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PARTIAL ORDERS ON THE TYPES IN βN

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Abstract. Three partial orders on the types of points in βN are defined and studied in this paper. Their relation to the types of points in $\beta N - N$ is also described.

Several natural partial orders can be given to the types of points in βN. The purpose of this paper is to give some of these orders wider publicity. I feel these orders are fundamental in the study of ultrafilters on the integers. I had hoped these orders would lead to a classification of the types of points in N^*. I no longer feel this is true, but connections with this important unsolved problem are discussed.

I. Let N denote the set of all positive integers and S the set of all subsets of N. Let βN denote the set of all ultrafilters on N and N^* the set of all free ultrafilters on N. For $M \subset N$ let $W(M)$ be the set of all terms of βN to which M belongs. Then the set of all $W(M)$ for $M \subset N$ forms a basis for a topology on βN and the resulting space is topologically the Čech compactification of the integers and N^* is topologically $\beta N - N$. To avoid ambiguity let n' be the ultrafilter to which the integer n belongs and N' the set of all fixed ultrafilters; thus $N^* = \beta N - N'$.

If p and q are points of a topological space X, p and q are *of the same type in* X provided there is a homeomorphism of X onto itself taking p into q. It is easy to see [1] that two ultrafilters on N are of the same type in βN if and only if there is a permutation of N which takes the members of one onto the members of the other. That Ω and θ are of the same type in βN will be denoted by $\Omega \sim \theta$ and $[\Omega]$ will denote the set of all ultafilters on N which are of the same type as Ω. Clearly $\sim$ is an equivalence relation and $[\Omega]$ has c members.

The problem of characterizing the types of points in N^* is the problem of finding reasonable necessary and sufficient conditions on terms Ω and θ of N^* so that one can construct a permutation of S which preserves *infinite* intersections and takes the members of Ω onto the members of θ.

A term Ω of N^* is called a P-point provided, for every countable subcollection $\{E_n\}_{n \in N}$ of Ω, there is a term E of Ω such that $E - E_n$ is finite for all n. In [1] Walter Rudin proves that the continuum hypothesis [CH] implies the existence of P-points in N^* and that all P-points are of the same type in N^*. Booth [3] has shown, using Martin's axiom rather than [CH], that there are P-points in N^* without an $\aleph_1$ base.

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In the light of these results, classification of the types in N^* seems hopeless without some set theoretic assumptions. The results of this paper frequently use [CH] and the strong structure of P-points this implies. The required background reading is [1].

II. A sequence $\{\rho_n\}_{n\in N}$ of terms of βN is called discrete if there exists a sequence $\{E_n\}_{n\in N}$ of disjoint subsets of N such that $E_n \in \rho_j$ if and only if $n=j$. Let D be the set of all such discrete countable sequences of terms of βN. If $X=\{\rho_n\}_{n\in N} \in D$ and $\theta \in \beta N$, define $\theta_X=\{M\subset N \mid \{n \mid M \in \rho_n\} \in \theta\}$. That is θ_X is the image of θ under the natural homeomorphism of βN onto the closure of X which takes n to ρ_n. Observe that, if X belongs to D and Ω to the closure of X, there is a unique θ such that $\theta_X=\Omega$.

In [2] Z. Frolik says θ *produces* θ_X. Then he proves that $\theta \in \beta N$ implies that θ produces 2^c terms of βN but is produced by at most c terms. Frolik also observes that if $X\in D$ and $X\subset N^*$, then there are at most c terms of $\bar{X}$ which have the same type in N^* and hence there are 2^c types of points in N^*.

A. Let us prove that Frolik's producing relation is a partial ordering of the types in βN.

1. *$\theta\sim\Omega$ implies that θ produces Ω.* For, if π is a permutation of N such that $M\in\Omega$ if and only if $\pi(M)\in\theta$, then $X=\{\pi(n)\}_{n\in N}$ is such that $\theta_X=\Omega$.

2. *If ϕ produces θ and θ produces Ω, then ϕ produces Ω.* Suppose $X\in D$ and $\{\rho_n\}_{n\in N}=Y\in D$ and $\phi_Y=\theta$ and $\theta_X=\Omega$. For each $n\in N$ define $\mu_n=(\rho_n)_X$ and let $Z=\{\mu_n\}_{n\in N}$. Then $\Omega=\phi_Z$.

3. *Suppose Ω produces θ and θ produces Ω. Then $\theta\sim\Omega$.* Suppose $X=\{\eta_n\}_{n\in N}\in D$ and $Y=\{\rho_n\}_{n\in N}\in D$ and $\theta_X=\Omega$ and $\Omega_Y=\theta$. For $n\in N$, define $\mu_n=(\rho_n)_X$ and let $Z=\{\mu_n\}_{n\in N}$. As in 2, $M\in\Omega$ if and only if $\{n \mid M\in\mu_n\}\in\Omega$. Let $\{E_n\}_{n\in N}$ be a set of disjoint subsets of N with $E_n\in\mu_n$. Define a two-valued function $f\colon N\to\{0,1\}$ as follows. Define $f(1)=0$ and, if $n\in E_1$ and $n>1$, define $f(n)=1$. Assume $i>1$ and $f(n)$ has been defined for all $n<i$ and all $n\in E_j$ where $j<i$. If $f(i)$ has been defined as 1 and $n>i$ and $n\in E_i$, define $f(n)=0$. If $f(i)$ has *not* been defined as 1, define $f(i)=0$ and, if $n>i$ and $n\in E_i$, define $f(n)=1$. Exactly one of $f^{-1}(0)$ and $f^{-1}(1)$ belongs to Ω. Suppose $f^{-1}(0)\in\Omega$. Let $M=f^{-1}(0)\cap\{n \mid f^{-1}(0)\in\mu_n\}$; $M\in\Omega$. If $n\in M$, then the finite set $f^{-1}(0)\cap E_n\in\mu_n$; hence μ_n is not free. But μ_n is not free implies $\mu_n\in X$, and we can define $\pi\colon M\to N$ by $\mu_n=\eta_{\pi(n)}$. We can find $E\subset M$ such that $E\in\Omega$, $N-E$ is infinite, and $N-\pi(E)$ is infinite. Then π/E can be extended to a permutation p of N onto N. It is easy to check that $B\in\Omega$ if and only if $p(B)\in\theta$. Thus $\theta\sim\Omega$.

