TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 155, Number 2, April 1971

PARTIAL ORDERS ON THE TYPES IN βN

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Abstract. Three partial orders on the types of points in βN are defined and studied in this paper. Their relation to the types of points in $\beta N-N$ is also described.

Several natural partial orders can be given to the types of points in βN. The purpose of this paper is to give some of these orders wider publicity. I feel these orders are fundamental in the study of ultrafilters on the integers. I had hoped these orders would lead to a classification of the types of points in N^*. I no longer feel this is true, but connections with this important unsolved problem are discussed.

I. Let N denote the set of all positive integers and S the set of all subsets of N. Let βN denote the set of all ultrafilters on N and N^* the set of all free ultrafilters on N. For $M \subset N$ let $W(M)$ be the set of all terms of βN to which M belongs. Then the set of all $W(M)$ for $M \subset N$ forms a basis for a topology on βN and the resulting space is topologically the Čech compactification of the integers and N^* is topologically $\beta N-N$. To avoid ambiguity let n' be the ultrafilter to which the integer n belongs and N' the set of all fixed ultrafilters; thus $N^*=\beta N-N'$.

If p and q are points of a topological space X, p and q are *of the same type in X* provided there is a homeomorphism of X onto itself taking p into q. It is easy to see [1] that two ultrafilters on N are of the same type in βN if and only if there is a permutation of N which takes the members of one onto the members of the other. That Ω and θ are of the same type in βN will be denoted by $\Omega \sim \theta$ and $[\Omega]$ will denote the set of all ultafilters on N which are of the same type as Ω. Clearly $\sim$ is an equivalence relation and $[\Omega]$ has c members.

The problem of characterizing the types of points in N^* is the problem of finding reasonable necessary and sufficient conditions on terms Ω and θ of N^* so that one can construct a permutation of S which preserves *infinite* intersections and takes the members of Ω onto the members of θ.

A term Ω of N^* is called a P-point provided, for every countable subcollection $\{E_n\}_{n\in N}$ of Ω, there is a term E of Ω such that $E-E_n$ is finite for all n. In [1] Walter Rudin proves that the continuum hypothesis [CH] implies the existence of P-points in N^* and that all P-points are of the same type in N^*. Booth [3] has shown, using Martin's axiom rather than [CH], that there are P-points in N^* without an $\aleph_1$ base.

Received by the editors March 13, 1970.

AMS 1969 *subject classifications.* Primary 5453, 0415.

Key words and phrases. Čech compactification of the integers, partial order, types of points, βN, ultrafilter on the integers.

In the light of these results, classification of the types in N^* seems hopeless without some set theoretic assumptions. The results of this paper frequently use [CH] and the strong structure of P-points this implies. The required background reading is [1].

II. A sequence $\{\rho_n\}_{n\in N}$ of terms of βN is called discrete if there exists a sequence $\{E_n\}_{n\in N}$ of disjoint subsets of N such that $E_n \in \rho_j$ if and only if $n=j$. Let D be the set of all such discrete countable sequences of terms of βN. If $X=\{\rho_n\}_{n\in N} \in D$ and $\theta \in \beta N$, define $\theta_X=\{M\subset N \mid \{n \mid M \in \rho_n\} \in \theta\}$. That is θ_X is the image of θ under the natural homeomorphism of βN onto the closure of X which takes n to ρ_n. Observe that, if X belongs to D and Ω to the closure of X, there is a unique θ such that $\theta_X=\Omega$.

In [2] Z. Frolik says θ *produces* θ_X. Then he proves that $\theta \in \beta N$ implies that θ produces 2^c terms of βN but is produced by at most c terms. Frolik also observes that if $X \in D$ and $X\subset N^*$, then there are at most c terms of $\bar{X}$ which have the same type in N^* and hence there are 2^c types of points in N^*.

A. Let us prove that Frolik's producing relation is a partial ordering of the types in βN.

1. $\theta \sim \Omega$ *implies that* θ *produces* Ω. For, if π is a permutation of N such that $M \in \Omega$ if and only if $\pi(M) \in \theta$, then $X=\{\pi(n)\}_{n\in N}$ is such that $\theta_X=\Omega$.

2. *If* ϕ *produces* θ *and* θ *produces* Ω, *then* ϕ *produces* Ω. Suppose $X \in D$ and $\{\rho_n\}_{n\in N}= Y \in D$ and $\phi_Y=\theta$ and $\theta_X=\Omega$. For each $n \in N$ define $\mu_n=(\rho_n)_X$ and let $Z=\{\mu_n\}_{n\in N}$. Then $\Omega=\phi_Z$.

