference of solutions even when V_1 is not linear.

THEOREM 11. *Let each of V_1 and V_2 be in OA, with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Suppose ${}_a\sum{}^b N_3[V_1[I + V_2] - V_1] = 0$ whenever (a, b) is in $S \times S$. Let λ_1, β, and λ_2 be given on $S \times S$ by $\lambda_1(a, b) = {}_a\prod{}^b N_2[W_1]$, $\beta(a, b) = {}_a\sum{}^b N_3[V_2]$, and $\lambda_2(a, b) = {}_a\prod{}^b N_3[W_2]$. Then whenever (a, b) is in $S \times S$,*

$$N_3[E[V_1 + V_2](a, b) - W_1(a, b)] \leqq \lambda_1(a, b) \cdot (R)\int_a^b \beta\lambda_2(\ , b).$$

REMARK. It should be noted that λ_1 is given by products of $N_2[W_1]$, not $N_3[W_1]$ as before. Theorem 11 follows from Lemmas 4 and 5.

VI. **Applications.** We shall show that our results include a slight modification of a theorem of Coppel, [1, p. 64], and we shall obtain a bound for perturbations of nonlinear differential equations. We shall also use our results to solve both linear and nonlinear interface problems for Stieltjes integral equations. Interface problems have been studied by Stallard, [11] and [12], and Zettl, [13], for differential equations.

THEOREM 12. *Let V_1 be in OAI, V_2 in OA, and $W_1 = E[V_1]$. Suppose there are numbers K and M so that $N_3[W_1(t, s)] \leq K$ and $_c\sum^t N_3[V_2] \leq M$ for $c \leq s \leq t$. Let $h = E[V_1 + V_2](\ , c)$. Then if $\varepsilon > 0$ there is $\delta > 0$ so that whenever p is in X and $N_1[p] < \delta$, then $N_1[h(t)[p]] < \varepsilon$ for each t in S. Furthermore, if $N_3 - \lim_{t \to \infty} W_1(t, s) = 0$ for each s in S, then $N_1 - \lim_{t \to \infty} h(t)[p] = 0$ for each p in S.*

REMARK. This theorem includes Coppel's theorem in the case in which the perturbation is Lipschitz.

Proof. From Lemma 7, $h(t)[p] = W_1(t, c)[p] + (L, R) \int_t^c W_1(t, \ )V_2h[p]$ for each (t, p) in $S \times X$. Let β be given by $\beta(a, b) = _a\sum^b N_3[V_2]$. Let (t, p) be in $S \times X$. Now

$$N_1[h(t)[p]] \leq KN_1[p] + (R) \int_t^c K\beta N_1[h(\ )[p]].$$

Let Q be given on S by

$$Q(t) = KN_1[p] + (R) \int_t^c K\beta N_1[h(\ )[p]] - N_1[h(t)[p]].$$

Now Q has only nonnegative values and, if t is in S,

$$(*) \qquad N_1[h(t)[p]] = KN_1[p] - Q(t) + (R) \int_t^c K\beta N_1[h(\ )[p]].$$

Let F be that nonnegative-valued member of BVH so that

$$(**) \qquad\qquad F(t) = KN_1[p] + (R) \int_t^c K\beta F$$

for each t in S. Now each of $(*)$ and $(**)$ can be solved by iteration [6, Theorem 2], and $-Q(t) \leq 0$ for each t in S, so $N_1[h(t)[p]] \leq F(t)$ for each t in S. But

$$F(t) = _t\prod^c [1 + K\beta]KN_1[p] \leq (\exp [K\beta(t, c)])KN_1[p]$$
$$\leq KN_1[p](\exp [KM])$$

for each t in S. This proves the first part of the theorem.

Note that in the first part of the proof we in fact showed that whenever p is in X, then $h(\ )[p]$ is bounded. Now let p be in S. If $t \geq s$, and B is such that $N_1[h(r)[p]] \leq B$ for each r in S, then

$$N_1\left[(R) \int_t^s W_1(t, \ )V_2h[p]\right] \leq KB\beta(t, s).$$

Let $\varepsilon > 0$ and find s_0 so that $\beta(t, s_0) < \varepsilon/(KB+1)$. Now if $t > s_0$,

$$N_1[h(t)[p]] \leqq N_3[W_1(t, c)]N_1[p] + (L, R) \int_{s_0}^c N_3[W_1(t,\ )]\beta N_1[h(\ )[p]] + \varepsilon$$

$$\leqq N_3[W_1(t, c)]N_1[p]$$

$$+ N_3[W_1(t, s_0)]\cdot(L, R) \int_{s_0}^c N_3[W_1(s_0,\ )]\beta N_1[h(\ )[p]] + \varepsilon.$$

The proof is now clear.