Now for Ω and θ in βN, define $[\theta]\leqq[\Omega]$ if θ produces Ω. By 1, 2, and 3, $\leqq$ is a partial order on the types in βN.

B. *If $\Omega\in\beta N$, the set of all types in βN which precede* $[\Omega]$ *in* $\leqq$ *is totally ordered by* $\leqq$. For suppose $[\phi]\leqq[\Omega]$ and $[\theta]\leqq[\Omega]$. There is $X=\{\rho_n\}_{n\in N}\in D$ and $Y\in D$ such that $\phi_Y=\theta_X=\Omega$. Temporarily ignore the order of terms of D and just use them as sets with closures in βN. And use $\bar{X}$ for the closure of X in βN. We need the fact [4] that if $V \in D$ and $\Omega \in \beta N$ and $\Omega \in \overline{V}$ and $V_1 \cap V_2 = \varnothing$ and $V_1 \cup V_2 = V$, then Ω belongs to one and only one of $\overline{V}_1$ and $\overline{V}_2$. In our case $\Omega \in \overline{X}$ and $X = (X \cap \overline{Y}) \cup (X - \overline{Y})$ and $\Omega \in \overline{Y}$ and $Y = (Y \cap \overline{X}) \cup (Y - \overline{X})$. Since $V = (X - \overline{Y}) \cup (Y - \overline{X})$ is countable and discrete, using the fact again we have that Ω belongs to the closure of $X \cap \overline{Y}$ or $Y \cap \overline{X}$. Say Ω belongs to the closure of $X \cap \overline{Y}$. Let $L = \{n \mid \rho_n \in \overline{Y}\}$; then $L \in \theta$. For $n \in L$, define μ_n to be the unique term of βN such that $(\mu_n)_Y = \rho_n$; if $n \in N - L$ define $\mu_n = n'$. Then $Z = \{\mu_n\}_{n \in N} \in D$ and $\phi_Z = \theta$.

C. The Frolik order $\leqq$ has been studied extensively and we have a great deal of information. The fixed ultrafilters form a type in βN which precedes all other types. In N^*, types of P-points are obviously minimal in this order. For $X \subset \beta N$, let $X^* = \overline{X} - X$. K. Kunen [5] has shown that [CH] there is a non-P-point in N^* not in X^* for any countable subset X of N^* and [CH] there is a countable subset X of N^* and a point of X^* which is not in Y^* for any Y in D. Types of both of these points are clearly minimal in N^* under $\leqq$. By B the order is tree-like and knowing that any term is preceded by at most c terms but followed by 2^c terms gives us the picture of a fast branching tree.

In his thesis [3] Booth defines the product $[\theta] \cdot [\Omega]$ for Ω and θ in βN to be $[\theta_X]$ where $X = \{\rho_n\}_{n \in N} \in D$ and $\rho_n \sim \Omega$. Obviously $[\theta] \leqq [\theta] \cdot [\Omega]$ and it is easy to show that $[\theta] \cdot [\Omega]$ is well defined. For $n > 1$, $[\theta]^n$ is defined inductively as $[\theta] \cdot [\theta]^{n-1}$; then $[\theta]^a \cdot [\theta]^b = [\theta]^{a+b}$. Among other things Booth uses these definitions to construct infinite well-ordered increasing and well-ordered decreasing sequences in $\leqq$.

If Ω and θ are of the same type in N^*, then the types which precede $[\Omega]$ by $\leqq$ must be precisely those which precede $[\theta]$ by $\leqq$. In [4] I showed that the condition is also sufficient if both Ω and θ are limit points of countable sets of P-points [CH]. But by Kunen's examples there are minimal elements $[\theta]$ and $[\phi]$ in N^* such that θ and ϕ are of different types in N^*. So the condition is clearly not sufficient. Also if Λ is any member of N^*, it is easy to show, using the methods of B, that $[\Lambda]$ is maximal in the set of predecessors of both $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$. Hence the predecessors of $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$ are exactly the same but none of their terms have the same type in N^*.

III. A more general partial order on the types of βN will now be discussed. Let F be the set of all functions from N onto N. If $\Omega \in \beta N$ and $f \in F$, define $f(\Omega) = \{f(M) \mid M \in \Omega\}$.

A. Let us prove that F induces a partial order on the types of points in βN.

1. *If Ω and θ belong to βN and $\Omega \sim \theta$, then there is an $f \in F$ such that $f(\Omega) = \theta$.* For, by definition, there is a permutation f of N such that $M \in \Omega$ if and only if $f(M) \in \theta$.