3. *Suppose* Ω *produces* θ *and* θ *produces* Ω. *Then* $\theta\sim\Omega$. Suppose $X=\{\eta_n\}_{n\in N} \in D$ and $Y=\{\rho_n\}_{n\in N} \in D$ and $\theta_X=\Omega$ and $\Omega_Y=\theta$. For $n \in N$, define $\mu_n=(\rho_n)_X$ and let $Z=\{\mu_n\}_{n\in N}$. As in 2, $M \in \Omega$ if and only if $\{n \mid M \in \mu_n\} \in \Omega$. Let $\{E_n\}_{n\in N}$ be a set of disjoint subsets of N with $E_n \in \mu_n$. Define a two-valued function $f\colon N \to \{0, 1\}$ as follows. Define $f(1)=0$ and, if $n \in E_1$ and $n>1$, define $f(n)=1$. Assume $i>1$ and $f(n)$ has been defined for all $n<i$ and all $n \in E_j$ where $j<i$. If $f(i)$ has been defined as 1 and $n>i$ and $n \in E_i$, define $f(n)=0$. If $f(i)$ has *not* been defined as 1, define $f(i)=0$ and, if $n>i$ and $n \in E_i$, define $f(n)=1$. Exactly one of $f^{-1}(0)$ and $f^{-1}(1)$ belongs to Ω. Suppose $f^{-1}(0) \in \Omega$. Let $M=f^{-1}(0) \cap \{n \mid f^{-1}(0) \in \mu_n\}$; $M \in \Omega$. If $n \in M$, then the finite set $f^{-1}(0) \cap E_n \in \mu_n$; hence μ_n is not free. But μ_n is not free implies $\mu_n \in X$, and we can define $\pi\colon M \to N$ by $\mu_n=\eta_{\pi(n)}$. We can find $E\subset M$ such that $E \in \Omega$, $N-E$ is infinite, and $N-\pi(E)$ is infinite. Then π/E can be extended to a permutation p of N onto N. It is easy to check that $B \in \Omega$ if and only if $p(B) \in \theta$. Thus $\theta\sim\Omega$.

Now for Ω and θ in βN, define $[\theta]\leqq[\Omega]$ if θ produces Ω. By 1, 2, and 3, $\leqq$ is a partial order on the types in βN.

B. *If* $\Omega \in \beta N$, *the set of all types in* βN *which precede* $[\Omega]$ *in* $\leqq$ *is totally ordered by* $\leqq$. For suppose $[\phi]\leqq[\Omega]$ and $[\theta]\leqq[\Omega]$. There is $X=\{\rho_n\}_{n\in N} \in D$ and $Y \in D$ such that $\phi_Y=\theta_X=\Omega$. Temporarily ignore the order of terms of D and just use them as sets with closures in βN. And use $\bar{X}$ for the closure of X in βN. We need the fact [4] that if $V \in D$ and $\Omega \in \beta N$ and $\Omega \in \bar{V}$ and $V_1 \cap V_2 = \varnothing$ and $V_1 \cup V_2 = V$, then Ω belongs to one and only one of $\bar{V}_1$ and $\bar{V}_2$. In our case $\Omega \in \bar{X}$ and $X = (X \cap \bar{Y}) \cup (X - \bar{Y})$ and $\Omega \in \bar{Y}$ and $Y = (Y \cap \bar{X}) \cup (Y - \bar{X})$. Since $V = (X - \bar{Y}) \cup (Y - \bar{X})$ is countable and discrete, using the fact again we have that Ω belongs to the closure of $X \cap \bar{Y}$ or $Y \cap \bar{X}$. Say Ω belongs to the closure of $X \cap \bar{Y}$. Let $L = \{n \mid \rho_n \in \bar{Y}\}$; then $L \in \theta$. For $n \in L$, define μ_n to be the unique term of βN such that $(\mu_n)_Y = \rho_n$; if $n \in N - L$ define $\mu_n = n'$. Then $Z = \{\mu_n\}_{n \in N} \in D$ and $\phi_Z = \theta$.

C. The Frolik order $\leqq$ has been studied extensively and we have a great deal of information. The fixed ultrafilters form a type in βN which precedes all other types. In N^*, types of P-points are obviously minimal in this order. For $X \subset \beta N$, let $X^* = \bar{X} - X$. K. Kunen [5] has shown that [CH] there is a non-P-point in N^* not in X^* for any countable subset X of N^* and [CH] there is a countable subset X of N^* and a point of X^* which is not in Y^* for any Y in D. Types of both of these points are clearly minimal in N^* under $\leqq$. By B the order is tree-like and knowing that any term is preceded by at most c terms but followed by 2^c terms gives us the picture of a fast branching tree.

In his thesis [3] Booth defines the product $[\theta] \cdot [\Omega]$ for Ω and θ in βN to be $[\theta_X]$ where $X = \{\rho_n\}_{n \in N} \in D$ and $\rho_n \sim \Omega$. Obviously $[\theta] \leqq [\theta] \cdot [\Omega]$ and it is easy to show that $[\theta] \cdot [\Omega]$ is well defined. For $n > 1$, $[\theta]^n$ is defined inductively as $[\theta] \cdot [\theta]^{n-1}$; then $[\theta]^a \cdot [\theta]^b = [\theta]^{a+b}$. Among other things Booth uses these definitions to construct infinite well-ordered increasing and well-ordered decreasing sequences in $\leqq$.

If Ω and θ are of the same type in N^*, then the types which precede $[\Omega]$ by $\leqq$ must be precisely those which precede $[\theta]$ by $\leqq$. In [4] I showed that the condition is also sufficient if both Ω and θ are limit points of countable sets of P-points [CH]. But by Kunen's examples there are minimal elements $[\theta]$ and $[\phi]$ in N^* such that θ and ϕ are of different types in N^*. So the condition is clearly not sufficient. Also if Λ is any member of N^*, it is easy to show, using the methods of B, that $[\Lambda]$ is maximal in the set of predecessors of both $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$. Hence the predecessors of $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$ are exactly the same but none of their terms have the same type in N^*.