Let M be that real-valued function on H given by

$$M[A] = \lim_{t \to 0+} (1/t)[N_3[I + tA] - 1].$$

Suppose B is an N_2-continuous function from S to H, and let V be given by $V(s, t) = \int_s^t B(r)\, dr$, if $s \leqq t$, and $V(s, t) = V(t, s)$ if $t \leqq s$. Let γ be given by $\gamma(a, b) = {}_a\sum^b [N_3[I + V] - 1]$. Martin, [8, Example 3.1] (see also [7]), has shown that if $s \leqq t$, then $\gamma(s, t) = \int_s^t M[B(r)]\, dr$ and $\gamma(s, t) = \gamma(t, s)$ if $t \leqq s$. Noting these results, we state without proof the following consequence of Theorem 11.

THEOREM 13. *Let each of A and B be an N_2-continuous function from S to H. Let W be that function from $S \times S$ to H such that if (s, p) is in $S \times X$, then $W(\ , s)[p]$ is the solution of $h'(t) = A(t)[h(t)]$, $h(s) = p$. Let λ be given by $\lambda(a, b) = {}_a\prod^b N_2[W]$. Let p be in X, and let h and g be continuously differentiable members of BVX such that $h(c) = g(c) = p$, $h'(t) = A(t)[h(t)] + B(t)[h(t)]$, and $g'(t) = A(t)[g(t)]$ whenever t is in S. Then, if t is in S,*

$$N_1[h(t) - g(t)] \leqq N_1[p]\lambda(t, c) \int_c^t N_3[B(s)] \exp\left[\int_c^s M[B(r)]\, dr \right] ds.$$

In [12, §3], Stallard showed how to solve the following problem: Let P be a Lebesgue summable matrix-valued function on an interval $[a, b]$. Let $(A_k)_{k=1}^n$ be a matrix-valued sequence, and let $a < t_1 < t_2 < \cdots < t_n < b$. Find a matrix-valued function Y on $[a, b]$ so that

  (i) $Y(a) = I$, where I is the identity matrix,

  (ii) $Y(t_k^+) = A_k Y(t_k^-)$ for $k = 1, 2, \ldots, n$,

  (iii) Y is absolutely continuous on $[a, t_1)$, $(t_n, b]$, and (t_{k-1}, t_k) for $k = 2, 3, \ldots, n$, and

  (iv) $Y'(t) = P(t) Y(t)$, almost everywhere in (a, t_1), (t_n, b), and (t_{k-1}, t_k) for $k = 2, 3, \ldots, n$.

We shall extend this problem to Stieltjes integral equations, and show that under certain circumstances we can write a formula for the solution.

THEOREM 14. *Let V be in OA, and let K be a countable subset of S, c not in K. Let each of A and B be a function from K to H such that if M is a bounded subset of K, then each of $\sum_{t \text{ in } M} N_2[A(t) - I - V(t, t^-)]$ and $\sum_{t \text{ in } M} N_2[B(t) - I - V(t^+, t)]$ is*

finite. Then whenever p is in X there is a member h of BVX such that each of (i), (ii), *and* (iii) *is true.*

(i) $h(c)=p$.

(ii) *If t is in K, then* $h(t)=A(t)[h(t^-)]$ *and* $h(t^+)=B(t)[h(t)]$.

(iii) *If N is an open connected subset of S in the complement of K, then there is q in X so that*

$$h(t) = q+(R)\int_t^c V[h]$$

whenever t is in N.

REMARK. It should be noted that, if K has no limit point, the conditions on A and B are satisfied a priori. In this circumstance it is clear that h is unique, and hence this theorem includes Stallard's result [12]. In other circumstances, additional conditions on A, B, and K may be necessary to ensure uniqueness (see [11, pp. 65–66]).

Proof. Let U_1 and U_2 be members of OA such that if $[a, b]$ is an interval in S then $U_1(b, a)=0$ if $(a, b]$ does not intersect K, $U_2(b, a)=0$ if $[a, b)$ does not intersect K,

$$U_1(b, a) = \sum_{s \text{ in } K,\, a < s \leq b} [A(s)-I-V(s, s^-)]$$

if $(a, b]$ does intersect K, and

$$U_2(b, a) = \sum_{s \text{ in } K,\, a \leq s < b} [B(s)-I-V(s^+, s)]$$

if $[a, b)$ does intersect K.