2. *If Ω, ϕ and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \phi$, then $g \circ f(\Omega) = \phi$.*

3. *Suppose that Ω and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \Omega$. We prove $\Omega \sim \theta$.* Let $L = \{n \mid (g \circ f)(n) > n\}$, $M = \{n \mid (g \circ f)(n) = n\}$ and

$Q=\{n \mid (g\circ f)(n)<n\}$. Suppose $M\in\Omega$; then f/M is one-to-one. There is a subset E of M belonging to Ω such that $N-E$ and $N-f(E)$ are infinite, and f/E can be extended to a permutation p of N onto N. Thus $B\in\Omega$ if and only if $p(B)\in\theta$ and $\Omega\sim\theta$.

If $M\notin\Omega$ one of L and Q must belong to Ω. Suppose $L\in\Omega$. If n and k belong to L, let us say nek provided that, for some nonnegative integers i and j, $(g\circ f)^i(n)=(g\circ f)^j(k)$ where $(g\circ f)^0$ is the identity map. Clearly e is an equivalence relation. Let E be the set of all equivalence classes of subsets of L related by e. From each $A\in E$ select $A_0\in A$. Then define a two-valued function $t\colon L\to\{0,1\}$ as follows. If $n\in A\in E$ and $(g\circ f)^i(n)=(g\circ f)^j(A_0)$ then define $t(n)$ as 0 if $|i-j|$ is even and 1 if $|i-j|$ is odd. The function t is well defined and only one of $t^{-1}(0)$ and $t^{-1}(1)$ belongs to Ω. Suppose $t^{-1}(0)\in\Omega$. By 2, $(g\circ f)(t^{-1}(0))\in\Omega$. But

$$(g\circ f)(t^{-1}(0))\subset t^{-1}(1)$$

and this is a contradiction.

As before, if Ω and θ belong to βN, define $[\Omega]\succeq[\theta]$ provided there is an $f\in F$ such that $f(\Omega)=\theta$. By 1, 2, and 3, $\succeq$ is a partial order on the types of points in βN.

B. We make several very simple observations.

1. *If Ω and θ belong to βN and $[\theta]\leqq[\Omega]$, then* $[\Omega]\succeq[\theta]$. For $[\theta]\leqq[\Omega]$ implies there is $X=\{\rho_n\}_{n\in N}\in D$ such that $\theta_X=\Omega$. And $X\in D$ implies there is a set $\{E_n\}_{n\in N}$ of disjoint subsets of N such that $E_n\in\rho_n$. If $n>1$ and $i\in E_n$, define $f(i)=n$; and if $i\notin E_n$ for any $n>1$, define $f(i)=1$. Then $f\in F$ and $f(\Omega)=\theta$.

2. *If $\theta\in\beta N$, in $\succeq$ $[\theta]$ is greater than at most c types but less than 2^c types.* The first follows from the cardinality of F being c. The second follows from 1 and Frolik's result in §II. In fact using Frolik's proof one shows that if *both* $\theta\in\beta N$ and $f\in F$ are given and, for all n, $f^{-1}(n)$ is infinite, there are 2^c terms Ω of βN such that $f(\Omega)=\theta$. By contrast recall that if $\theta\in\beta N$ and $X\in D$ are given there is a unique Ω such that $\theta_X=\Omega$.

3. *If θ and ϕ belong to βN, there is an Ω in βN such that $[\Omega]\succeq[\theta]$ and $[\Omega]\succeq[\phi]$.* Select $f\in F$ such that $f^{-1}(i)$ is infinite for each $i\in N$. Now select $g\in F$ such that, for each i and j in N, $g^{-1}(j)\cap f^{-1}(i)$ is infinite. Then select $\rho_{ij}\in N^*$ such that $g^{-1}(j)\cap f^{-1}(i)\in\rho_{ij}$. For $i\in N$ define $\rho_i=\phi_{X_i}$ where $X_i=\{\rho_{ij}\}_{j\in N}$; and for $X=\{\rho_i\}_{i\in N}$ let $\Omega=\theta_X$. Then $f(\Omega)=\theta$ and $g(\Omega)=\phi$.

4. *Suppose θ and ϕ belong to βN. Let $B=\{[\Omega]\mid[\Omega]\succeq[\theta]$ and $[\Omega]\succeq[\phi]\}$. Then $[\Omega]\in B$ is minimal in B if and only if, for all f and g in F such that $f(\Omega)=\theta$ and $g(\Omega)=\phi$, there is an $M\in\Omega$ such that for i and j in N, $f^{-1}(i)\cap g^{-1}(j)\cap M$ is at most a singleton.* To prove the *only if,* suppose f and g are given and let $Q=\{(i,j)\in N\times N \mid f^{-1}(i)\cap g^{-1}(j)\neq\varnothing\}$. If Q is finite, $[\theta]$ and $[\phi]$ and $[\Omega]$ are N'; if $\Omega=m'$, $M=\{m\}$ has the desired properties. If Q is infinite there is a one-to-one function q from N onto Q; define $k\in F$ by $k^{-1}(n)=f^{-1}(i)\cap g^{-1}(j)$ where $(i,j)=q(n)$. Hence if, for $n\in N$, we define $f^*(n)=f(k^{-1}(n))$ and $g^*(n)=g(k^{-1}(n))$, then f^* and g^* belong to F and $f^*(k(\Omega))=\theta$ and $g^*(k(\Omega))=\phi$. But $k(\Omega)\sim\Omega$ only if there

is an $M \in \Omega$ such that k/M is one-to-one. Now to prove *if*, suppose f, g and h belong to F and $f(h(\Omega)) = \theta$ and $g(h(\Omega)) = \phi$. Then $f \circ h(\Omega) = \theta$ and $g \circ h(\Omega) = \phi$. So assume also that there is an M such that, for all i and j in N,

$$(f \circ h)^{-1}(i) \cap (g \circ h)^{-1}(j) \cap M$$

is at most a singleton. Then h/M is one-to-one and thus $h(\Omega) \sim \Omega$.