III. A more general partial order on the types of βN will now be discussed. Let F be the set of all functions from N onto N. If $\Omega \in \beta N$ and $f \in F$, define $f(\Omega) = \{f(M) \mid M \in \Omega\}$.

A. Let us prove that F induces a partial order on the types of points in βN.

1. *If Ω and θ belong to βN and $\Omega \sim \theta$, then there is an $f \in F$ such that $f(\Omega) = \theta$.* For, by definition, there is a permutation f of N such that $M \in \Omega$ if and only if $f(M) \in \theta$.

2. *If Ω, ϕ and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \phi$, then $g \circ f(\Omega) = \phi$.*

3. *Suppose that Ω and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \Omega$. We prove $\Omega \sim \theta$.* Let $L = \{n \mid (g \circ f)(n) > n\}$, $M = \{n \mid (g \circ f)(n) = n\}$ and

$Q=\{n \mid (g\circ f)(n)<n\}$. Suppose $M\in\Omega$; then f/M is one-to-one. There is a subset E of M belonging to Ω such that $N-E$ and $N-f(E)$ are infinite, and f/E can be extended to a permutation p of N onto N. Thus $B\in\Omega$ if and only if $p(B)\in\theta$ and $\Omega\sim\theta$.

If $M\notin\Omega$ one of L and Q must belong to Ω. Suppose $L\in\Omega$. If n and k belong to L, let us say nek provided that, for some nonnegative integers i and j, $(g\circ f)^i(n)=(g\circ f)^j(k)$ where $(g\circ f)^0$ is the identity map. Clearly e is an equivalence relation. Let E be the set of all equivalence classes of subsets of L related by e. From each $A\in E$ select $A_0\in A$. Then define a two-valued function $t: L\to\{0,1\}$ as follows. If $n\in A\in E$ and $(g\circ f)^i(n)=(g\circ f)^j(A_0)$ then define $t(n)$ as 0 if $|i-j|$ is even and 1 if $|i-j|$ is odd. The function t is well defined and only one of $t^{-1}(0)$ and $t^{-1}(1)$ belongs to Ω. Suppose $t^{-1}(0)\in\Omega$. By 2, $(g\circ f)(t^{-1}(0))\in\Omega$. But

$$(g\circ f)(t^{-1}(0)) \subset t^{-1}(1)$$

and this is a contradiction.

As before, if Ω and θ belong to βN, define $[\Omega]\succeq[\theta]$ provided there is an $f\in F$ such that $f(\Omega)=\theta$. By 1, 2, and 3, $\succeq$ is a partial order on the types of points in βN.

B. We make several very simple observations.

1. *If Ω and θ belong to βN and $[\theta]\leqq[\Omega]$, then $[\Omega]\succeq[\theta]$.* For $[\theta]\leqq[\Omega]$ implies there is $X=\{\rho_n\}_{n\in N}\in D$ such that $\theta_X=\Omega$. And $X\in D$ implies there is a set $\{E_n\}_{n\in N}$ of disjoint subsets of N such that $E_n\in\rho_n$. If $n>1$ and $i\in E_n$, define $f(i)=n$; and if $i\notin E_n$ for any $n>1$, define $f(i)=1$. Then $f\in F$ and $f(\Omega)=\theta$.

2. *If $\theta\in\beta N$, in $\succeq$ $[\theta]$ is greater than at most c types but less than 2^c types.* The first follows from the cardinality of F being c. The second follows from 1 and Frolik's result in §II. In fact using Frolik's proof one shows that if *both* $\theta\in\beta N$ and $f\in F$ are given and, for all n, $f^{-1}(n)$ is infinite, there are 2^c terms Ω of βN such that $f(\Omega)=\theta$. By contrast recall that if $\theta\in\beta N$ and $X\in D$ are given there is a unique Ω such that $\theta_X=\Omega$.

3. *If θ and ϕ belong to βN, there is an Ω in βN such that $[\Omega]\succeq[\theta]$ and $[\Omega]\succeq[\phi]$.* Select $f\in F$ such that $f^{-1}(i)$ is infinite for each $i\in N$. Now select $g\in F$ such that, for each i and j in N, $g^{-1}(j)\cap f^{-1}(i)$ is infinite. Then select $\rho_{ij}\in N^*$ such that $g^{-1}(j)\cap f^{-1}(i)\in\rho_{ij}$. For $i\in N$ define $\rho_i=\phi_{X_i}$ where $X_i=\{\rho_{ij}\}_{j\in N}$; and for $X=\{\rho_i\}_{i\in N}$ let $\Omega=\theta_X$. Then $f(\Omega)=\theta$ and $g(\Omega)=\phi$.

4. *Suppose θ and ϕ belong to βN. Let $B=\{[\Omega]\mid[\Omega]\succeq[\theta]$ and $[\Omega]\succeq[\phi]\}$. Then $[\Omega]\in B$ is minimal in B if and only if, for all f and g in F such that $f(\Omega)=\theta$ and $g(\Omega)=\phi$, there is an $M\in\Omega$ such that for i and j in N, $f^{-1}(i)\cap g^{-1}(j)\cap M$ is at most a singleton.* To prove the *only if*, suppose f and g are given and let $Q=\{(i,j)\in N\times N\mid f^{-1}(i)\cap g^{-1}(j)\neq\varnothing\}$. If Q is finite, $[\theta]$ and $[\phi]$ and $[\Omega]$ are N'; if $\Omega=m'$, $M=\{m\}$ has the desired properties. If Q is infinite there is a one-to-one function q from N onto Q; define $k\in F$ by $k^{-1}(n)=f^{-1}(i)\cap g^{-1}(j)$ where $(i,j)=q(n)$. Hence if, for $n\in N$, we define $f^*(n)=f(k^{-1}(n))$ and $g^*(n)=g(k^{-1}(n))$, then f^* and g^* belong to F and $f^*(k(\Omega))=\theta$ and $g^*(k(\Omega))=\phi$. But $k(\Omega)\sim\Omega$ only if there