Now if t is in K,

$$U_1(t, t^-) = A(t)-I-V(t, t^-), \qquad U_1(t^+, t) = 0,$$
$$U_2(t, t^-) = 0, \qquad\qquad\qquad U_2(t^+, t) = B(t)-I-V(t^+, t).$$

Let p be in X, and let h be given by $h(t)=E[V+U_1+U_2](t, c)[p]$. Now h is in BVX, and if t is in S then

$$h(t) = p+(R)\int_t^c V[h]+(R)\int_t^c U_1[h]+(R)\int_t^c U_2[h].$$

Hence (i) and (iii) follow immediately. Also, if t is in S, $t>c$, then

$$h(t) = [I+V+U_1+U_2](t, t^-)[h(t^-)]$$

and

$$h(t^+) = [I+V+U_1+U_2](t^+, t)[h(t)].$$

Thus, if t is in K, $h(t)=A(t)[h(t^-)]$ and $h(t^+)=B(t)[h(t)]$, and the proof is complete.

COROLLARY 2. *Let V be in OAI, $W=E[V]$, and let K, A, and B be as in Theorem 14. Let p be in X, and let h be the function determined in the proof of Theorem 14.*

Then if t is in S, not in K, and $[c, t]$ *intersects K, then*

$$h(t) = W(t, c)\Big[{}_t\prod_{(s \text{ in } K)}^c [W(s^+, c)^{-1}B(s)A(s)W(s^-, c)]\Big][p].$$

Proof. Let U_1 and U_2 be as in the proof of Theorem 14. Let T be a member of OA such that if $[a, b]$ is an interval in S and q is in X, then

$$T(b, a)[q] = (L, R) \int_b^a W(\ , c)^{-1}[U_1 + U_2]W(\ , c)[q].$$

Now, by Theorem 1, h is given by $h(t) = W(t, c)[{}_t\prod^c [I + T][p]]$. If t is in S, not in K, and $[c, t]$ intersects K, then

$$_t\prod^c [I + T][p] = {}_t\prod_{(s \text{ in } K)}^c [[I + T(s^+, s)][I + T(s, s^-)]][p].$$

But, if s is in K,

$$\begin{aligned}
T(s^+, s) &= W(s^+, c)^{-1}U_2(s^+, s)W(s, c) \\
&= W(s^+, c)^{-1}[B(s) - I - V(s^+, s)]W(s, c) \\
&= W(s^+, c)^{-1}B(s)W(s, c) - W(s^+, c)^{-1}W(s^+, c) \\
&= W(s^+, c)^{-1}B(s)W(s, c) - I.
\end{aligned}$$

Similarly, if s is in K,

$$T(s, s^-) = W(s, c)^{-1}A(s)W(s^-, c) - I.$$

This completes the proof.

ACKNOWLEDGEMENT. The author gratefully acknowledges the helpful recommendations of the referee with respect to Theorem 14 and Corollary 2.

REFERENCES

1. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, Mass., 1965. MR **32** #7875.

2. F. R. Gantmacher, *The theory of matrices*, GITTL, Moscow, 1953; English transl. Vol. 2, Chelsea, New York, 1959. MR **16**, 438; MR **21** #6372c.

3. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math. **16** (1966), 297–322. MR **32** #6167.

4. J. V. Herod, *Multiplicative inverses of solutions for Volterra-Stieltjes integral equations*, Proc. Amer. Math. Soc. **22** (1969), 650–656.

5. J. S. Mac Nerney, *Integral equations and semigroups*, Illinois J. Math. **7** (1963), 148–173. MR **26** #1726.

6. ———, *A nonlinear integral operation*, Illinois J. Math. **8** (1964), 621–638. MR **29** #5082.

7. R. H. Martin, Jr., *A bound for solutions of Volterra-Stieltjes integral equations*, Proc. Amer. Math. Soc. **23** (1969), 506–512. MR **40** #662.

8. ———, *Bounds for solutions to a class of nonlinear integral equations* (submitted for publication).

9. J. W. Neuberger, *Continuous products and nonlinear integral equations*, Pacific J. Math. **8** (1958), 529–549. MR **21** #1509.

10. C. C. Ross, Jr., *On the multiplication of solutions of homogeneous linear differential systems*, J. Math. Anal. Appl. **25** (1969), 266–271. MR **38** #1311.

11. F. W. Stallard, *Differential systems with interface conditions*, Oak Ridge National Laboratory Publication #1876 (Physics) 1955.

12. ———, *Functions of bounded variation as solutions of differential systems*, Proc. Amer. Math. Soc. **13** (1962), 366–373. MR **25** #2276.

13. A. Zettl, *Adjoint and self-adjoint boundary value problems with interface conditions*, MRC Technical Report #827, 1967.

GEORGIA INSTITUTE OF TECHNOLOGY,
ATLANTA, GEORGIA 30332