5. K. Kunen has a beautiful proof [5] that $\succeq$ is not a total order. The same proof shows that there are c types no pair of which are ordered. And using the continuum hypothesis it is easy to show that there are 2^c pairwise unordered types in $\succeq$, even 2^c minimal in N^* types.

6. Together 3 and 5 imply that, unlike II B, this order is not treelike. That is, there are types in βN whose predecessors are not totally ordered by $\succeq$. In fact $\succeq$ is more rootlike; that is, things get together near the top.

C. In addition to the facts in B, what can we say about $\succeq$? Again N', the type of all fixed ultrafilters, is less than all other types. If $[\Omega]$ is minimal in N^* under $\succeq$, then Ω is a P-point by definition. If θ is a P-point and $[\theta] \succeq [\phi]$, then ϕ is a P-point or a fixed ultrafilter. It is not hard to prove [CH] that there are types which are minimal in N^*. This was first proved by J. Keisler [6] and will be a corollary of the example given in IV C. We prove [CH] that above every P-point type is another P-point type. Thus two types of the same type in N^* may or may not be ordered under $\succeq$; for two P-point types which are minimal under $\succeq$ are not ordered.

Suppose $\theta \in \beta N$ is a P-point. Then [CH] *there is a P-point Ω such that* $[\Omega] \succeq [\theta]$ *but* $[\Omega] \neq [\theta]$.

Proof. Clearly [CH] implies that both F and S have cardinality $\aleph_1$; hence let $F = \{f_\alpha\}_{\alpha < \omega_1}$ and $S = \{S_\alpha\}_{\alpha < \omega_1}$. Let f be a term of F such that, for each n, $f^{-1}(n)$ has n terms. We build Ω so that $f(\Omega) = \theta$ by induction on the countable ordinals. Let $\mathscr{A}$ be the set of all $A \subset N$ such that for some $a \in N$ and all $n \in N$, the number of terms of $f^{-1}(n) \cap A$ is less than a; observe that $\mathscr{A}$ is closed under finite union.

For each $\alpha \in \omega_1$, we define a countable subset Ω_α of subsets of N such that

(1) For $\beta < \alpha$, $\Omega_\beta \subset \Omega_\alpha$. Also Ω_α is closed under finite intersection.

(2) If $\alpha = \beta + 1$, there is a term X of Ω_α such that (a) $X \subset S_\beta$ or $X \subset N - S_\beta$ and (b) for some $n \in N$, $X \subset f_\beta^{-1}(n)$, or, for all $n \in N$, $X \cap f_\beta^{-1}(n)$ is finite.

(3) For $E \in \theta$, $M \in \Omega_\alpha$, and $A \in \mathscr{A}$, $M \cap f^{-1}(E) \not\subset A$. By (1), there exists $\Omega \in \beta N$ such that $\Omega \supset \bigcup_{\alpha < \omega_1} \Omega_\alpha$. By (2(a)), $L \in \Omega$ implies $L \supset M \in \Omega_\alpha$ for some $\alpha < \omega_1$. By (3), $f(\Omega) = \theta$ but $\Omega \nsim \theta$. By (2(b)), Ω is a P-point (or a fixed ultrafilter but this is impossible since $f(\Omega) = \theta$).

So it will suffice to define the Ω_α.

Define $\Omega_0 = \{N\}$ and, for limit ordinals α, define $\Omega_\alpha = \bigcup_{\beta < \alpha} \Omega_\beta$. Then (1), (2), and (3) are trivially satisfied.

Suppose $\alpha = \beta + 1$ for some $\beta < \omega_1$. We find an $X \subset N$ satisfying (2(b)) such that, for all $L \in \Omega_\beta$ and $M = L \cap X$, (3) is satisfied. If for all $L \in \Omega_\beta$, (3) is satisfied with $M = L \cap X \cap S_\beta$ define $Y = X \cap S_\beta$. Otherwise since Ω_β and θ are closed under finite intersection and $\mathscr{A}$ under finite union, (3) is satisfied with $M=L\cap X\cap(N-S_\beta)$; and in this case define $Y=X\cap(N-S_\beta)$. We then define $\Omega_\alpha=\Omega_{\alpha-1}\cup\{Y\cap L\mid L\in\Omega_{\alpha-1}\}$ and (1), (2), and (3) are all satisfied. We define X by cases.

Case 1. There is an $n\in N$ such that, for all $L\in\Omega_\beta$, $E\in\theta$, and $A\in\mathscr{A}$, $f_\beta^{-1}(n)\cap L\cap f^{-1}(E)\not\subset A$; then $X=f_\beta^{-1}(n)$ has the desired properties.