is an $M \in \Omega$ such that k/M is one-to-one. Now to prove *if*, suppose f, g and h belong to F and $f(h(\Omega)) = \theta$ and $g(h(\Omega)) = \phi$. Then $f \circ h(\Omega) = \theta$ and $g \circ h(\Omega) = \phi$. So assume also that there is an M such that, for all i and j in N,

$$(f \circ h)^{-1}(i) \cap (g \circ h)^{-1}(j) \cap M$$

is at most a singleton. Then h/M is one-to-one and thus $h(\Omega) \sim \Omega$.

5. K. Kunen has a beautiful proof [5] that $\succeq$ is not a total order. The same proof shows that there are c types no pair of which are ordered. And using the continuum hypothesis it is easy to show that there are 2^c pairwise unordered types in $\succeq$, even 2^c minimal in N^* types.

6. Together 3 and 5 imply that, unlike II B, this order is not treelike. That is, there are types in βN whose predecessors are not totally ordered by $\succeq$. In fact $\succeq$ is more rootlike; that is, things get together near the top.

C. In addition to the facts in B, what can we say about $\succeq$? Again N', the type of all fixed ultrafilters, is less than all other types. If $[\Omega]$ is minimal in N^* under $\succeq$, then Ω is a P-point by definition. If θ is a P-point and $[\theta] \succeq [\phi]$, then ϕ is a P-point or a fixed ultrafilter. It is not hard to prove [CH] that there are types which are minimal in N^*. This was first proved by J. Keisler [6] and will be a corollary of the example given in IV C. We prove [CH] that above every P-point type is another P-point type. Thus two types of the same type in N^* may or may not be ordered under $\succeq$; for two P-point types which are minimal under $\succeq$ are not ordered.

Suppose $\theta \in \beta N$ is a P-point. Then [CH] *there is a P-point Ω such that* $[\Omega] \succeq [\theta]$ *but* $[\Omega] \neq [\theta]$.

Proof. Clearly [CH] implies that both F and S have cardinality $\aleph_1$; hence let $F = \{f_\alpha\}_{\alpha < \omega_1}$ and $S = \{S_\alpha\}_{\alpha < \omega_1}$. Let f be a term of F such that, for each n, $f^{-1}(n)$ has n terms. We build Ω so that $f(\Omega) = \theta$ by induction on the countable ordinals. Let $\mathscr{A}$ be the set of all $A \subset N$ such that for some $a \in N$ and all $n \in N$, the number of terms of $f^{-1}(n) \cap A$ is less than a; observe that $\mathscr{A}$ is closed under finite union.

For each $\alpha \in \omega_1$, we define a countable subset Ω_α of subsets of N such that

(1) For $\beta < \alpha$, $\Omega_\beta \subset \Omega_\alpha$. Also Ω_α is closed under finite intersection.

(2) If $\alpha = \beta + 1$, there is a term X of Ω_α such that (a) $X \subset S_\beta$ or $X \subset N - S_\beta$ and (b) for some $n \in N$, $X \subset f_\beta^{-1}(n)$, or, for all $n \in N$, $X \cap f_\beta^{-1}(n)$ is finite.

(3) For $E \in \theta$, $M \in \Omega_\alpha$, and $A \in \mathscr{A}$, $M \cap f^{-1}(E) \not\subset A$. By (1), there exists $\Omega \in \beta N$ such that $\Omega \supset \bigcup_{\alpha < \omega_1} \Omega_\alpha$. By (2(a)), $L \in \Omega$ implies $L \supset M \in \Omega_\alpha$ for some $\alpha < \omega_1$. By (3), $f(\Omega) = \theta$ but $\Omega \nsim \theta$. By (2(b)), Ω is a P-point (or a fixed ultrafilter but this is impossible since $f(\Omega) = \theta$).

So it will suffice to define the Ω_α.

Define $\Omega_0 = \{N\}$ and, for limit ordinals α, define $\Omega_\alpha = \bigcup_{\beta < \alpha} \Omega_\beta$. Then (1), (2), and (3) are trivially satisfied.

Suppose $\alpha = \beta + 1$ for some $\beta < \omega_1$. We find an $X \subset N$ satisfying (2(b)) such that, for all $L \in \Omega_\beta$ and $M = L \cap X$, (3) is satisfied. If for all $L \in \Omega_\beta$, (3) is satisfied with $M = L \cap X \cap S_\beta$ define $Y = X \cap S_\beta$. Otherwise since Ω_β and θ are closed under finite intersection and $\mathscr{A}$ under finite union, (3) is satisfied with $M=L \cap X \cap (N-S_\beta)$; and in this case define $Y=X \cap (N-S_\beta)$. We then define $\Omega_\alpha=\Omega_{\alpha-1} \cup \{Y \cap L \mid L \in \Omega_{\alpha-1}\}$ and (1), (2), and (3) are all satisfied. We define X by cases.