Case 2. For each $n\in N$ there exists $L_n\in\Omega_\beta$, $E_n\in\theta$ and $A_n\in\mathscr{A}$ such that $f_\beta^{-1}(n)\cap L_n\cap f^{-1}(E_n)\subset A_n$. Without loss of generality we assume that $A_n\subset A_{n+1}$, $E_n\supset E_{n+1}$, $L_n\supset L_{n+1}$ and, for $L\in\Omega_\beta$, there is an n such that $L\supset L_n$. For $j\in N$, define

$$D_j=\{e\in N\mid f^{-1}(e)\cap L_j-f_\beta^{-1}(1,2,\ldots,j)\text{ has more than } j \text{ terms}\}.$$

Observe that $D_j\in\theta$. Otherwise $D'=(N-D_j)\cap E_j\in\theta$. And by the definition of D_j there is a term A of $\mathscr{A}$ such that $f^{-1}(N-D_j)\cap L_j-f_\beta(1,\ldots,j)\subset A$. But by our assumption $f^{-1}(E_j)\cap L_j\cap f_\beta^{-1}(1,\ldots,j)\subset A_j$. Hence $f^{-1}(D')\cap L_j\subset(A_j\cup A)\in\mathscr{A}$. But by (3) of our induction hypotheses, if $D'\in\theta, f^{-1}(D')\cap L_j$ is not a subset of any term of $\mathscr{A}$. Hence, since θ is a P-point, there is a $D\in\theta$ such that, for all $j\in N$, $D-D_j$ is finite. If $e\in D\cap D_1$, select $x_{e1}\in f^{-1}(e)\cap L_1\cap f_\beta^{-1}(n)$ with n maximal. And for $j>1$ and $j\in N$, if $e\in D\cap D_j$ select

$$x_{ej}\in f^{-1}(e)\cap L_j\cap f_\beta^{-1}(n)-(x_{e1},x_{e2},\ldots,x_{e,j-1})$$

with n maximal. Let $X=\{x_{ej}\}$.

Fix $n\in N$ and let us show that $X\cap f_\beta^{-1}(n)$ is finite. By the definition of D_j and x_{ej}, if $e\in D_j$ and $x_{ej}\in f_\beta^{-1}(n)$, then $n>j$. Similarly, if $e\in D_n$ and x_{ej} is defined, $x_{ej}\in f_\beta^{-1}(m)$ for some $m>n$. So since x_{ej} is only defined for $e\in D$ and $D-D_n$ is finite, there are at most finitely many j and e such that $x_{ej}\in f_\beta^{-1}(n)$.

Now suppose $E\in\theta$ and $L\in\Omega_\beta$. Clearly $E\supset D\cap E$ and, for some i, $L\supset L_i$. For some $j>i$, let $A=\{x_{ek}\mid k<j\}$; then $A\in\mathscr{A}$. By (3) of our induction hypotheses $f^{-1}(D\cap E)\cap L_j\not\subset A$. But this implies that $\{e\in D\cap E\mid x_{ej}\text{ is defined}\}\neq\varnothing$ for any $j>i$. And this implies that $f^{-1}(D\cap E)\cap L_i\cap X$ is not a subset of any term of $\mathscr{A}$. Hence $f^{-1}(E)\cap L\cap X$ is not a subset of any term of $\mathscr{A}$ and (3) is satisfied with $M=L\cap X$.

IV. Let us describe a third partial order on βN which is between the other two. For Ω and θ in βN, let us say that Ω is *essentially greater than θ through f* if there is an $f\in F$ such that $f(\Omega)=\theta$ and, for $M\in\Omega$, $\{n\in N\mid f^{-1}(n)\cap M\text{ is infinite}\}\neq\varnothing$.

A. 1. *Suppose $\Omega\sim\Lambda$ and $\theta\sim\phi$ and Ω is essentially greater than θ through f.* Let π and p be permutations of N such that $\pi(\Lambda)=\Omega$ and $p(\theta)=\phi$. Then $p\circ f\circ\pi\in F$, $p\circ f\circ\pi(\Lambda)=\phi$ and, for $L\in\Lambda$, $\{n\in N\mid\pi^{-1}\circ f^{-1}\circ p^{-1}(n)\cap L$ is infinite$\}\neq\varnothing$. Hence Λ *is essentially greater than* ϕ.

2. For Ω and θ in βN, define $[\Omega]\sqsupseteq[\theta]$ if either $[\Omega]=[\theta]$ or Ω is essentially greater than θ. By 1, $\sqsupseteq$ is well defined. Clearly $\sqsupseteq$ is transitive and, by III A3, it is antisymmetric. Hence $\sqsupseteq$ *is a partial order on the types in βN.*

B. 1. *Suppose Ω and θ belong to βN. Then $[\Omega]\geqq[\theta]$ implies $[\Omega]\sqsupseteq[\theta]$ which implies $[\Omega]\succeq[\theta]$.*

2. By almost the same proofs, Theorems III B 2, 3, 5 and 6 are true with $\sqsupseteq$ replacing $\succeq$. However III B4 is false.