Case 1. There is an $n \in N$ such that, for all $L \in \Omega_\beta$, $E \in \theta$, and $A \in \mathscr{A}$, $f_\beta^{-1}(n) \cap L \cap f^{-1}(E) \not\subset A$; then $X=f_\beta^{-1}(n)$ has the desired properties.

Case 2. For each $n \in N$ there exists $L_n \in \Omega_\beta$, $E_n \in \theta$ and $A_n \in \mathscr{A}$ such that $f_\beta^{-1}(n) \cap L_n \cap f^{-1}(E_n) \subset A_n$. Without loss of generality we assume that $A_n \subset A_{n+1}$, $E_n \supset E_{n+1}$, $L_n \supset L_{n+1}$ and, for $L \in \Omega_\beta$, there is an n such that $L \supset L_n$. For $j \in N$, define

$$D_j = \{e \in N \mid f^{-1}(e) \cap L_j - f_\beta^{-1}(1, 2, \dots, j) \text{ has more than } j \text{ terms}\}.$$

Observe that $D_j \in \theta$. Otherwise $D'=(N-D_j) \cap E_j \in \theta$. And by the definition of D_j there is a term A of $\mathscr{A}$ such that $f^{-1}(N-D_j) \cap L_j - f_\beta(1, \dots, j) \subset A$. But by our assumption $f^{-1}(E_j) \cap L_j \cap f_\beta^{-1}(1, \dots, j) \subset A_j$. Hence $f^{-1}(D') \cap L_j \subset (A_j \cup A) \in \mathscr{A}$. But by (3) of our induction hypotheses, if $D' \in \theta, f^{-1}(D') \cap L_j$ is not a subset of any term of $\mathscr{A}$. Hence, since θ is a P-point, there is a $D \in \theta$ such that, for all $j \in N$, $D-D_j$ is finite. If $e \in D \cap D_1$, select $x_{e1} \in f^{-1}(e) \cap L_1 \cap f_\beta^{-1}(n)$ with n maximal. And for $j>1$ and $j \in N$, if $e \in D \cap D_j$ select

$$x_{ej} \in f^{-1}(e) \cap L_j \cap f_\beta^{-1}(n) - (x_{e1}, x_{e2}, \dots, x_{e,j-1})$$

with n maximal. Let $X=\{x_{ej}\}$.

Fix $n \in N$ and let us show that $X \cap f_\beta^{-1}(n)$ is finite. By the definition of D_j and x_{ej}, if $e \in D_j$ and $x_{ej} \in f_\beta^{-1}(n)$, then $n>j$. Similarly, if $e \in D_n$ and x_{ej} is defined, $x_{ej} \in f_\beta^{-1}(m)$ for some $m>n$. So since x_{ej} is only defined for $e \in D$ and $D-D_n$ is finite, there are at most finitely many j and e such that $x_{ej} \in f_\beta^{-1}(n)$.

Now suppose $E \in \theta$ and $L \in \Omega_\beta$. Clearly $E \supset D \cap E$ and, for some i, $L \supset L_i$. For some $j>i$, let $A=\{x_{ek} \mid k<j\}$; then $A \in \mathscr{A}$. By (3) of our induction hypotheses $f^{-1}(D \cap E) \cap L_j \not\subset A$. But this implies that $\{e \in D \cap E \mid x_{ej} \text{ is defined}\} \neq \varnothing$ for any $j>i$. And this implies that $f^{-1}(D \cap E) \cap L_i \cap X$ is not a subset of any term of $\mathscr{A}$. Hence $f^{-1}(E) \cap L \cap X$ is not a subset of any term of $\mathscr{A}$ and (3) is satisfied with $M=L \cap X$.

IV. Let us describe a third partial order on βN which is between the other two. For Ω and θ in βN, let us say that Ω is *essentially greater than* θ *through* f if there is an $f \in F$ such that $f(\Omega)=\theta$ and, for $M \in \Omega$, $\{n \in N \mid f^{-1}(n) \cap M \text{ is infinite}\} \neq \varnothing$.

A. 1. *Suppose* $\Omega \sim \Lambda$ *and* $\theta \sim \phi$ *and* Ω *is essentially greater than* θ *through* f. Let π and p be permutations of N such that $\pi(\Lambda)=\Omega$ and $p(\theta)=\phi$. Then $p \circ f \circ \pi \in F$, $p \circ f \circ \pi(\Lambda)=\phi$ and, for $L \in \Lambda$, $\{n \in N \mid \pi^{-1} \circ f^{-1} \circ p^{-1}(n) \cap L \text{ is infinite}\} \neq \varnothing$. Hence Λ *is essentially greater than* ϕ.

2. For Ω and θ in βN, define $[\Omega] \sqsupseteq [\theta]$ if either $[\Omega]=[\theta]$ or Ω is essentially greater than θ. By 1, $\sqsupseteq$ is well defined. Clearly $\sqsupseteq$ is transitive and, by III A3, it is antisymmetric. Hence $\sqsupseteq$ *is a partial order on the types in* βN.

B. 1. *Suppose* Ω *and* θ *belong to* βN. *Then* $[\Omega] \geqq [\theta]$ *implies* $[\Omega] \sqsupseteq [\theta]$ *which implies* $[\Omega] \succeq [\theta]$.