3. Observe that, *if θ and Ω belong to βN and f and g to F and $f(\Omega)=g(\Omega)=\theta$, Ω may be essentially greater than θ through f but not through g.* To see this choose any $\theta \in N^*$ and select a $g \in F$ such that $g^{-1}(n)$ has precisely n terms $x_{1n}, x_{2n}, \ldots, x_{nn}$. Define $f \in F$ by $f^{-1}(i)=\{x_{in} \mid n \in N\}$. Recall that $x \in N$ implies x' is the fixed ultrafilter to which x belongs. Define $X_1=\{x'_{1n}\}_{n\in N} \in D$ and for $n>1$ define $X_n \in D$ as $x'_{11}, x'_{22}, \ldots, x'_{nn}, x'_{n,n+1}, x'_{n,n+2}, \ldots$. For each $n \in N$, let $\rho_n=\theta_{X_n}$ and $X=\{\rho_n\}_{n\in N} \in D$ and $\Omega=\theta_X$. Then $g(\Omega)=\theta$ and $f(\Omega)=\theta$ and Ω is essentially greater than θ through f but not through g.

4. By definition $[\Omega]$ *is minimal in N^* under $\sqsupseteq$ if and only if Ω is a P-point.* Clearly N' is again minimal under $\sqsupseteq$ in βN.

5. The general character of $\sqsupseteq$ is more like that of $\succeq$ than that of $\geqq$. However it has one nice property of $\geqq$. *If Ω and θ are of the same type in N^*, then the set of all predecessors of Ω under $\sqsupseteq$ is precisely the set of all predecessors of θ.*

C. Together B 4 and 5 raised hope that the position in $\sqsupseteq$ of a type in βN might determine its type in N^*; 2 destroys this hope. It also gives a constructive method of finding non-P-point types minimal in N^* under $\geqq$. Using $\succeq$ and $\sqsupseteq$ together does not look useful as seen in 1.

1. Suppose Ω and θ are P-points in N^*. Then [CH] Ω and θ have the same type in N^*. And neither $[\Omega]$ nor $[\theta]$ has any predecessors under $\sqsupseteq$. But [CH] $[\Omega]$ and $[\theta]$ may be ordered by $\succeq$ or not ordered by $\succeq$. One can use sequences of P-points to show [CH] that there are two types in βN which (a) are of the same type in N^*, (b) have the same nonempty set of predecessors under $\sqsupseteq$, and (c) are comparable under $\succeq$; by the same method one can construct two types which satisfy (a), (b), and *not* (c).

2. *There exist* [CH] *terms Ω, θ, and Δ of N^* such that $[\Delta]$ is minimal in N^* in $\succeq$, Δ is the only term of N^* essentially less than Ω and the only term of N^* essentially less than θ, but Ω and θ are not of the same type in N^*. In fact θ is a limit point of a countable discrete sequence of P-points, but Ω is not a limit point of any countable subset of N^*.*

Proof. Choose $f \in F$ such that $f^{-1}(n)$ is infinite for each $n \in N$.

For $0<\alpha<\omega_1$ and $n \in N$ select $\alpha_n \in \omega_1$ in such a way that, if α is not a limit ordinal, $\alpha_n=\alpha-1$, and if α is a limit ordinal, $\{\alpha_n\}_{n\in N}=\{\beta \mid \beta<\alpha\}$.

By [CH], F and S can be indexed so $F=\{f_\alpha\}_{\alpha<\omega_1}$ and $S=\{S_\alpha\}_{\alpha<\omega_1}$. By a complicated induction on the countable ordinals, we define various subsets of N and points of N^* which in turn allow us to define Ω, θ, and $f(\theta)=f(\Omega)=\Delta$ with the desired properties.

For each countable ordinal α we wish to select

(a) an infinite subset M_α of N,

(b) a countable ordinal $\alpha^* \geqq \alpha$,

(c) for each $n \in M_\alpha$ and $\beta \in \omega_1$, a subset $E_{\alpha n\beta}$ of $f^{-1}(n)$.

The following conditions are satisfied for all $n \in N$:

1. There is a P-point $p_{\alpha n}=\{U\subset N \mid \text{for some } \beta \in \omega_1, U\supset E_{\alpha n\beta}\}$ and $\delta<\beta<\omega_1$ implies that $E_{\alpha n\beta}-E_{\alpha n\delta}$ is finite and $E_{\alpha n\delta}-E_{\alpha n\beta}$ is infinite.

2. If $\gamma<\alpha$ and $E_{\alpha n0} \cap E_{\gamma n\delta}$ is infinite, then $E_{\alpha n0}-E_{\gamma n\delta}$ is finite.

3. If $\gamma<\alpha$, then $\gamma^*<\alpha^*$ and $E_{\gamma n(\alpha^*+1)} \cap E_{\alpha n0}$ is finite, but there exists a $\delta<\alpha$ such that $E_{\alpha n0}-E_{\delta n\alpha^*}$ is finite.

4. If $\alpha>0$ and n is the ith term of M_α, then

$$n \in M_{\alpha_1} \cap M_{\alpha_2} \cap \cdots \cap M_{\alpha_i} \quad \text{and} \quad E_{\alpha_1 n0} \cap E_{\alpha_2 n0} \cap \cdots \cap E_{\alpha_i n0} \cap E_{\alpha n0}$$

is infinite.

In all cases, once $E_{\alpha n0}$ has been chosen, choose $E_{\alpha n\beta}$ and $p_{\alpha n}$ in accordance with 1.

Let $M_0=N$, $0^*=0$ and, for all $n \in N$, $E_{0n0}=f^{-1}(n)$.