2. By almost the same proofs, Theorems III B 2, 3, 5 and 6 are true with $\sqsupseteq$ replacing $\succeq$. However III B4 is false.

3. Observe that, *if θ and Ω belong to βN and f and g to F and $f(\Omega)=g(\Omega)=\theta$, Ω may be essentially greater than θ through f but not through g.* To see this choose any $\theta \in N^*$ and select a $g \in F$ such that $g^{-1}(n)$ has precisely n terms $x_{1n}, x_{2n}, \ldots, x_{nn}$. Define $f \in F$ by $f^{-1}(i)=\{x_{in} \mid n \in N\}$. Recall that $x \in N$ implies x' is the fixed ultrafilter to which x belongs. Define $X_1=\{x'_{1n}\}_{n\in N} \in D$ and for $n>1$ define $X_n \in D$ as $x'_{11}, x'_{22}, \ldots, x'_{nn}, x'_{n,n+1}, x'_{n,n+2}, \ldots$. For each $n \in N$, let $\rho_n=\theta_{X_n}$ and $X=\{\rho_n\}_{n\in N} \in D$ and $\Omega=\theta_X$. Then $g(\Omega)=\theta$ and $f(\Omega)=\theta$ and Ω is essentially greater than θ through f but not through g.

4. By definition $[\Omega]$ *is minimal in N^* under $\sqsupseteq$ if and only if Ω is a P-point.* Clearly N' is again minimal under $\sqsupseteq$ in βN.

5. The general character of $\sqsupseteq$ is more like that of $\succeq$ than that of $\geqq$. However it has one nice property of $\geqq$. *If Ω and θ are of the same type in N^*, then the set of all predecessors of Ω under $\sqsupseteq$ is precisely the set of all predecessors of θ.*

C. Together B 4 and 5 raised hope that the position in $\sqsupseteq$ of a type in βN might determine its type in N^*; 2 destroys this hope. It also gives a constructive method of finding non-P-point types minimal in N^* under $\geqq$. Using $\succeq$ and $\sqsupseteq$ together does not look useful as seen in 1.

1. Suppose Ω and θ are P-points in N^*. Then [CH] Ω and θ have the same type in N^*. And neither $[\Omega]$ nor $[\theta]$ has any predecessors under $\sqsupseteq$. But [CH] $[\Omega]$ and $[\theta]$ may be ordered by $\succeq$ or not ordered by $\succeq$. One can use sequences of P-points to show [CH] that there are two types in βN which (a) are of the same type in N^*, (b) have the same nonempty set of predecessors under $\sqsupseteq$, and (c) are comparable under $\succeq$; by the same method one can construct two types which satisfy (a), (b), and *not* (c).

2. *There exist* [CH] *terms Ω, θ, and Δ of N^* such that $[\Delta]$ is minimal in N^* in $\succeq$, Δ is the only term of N^* essentially less than Ω and the only term of N^* essentially less than θ, but Ω and θ are not of the same type in N^*. In fact θ is a limit point of a countable discrete sequence of P-points, but Ω is not a limit point of any countable subset of N^*.*

Proof. Choose $f \in F$ such that $f^{-1}(n)$ is infinite for each $n \in N$.

For $0<\alpha<\omega_1$ and $n \in N$ select $\alpha_n \in \omega_1$ in such a way that, if α is not a limit ordinal, $\alpha_n=\alpha-1$, and if α is a limit ordinal, $\{\alpha_n\}_{n\in N}=\{\beta \mid \beta<\alpha\}$.

By [CH], F and S can be indexed so $F=\{f_\alpha\}_{\alpha<\omega_1}$ and $S=\{S_\alpha\}_{\alpha<\omega_1}$. By a complicated induction on the countable ordinals, we define various subsets of N and points of N^* which in turn allow us to define Ω, θ, and $f(\theta)=f(\Omega)=\Delta$ with the desired properties.

For each countable ordinal α we wish to select

(a) an infinite subset M_α of N,

(b) a countable ordinal $\alpha^* \geqq \alpha$,

(c) for each $n \in M_\alpha$ and $\beta \in \omega_1$, a subset $E_{\alpha n\beta}$ of $f^{-1}(n)$.

The following conditions are satisfied for all $n \in N$:

1. There is a P-point $p_{\alpha n} = \{U \subset N \mid \text{for some } \beta \in \omega_1, U \supset E_{\alpha n \beta}\}$ and $\delta < \beta < \omega_1$ implies that $E_{\alpha n \beta} - E_{\alpha n \delta}$ is finite and $E_{\alpha n \delta} - E_{\alpha n \beta}$ is infinite.

2. If $\gamma < \alpha$ and $E_{\alpha n 0} \cap E_{\gamma n \delta}$ is infinite, then $E_{\alpha n 0} - E_{\gamma n \delta}$ is finite.

3. If $\gamma < \alpha$, then $\gamma^* < \alpha^*$ and $E_{\gamma n(\alpha^*+1)} \cap E_{\alpha n 0}$ is finite, but there exists a $\delta < \alpha$ such that $E_{\alpha n 0} - E_{\delta n \alpha^*}$ is finite.

4. If $\alpha > 0$ and n is the ith term of M_α, then

$$n \in M_{\alpha_1} \cap M_{\alpha_2} \cap \cdots \cap M_{\alpha_i} \quad \text{and} \quad E_{\alpha_1 n 0} \cap E_{\alpha_2 n 0} \cap \cdots \cap E_{\alpha_i n 0} \cap E_{\alpha n 0}$$

is infinite.