Assume our choices have been made for all $\gamma<\alpha$.

First suppose α is a limit ordinal. Choose $n_1 \in M_{\alpha_1}$. And, for all $i>1$, choose $n_i \in M_{\alpha_1} \cap \cdots \cap M_{\alpha_i}$ such that $E_{\alpha_1 n_i 0} \cap \cdots \cap E_{\alpha_i n_i 0}$ is infinite and $n_i>n_{i-1}$. By 4, such n_i exist. Then let $M_\alpha=\{n_i\}_{i\in N}$ and α^* be the limit of $\{\gamma^* \mid \gamma<\alpha\}$. Let $E_{\alpha n0} =E_{\gamma n\alpha^*}-E_{\gamma n(\alpha^*+1)}$, where if $n=n_i$, γ is the largest of $\alpha_1, \ldots, \alpha_i$, and otherwise $\gamma=0$. One can check that 2, 3, and 4 are again satisfied.

Suppose $\alpha=\beta+1$ and let g denote f_β.

Case 1. $X=\{n \in M_\beta \mid p_{\beta n} \notin g^{-1}(j) \text{ for any } j \in N\}$ is infinite. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in X$ and $j \in N$, $E_{\beta n\delta} \cap g^{-1}(j)$ is finite. Let $M=X$.

Case 2. X is finite and there exists an $i \in N$ and an infinite subset Z of M_β such that $n \in Z$ implies $p_{\beta n} \in g^{-1}(i)$. In this case there is a $\delta \in \omega_1$ such that, for all $n \in Z$, $E_{\beta n\delta}-g^{-1}(i)$ is finite. Let $M=Z$.

Case 3. Neither Case 1 nor 2 holds. Then there exist infinite subsets W of M_β and $\{a_j\}_{j\in N}$ of N such that $j<k$ in W implies $p_{\beta j} \in g^{-1}(a_j)$ and $p_{\beta k} \in g^{-1}(a_k)$ and $a_j<a_k$. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in W$, $E_{\beta n\delta}-g^{-1}(a_n)$ is finite. Let $M=W$.

In all cases consider $g(M)$. If there is an infinite subset V of M and a $v \in N$ such that $g^{-1}(v)\supset V$, then let $M'=V$. Otherwise there is an infinite subset M' of M such that j and k belong to M' implies that $g(j)\neq g(k)$.

Choose $\alpha^*=\beta^*+\delta+1$.

For some infinite subset M'' of M', for all $n \in M''$, $Q=S_\beta \cap (E_{\beta n\alpha^*}-E_{\beta n(\alpha^*+1)})$ is infinite or $Q=(N-S_\beta) \cap (E_{\beta n\alpha^*}-E_{\beta n(\alpha^*+1)})$ is infinite. Let $M_\alpha=M''$.

If $n \notin M_\alpha$, let $E_{\alpha n0}=E_{\beta n\alpha^*}-E_{\beta n(\alpha^*+1)}$.

In Case 1, if $n \in M_\alpha$, let $E_{\alpha n0}=Q-g^{-1}(1, \ldots, n)$.

In Case 2, if $n \in M_\alpha$, let $E_{\alpha n0}=Q \cap g^{-1}(i)$.

In Case 3, if $n \in M_\alpha$, let $E_{\alpha n0}=Q \cap g^{-1}(a_n)$.

It is easy to check that 2, 3 and 4 are again satisfied.

Let $\Omega=\{E \in S \mid \text{for some } \alpha \in \omega_1, E\supset \bigcup_{n\in M_\alpha} E_{\alpha n0}\}$. By 4 and our selection of M'' and $E_{\alpha n0}$, Ω is a free ultrafilter on N.

Let $f(\Omega)=\Delta$; observe that $\Delta=\{M\subset N \mid \text{for some } \alpha \in \omega_1, M\supset M_\alpha\}$.

Suppose $g \in F$. Then $g = f_\beta$ for some $\beta \in \omega_1$. If $\alpha = \beta + 1$ and $V = \bigcup_{n \in M_\alpha} E_{\alpha n 0}$, then $V \in \Omega$. In Case 1, for $j \in N$, $V \cap g^{-1}(j)$ is finite. In Case 2, $g(V) = i$ so $g(\Omega) = i'$. And in Case 3, $g(\Omega) \sim f(\Omega)$. So if $\phi \in \beta N$ and Ω is essentially greater than ϕ through g, either $[\phi] = N'$ or $[\phi] = [\Delta]$. This means that $[\Delta]$ is minimal in $\succeq$ in N^* (hence Δ is a P-point) and $[\Delta]$ is the only type in N^* less than $[\Omega]$ in $\sqsupset$.

Select $X = \{x_n\}_{n \in N} \in D$ such that x_n is a P-point to which $f^{-1}(n)$ belongs. Let $\theta = \Delta_X$.