In all cases, once $E_{\alpha n 0}$ has been chosen, choose $E_{\alpha n \beta}$ and $p_{\alpha n}$ in accordance with 1.

Let $M_0 = N$, $0^* = 0$ and, for all $n \in N$, $E_{0n0} = f^{-1}(n)$.

Assume our choices have been made for all $\gamma < \alpha$.

First suppose α is a limit ordinal. Choose $n_1 \in M_{\alpha_1}$. And, for all $i > 1$, choose $n_i \in M_{\alpha_1} \cap \cdots \cap M_{\alpha_i}$ such that $E_{\alpha_1 n_i 0} \cap \cdots \cap E_{\alpha_i n_i 0}$ is infinite and $n_i > n_{i-1}$. By 4, such n_i exist. Then let $M_\alpha = \{n_i\}_{i \in N}$ and α^* be the limit of $\{\gamma^* \mid \gamma < \alpha\}$. Let $E_{\alpha n 0} = E_{\gamma n \alpha^*} - E_{\gamma n(\alpha^*+1)}$, where if $n = n_i$, γ is the largest of $\alpha_1, \ldots, \alpha_i$, and otherwise $\gamma = 0$. One can check that 2, 3, and 4 are again satisfied.

Suppose $\alpha = \beta + 1$ and let g denote f_β.

Case 1. $X = \{n \in M_\beta \mid p_{\beta n} \notin g^{-1}(j) \text{ for any } j \in N\}$ is infinite. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in X$ and $j \in N$, $E_{\beta n \delta} \cap g^{-1}(j)$ is finite. Let $M = X$.

Case 2. X is finite and there exists an $i \in N$ and an infinite subset Z of M_β such that $n \in Z$ implies $p_{\beta n} \in g^{-1}(i)$. In this case there is a $\delta \in \omega_1$ such that, for all $n \in Z$, $E_{\beta n \delta} - g^{-1}(i)$ is finite. Let $M = Z$.

Case 3. Neither Case 1 nor 2 holds. Then there exist infinite subsets W of M_β and $\{a_j\}_{j \in N}$ of N such that $j < k$ in W implies $p_{\beta j} \in g^{-1}(a_j)$ and $p_{\beta k} \in g^{-1}(a_k)$ and $a_j < a_k$. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in W$, $E_{\beta n \delta} - g^{-1}(a_n)$ is finite. Let $M = W$.

In all cases consider $g(M)$. If there is an infinite subset V of M and a $v \in N$ such that $g^{-1}(v) \supset V$, then let $M' = V$. Otherwise there is an infinite subset M' of M such that j and k belong to M' implies that $g(j) \neq g(k)$.

Choose $\alpha^* = \beta^* + \delta + 1$.

For some infinite subset M'' of M', for all $n \in M''$, $Q = S_\beta \cap (E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)})$ is infinite or $Q = (N - S_\beta) \cap (E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)})$ is infinite. Let $M_\alpha = M''$.

If $n \notin M_\alpha$, let $E_{\alpha n 0} = E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)}$.

In Case 1, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q - g^{-1}(1, \ldots, n)$.

In Case 2, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q \cap g^{-1}(i)$.

In Case 3, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q \cap g^{-1}(a_n)$.

It is easy to check that 2, 3 and 4 are again satisfied.

Let $\Omega = \{E \in S \mid \text{for some } \alpha \in \omega_1, E \supset \bigcup_{n \in M_\alpha} E_{\alpha n 0}\}$. By 4 and our selection of M'' and $E_{\alpha n 0}$, Ω is a free ultrafilter on N.

Let $f(\Omega) = \Delta$; observe that $\Delta = \{M \subset N \mid \text{for some } \alpha \in \omega_1, M \supset M_\alpha\}$.

Suppose $g \in F$. Then $g = f_\beta$ for some $\beta \in \omega_1$. If $\alpha = \beta + 1$ and $V = \bigcup_{n \in M_\alpha} E_{\alpha n 0}$, then $V \in \Omega$. In Case 1, for $j \in N$, $V \cap g^{-1}(j)$ is finite. In Case 2, $g(V) = i$ so $g(\Omega) = i'$. And in Case 3, $g(\Omega) \sim f(\Omega)$. So if $\phi \in \beta N$ and Ω is essentially greater than ϕ through g, either $[\phi] = N'$ or $[\phi] = [\Delta]$. This means that $[\Delta]$ is minimal in $\succeq$ in N^* (hence Δ is a P-point) and $[\Delta]$ is the only type in N^* less than $[\Omega]$ in $\sqsupset$.

Select $X = \{x_n\}_{n \in N} \in D$ such that x_n is a P-point to which $f^{-1}(n)$ belongs. Let $\theta = \Delta_X$.

Suppose $g \in F$. Let $Y = \{n \in N \mid \text{for all } k \in N,\ g^{-1}(k) \notin x_n\}$. For $n \in Y$ we can select $L_n \in x_n$ such that $L_n \cap g^{-1}(1, \ldots, n) = \varnothing$ and, for all $k \in N$, $L_n \cap g^{-1}(k)$ is finite. If $Y \in \Delta$, then $\bigcup_{n \in Y} L_n \in \theta$; so θ is not essentially greater than $g(\theta)$ through g if $Y \in \Delta$. If $Y \notin \Delta$, define $h \in F$ by $g^{-1}(h(n)) \in x_n$ for $n \in N - Y$ and $h(n) = 1$ for $n \in Y$. Then $h = f_\beta$ for some $\alpha - 1 = \beta \in \omega_1$ and $M_\alpha \cap (N - Y) \in \Delta$. By our definition of M', either $h(M') = v$ for some $v \in N$ or h restricted to M' is one-to-one; but $M' \supset M_\alpha$. So $Y \notin \Delta$ implies $g(\theta) = v' \in N'$ or $g(\theta) \sim f(\theta)$. Hence, if θ is essentially greater than $g(\theta)$ through g and $g(\theta) \in N^*$, $[g(\theta)] = [f(\theta)]$. Thus $[\Delta]$ is the one type contained in N^* which is essentially less than θ or Ω.

Now we show that Ω is not a limit point of any countable subset of N^*; one implication of this is that Ω and θ are not of the same type in N^*.

Suppose that $\{\rho_i\}_{i \in N}$ is a subset of $N^* - \{\Omega\}$. Let $A = \{i \in N \mid \text{for some } \alpha \in \omega_1 \text{ and } n \in N,\ \rho_i = p_{\alpha n}\}$. Let $B = \{i \in N \mid \text{for some } n \in N, f^{-1}(n) \in \rho_i \text{ but } \rho_i \neq p_{\alpha n} \text{ for any } \alpha \in \omega_1\}$. Let $C = \{i \in N \mid \text{for all } n \in N, f^{-1}(n) \notin \rho_i\}$. We find terms U, V, and W of Ω such that, for $i \in A$, $\rho_i \notin U$, for $i \in B$, $\rho_i \notin V$, and, for $i \in C$, $\rho_i \notin W$. Since $U \cap V \cap W \in \Omega$ and $A \cup B \cup C = N$, Ω is not a limit point of $\{\rho_i\}_{i \in N}$.

Choose α such that $\alpha > \gamma$ for all $\gamma \in \omega_1$ for which there are i and n in N such that $\rho_i = p_{\gamma n}$. Let $U = \bigcup_{n \in N} E_{\alpha n 0}$. If $\rho_i = p_{\gamma n}$, then $\gamma < \alpha$ and, by 1, $E_{\gamma n (\alpha^* + 1)} \in \rho_i$ and, by 3, $E_{\gamma n (\alpha^* + 1)} - E_{\alpha n 0} \in \rho_i$ and, by (c), $U \notin \rho_i$. Thus for $i \in A$, $U \in \Omega$ but $U \notin \rho_i$.

We want to choose a sequence $\{\beta^j\}_{j \in N} \subset \omega_1$, by induction. Let $\beta_1 = 0$. Suppose β^{j-1} has been selected. If $\gamma \in \omega_1$ and $n \in N$ and $i \in B$, by 1, there is a $\beta \in \omega_1$ such that $\beta \leqq \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Thus we can select $\beta^j \in \omega_1$ such that $\beta^{j-1} < \beta^j$ and, for all $\gamma \leqq \beta^{j-1}$ and $n \in N$ and $i \in B$, $\beta^j < \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Let α be the limit of $\{\beta^j\}_{j \in N}$ and let $V = \bigcup_{n \in N} E_{\alpha n 0} \in \Omega$. By 3, there is a $\gamma < \alpha$ such that $E_{\alpha n 0} - E_{\gamma n \alpha^*}$ is finite. So for $i \in B$, $E_{\alpha n 0} \notin \rho_i$. But for each $i \in B$ there is an $n \in N$ such that $f^{-1}(n) \in \rho_i$. Since $V \cap f^{-1}(n) = E_{\alpha n 0}$, $V \notin \rho_i$ for any $i \in B$.

For each $j \in N$, since $\Omega \neq \rho_j$, there is a $\delta^j \in \omega_1$ such that $\bigcup_{n \in M_{\delta^j}} E_{\delta^j n 0} \notin \rho_j$. Choose a limit ordinal α greater than δ^j for all $j \in N$. Suppose $j \in C$. There is an $i \in N$ such that $\delta^j = \alpha_i$. Let m be the ith term of M_α. Consider

$$W_j = \bigcup_{n \in M_\alpha} (E_{\alpha n 0} - E_{\alpha_i n 0}) - f^{-1}(1, \ldots, m).$$

If $\bigcup_{n \in M_\alpha} E_{\alpha n 0} \in \rho_j$ then $W_j \in \rho_j$ for $j \in C$ implies $f^{-1}(1, \ldots, m) \notin \rho_j$ and, by 4, $n \in M_\alpha$ and $n > m$ implies $n \in M_{\alpha_i}$ and $\delta^j = \alpha_i$ implies $\bigcup_{n \in M_{\alpha_i}} E_{\alpha_i n 0} \notin \rho_j$. Together 2 and 4 imply that, for $n \in M_\alpha$, $W_j \cap f^{-1}(n)$ is finite. For some $\beta \in \omega_1$, $\bigcup_{j \in C} W_j = S_\beta$;

since only finitely many W_j intersect $f^{-1}(n)$, $S_\beta \cap f^{-1}(n)$ is finite for all $n \in N$. But Ω is essentially greater than Δ through f, so $S_\beta \notin \Omega$. Thus $W = N - S_\beta \in \Omega$ but $W \notin \rho_j$ for any $j \in C$.

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