Suppose $g \in F$. Let $Y = \{n \in N \mid \text{for all } k \in N,\ g^{-1}(k) \notin x_n\}$. For $n \in Y$ we can select $L_n \in x_n$ such that $L_n \cap g^{-1}(1, \ldots, n) = \varnothing$ and, for all $k \in N$, $L_n \cap g^{-1}(k)$ is finite. If $Y \in \Delta$, then $\bigcup_{n \in Y} L_n \in \theta$; so θ is not essentially greater than $g(\theta)$ through g if $Y \in \Delta$. If $Y \notin \Delta$, define $h \in F$ by $g^{-1}(h(n)) \in x_n$ for $n \in N - Y$ and $h(n) = 1$ for $n \in Y$. Then $h = f_\beta$ for some $\alpha - 1 = \beta \in \omega_1$ and $M_\alpha \cap (N - Y) \in \Delta$. By our definition of M', either $h(M') = v$ for some $v \in N$ or h restricted to M' is one-to-one; but $M' \supset M_\alpha$. So $Y \notin \Delta$ implies $g(\theta) = v' \in N'$ or $g(\theta) \sim f(\theta)$. Hence, if θ is essentially greater than $g(\theta)$ through g and $g(\theta) \in N^*$, $[g(\theta)] = [f(\theta)]$. Thus $[\Delta]$ is the one type contained in N^* which is essentially less than θ or Ω.

Now we show that Ω is not a limit point of any countable subset of N^*; one implication of this is that Ω and θ are not of the same type in N^*.

Suppose that $\{\rho_i\}_{i \in N}$ is a subset of $N^* - \{\Omega\}$. Let $A = \{i \in N \mid \text{for some } \alpha \in \omega_1 \text{ and } n \in N,\ \rho_i = p_{\alpha n}\}$. Let $B = \{i \in N \mid \text{for some } n \in N, f^{-1}(n) \in \rho_i \text{ but } \rho_i \neq p_{\alpha n} \text{ for any } \alpha \in \omega_1\}$. Let $C = \{i \in N \mid \text{for all } n \in N, f^{-1}(n) \notin \rho_i\}$. We find terms U, V, and W of Ω such that, for $i \in A$, $\rho_i \notin U$, for $i \in B$, $\rho_i \notin V$, and, for $i \in C$, $\rho_i \notin W$. Since $U \cap V \cap W \in \Omega$ and $A \cup B \cup C = N$, Ω is not a limit point of $\{\rho_i\}_{i \in N}$.

Choose α such that $\alpha > \gamma$ for all $\gamma \in \omega_1$ for which there are i and n in N such that $\rho_i = p_{\gamma n}$. Let $U = \bigcup_{n \in N} E_{\alpha n 0}$. If $\rho_i = p_{\gamma n}$, then $\gamma < \alpha$ and, by 1, $E_{\gamma n (\alpha^* + 1)} \in \rho_i$ and, by 3, $E_{\gamma n (\alpha^* + 1)} - E_{\alpha n 0} \in \rho_i$ and, by (c), $U \notin \rho_i$. Thus for $i \in A$, $U \in \Omega$ but $U \notin \rho_i$.

We want to choose a sequence $\{\beta^j\}_{j \in N} \subset \omega_1$, by induction. Let $\beta_1 = 0$. Suppose β^{j-1} has been selected. If $\gamma \in \omega_1$ and $n \in N$ and $i \in B$, by 1, there is a $\beta \in \omega_1$ such that $\beta \leqq \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Thus we can select $\beta^j \in \omega_1$ such that $\beta^{j-1} < \beta^j$ and, for all $\gamma \leqq \beta^{j-1}$ and $n \in N$ and $i \in B$, $\beta^j < \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Let α be the limit of $\{\beta^j\}_{j \in N}$ and let $V = \bigcup_{n \in N} E_{\alpha n 0} \in \Omega$. By 3, there is a $\gamma < \alpha$ such that $E_{\alpha n 0} - E_{\gamma n \alpha^*}$ is finite. So for $i \in B$, $E_{\alpha n 0} \notin \rho_i$. But for each $i \in B$ there is an $n \in N$ such that $f^{-1}(n) \in \rho_i$. Since $V \cap f^{-1}(n) = E_{\alpha n 0}$, $V \notin \rho_i$ for any $i \in B$.

For each $j \in N$, since $\Omega \neq \rho_j$, there is a $\delta^j \in \omega_1$ such that $\bigcup_{n \in M_{\delta^j}} E_{\delta^j n 0} \notin \rho_j$. Choose a limit ordinal α greater than δ^j for all $j \in N$. Suppose $j \in C$. There is an $i \in N$ such that $\delta^j = \alpha_i$. Let m be the ith term of M_α. Consider

$$W_j = \bigcup_{n \in M_\alpha} (E_{\alpha n 0} - E_{\alpha_i n 0}) - f^{-1}(1, \ldots, m).$$

If $\bigcup_{n \in M_\alpha} E_{\alpha n 0} \in \rho_j$ then $W_j \in \rho_j$ for $j \in C$ implies $f^{-1}(1, \ldots, m) \notin \rho_j$ and, by 4, $n \in M_\alpha$ and $n > m$ implies $n \in M_{\alpha_i}$ and $\delta^j = \alpha_i$ implies $\bigcup_{n \in M_{\alpha_i}} E_{\alpha_i n 0} \notin \rho_j$. Together 2 and 4 imply that, for $n \in M_\alpha$, $W_j \cap f^{-1}(n)$ is finite. For some $\beta \in \omega_1$, $\bigcup_{j \in C} W_j = S_\beta$;

since only finitely many W_j intersect $f^{-1}(n)$, $S_\beta \cap f^{-1}(n)$ is finite for all $n \in N$. But Ω is essentially greater than Δ through f, so $S_\beta \notin \Omega$. Thus $W=N-S_\beta \in \Omega$ but $W \notin \rho_j$ for any $j \in C$.

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