SOME TRANSPLANTATION THEOREMS FOR THE GENERALIZED MEHLER TRANSFORM AND RELATED ASYMPTOTIC EXPANSIONS

BY

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Abstract. Let $P^m_{-1/2+ix}(z)$ be the associated Legendre function of order m and degree $-1/2+ix$. We give, here, two integral transforms G^m and H^m, arising naturally from the generalized Mehler transform, which is induced by $P^m_{-1/2+ix}(\cosh y)$, such that $H^m G^m =$ Identity (formally). We show that if $1 < p < \infty$, $-1/p < \alpha < 1-1/p$, $m \leq 1/2$ or $m = 1, 2, \ldots$, then $\|G^m f\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$ and $\|H^m f\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$, where "$\wedge$" denotes the Fourier cosine transform. We also prove that $G^m f$, $H^m f$ exist as limits in $L^{p,\alpha}$ of partial integrals, and we prove inequalities equivalent to the above pair: $\|G^m \hat{f}\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$ and $\|H^m \hat{f}\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$. These we dualize to $\|(H^m f)^{\wedge}\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$ and $\|(G^m f)^{\wedge}\|_{p,\alpha} \leq A^m_{p,\alpha}\|f\|_{p,\alpha}$.

G^m and H^m are given by $G^m(f; y) = \int_0^\infty f(x)K^m(x, y)\, dx$ and $H^m(f; x) = \int_0^\infty f(y)K^m(x, y)\, dy$ $(0 \leq y < \infty)$, where

$$K^m(x, y) = |\Gamma(1/2-m-ix)/\Gamma(-ix)|(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y).$$

The principal method of proving the inequalities involves getting asymptotic expansions for $K^m(x, y)$; these are in terms of sines and cosines for large y, and in terms of Bessel functions for y small. Then we can use Fourier and Hankel multiplier theorems.

The main consequences of our results are the typical ones for transplantation theorems: mean convergence and multiplier theorems. They can easily be restated in terms of the more usual Mehler transform pair

$$g(y) = \int_0^\infty f(x)P_{-1/2+ix}(y)\, dx$$

and

$$f(x) = \pi^{-1}x \sinh \pi x \cdot \Gamma(1/2-m+ix)\, \Gamma(1/2-m-ix) \int_0^\infty g(y)P_{-1/2+ix}(y)\, dy.$$

1. **Introduction.** Let $P^m_{-1/2+ix}(z)$ be the Legendre function of real order m and degree $-\frac{1}{2}+ix$ given by the formula [5, p. 122, 3.2(3)]

$$P^m_{-1/2+ix}(z) = \frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{m/2} F(\tfrac{1}{2}-ix, \tfrac{1}{2}+ix; 1-m; \tfrac{1}{2}-\tfrac{1}{2}z)$$

for $m \neq 1, 2, \ldots$, and by [5, p. 148, 3.6(1)]

$$\Gamma(\tfrac{1}{2}-m+ix)m!\, P^m_{-1/2+ix}(z) = 2^{-m}\, \Gamma(\tfrac{1}{2}+m+ix)(z^2-1)^{m/2}$$
$$\cdot F(\tfrac{1}{2}+m+ix, \tfrac{1}{2}+m-ix; 1+m; \tfrac{1}{2}-\tfrac{1}{2}z),$$

if $m = 1, 2, \ldots$; F is the classical hypergeometric function. Then $P^m_{-1/2+ix}(z)$ satisfies the second order differential equation [5, p. 121, 3.2(1)]

$$\frac{d}{dz}\left\{(1-z^2)\frac{d}{dz}u(z)\right\} - \left\{x^2 + \frac{1}{4} - \frac{m^2}{1-z^2}\right\}u(z) = 0.$$

Replacement of z by $\cosh y$, for x and y real, gives the conical functions, introduced by Mehler. They arise in the course of solving Laplace's equation in a hyperboloid of two sheets. For certain values of m they occur in a probabilistic connection, as spherical functions on some noncompact hyperbolic spaces.

From the differential equation satisfied by $(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y) = v(y)$,

$$v'' + \left\{x^2 + \frac{1-4m^2}{4\sinh^2 y}\right\}v = 0,$$

and the definition of $P^m_{-1/2+ix}(z)$, it is easy to show that

(1.1) $(\sinh y)^{1/2}P^{1/2}_{-1/2+ix}(\cosh y) = (2/\pi)^{1/2}\cos xy$

and

(1.2) $(\sinh y)^{1/2}P^{-1/2}_{-1/2+ix}(\cosh y) = (2/\pi)^{1/2}\sin xy.$

Rewriting the differential equation for $(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$ as

$$v'' + x^2 v = -\left\{\frac{1-4m^2}{4\sinh^2 y}\right\}v$$

and treating it as a nonhomogeneous second order differential equation leads us to seek trigonometric expansions for $(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$. We will, in fact, find functions $k_m(x)$ so that $k_m(x)(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$ behaves roughly like constant (i.e., independent of x) linear combinations of $\sin xy$ and $\cos xy$ (or $J_{-m}(xy)$, with J_{-m} a Bessel function). The asymptotic behavior of the error terms, as well as that of their partial derivatives, can be determined; they can be differentiated termwise.

The functions $k_m(x)(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$ form the kernels of two integral transforms. From the asymptotic expansions it will follow that the mapping from the cosine transform of a given function to either of these two integral transforms of the same function is, in some sense, bounded.

Let us be more specific; for f a Lebesgue measurable function on $[0, \infty)$, we formally define

(1.3) $$g(y) = \int_0^\infty f(x)P^m_{-1/2+ix}(y)\,dx.$$

With certain conditions of f, principally on its behavior near 0 and ∞, it can be recovered from g by means of the formula [15, p. 2C(3)]

(1.4) $$f(x) = \pi^{-1}x\sinh\pi x\Gamma(\tfrac{1}{2}-m+ix)\Gamma(\tfrac{1}{2}-m-ix)\int_0^\infty g(y)P^m_{-1/2+ix}(y)\,dy.$$

Rosenthal [19] has shown that this is the right inversion formula and has given conditions for its validity. This integral transform pair occurs in solving certain boundary value problems arising from conductivity questions. In order to rewrite (1.3) and (1.4) so that the cosine and sine transforms arise when $m = \pm\frac{1}{2}$, let

$$(1.5) \qquad k_m(x) = |\Gamma(\tfrac{1}{2} - m - ix)/\Gamma(-ix)|;$$

using an identity for the Γ-function [5, p. 3,1.2(5)] one gets

$$\pi^{-1}x \sinh \pi x \Gamma(\tfrac{1}{2} - m + ix)\Gamma(\tfrac{1}{2} - m - ix) = k_m^2(x).$$

Also, define

$$(1.6) \qquad K^m(x, y) = k_m(x)(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y).$$

Then $K^{1/2}(x, y) = (2/\pi)^{1/2}\cos xy$ and $K^{-1/2}(x, y) = (2/\pi)^{1/2}\sin xy$. We rewrite (1.3) and (1.4) as

$$g(\cosh y)(\sinh y)^{1/2} = \int_0^\infty \frac{f(x)}{k_m(x)} K^m(x, y)\, dx,$$

and

$$\frac{f(x)}{k_m(x)} = \int_0^\infty g(\cosh y)(\sinh y)^{1/2}K^m(x, y)\, dy.$$

Replacing $f(x)/k_m(x)$ by $F(x)$, and $g(\cosh y)(\sinh y)^{1/2}$ by $G(y)$ leads us to define two integral operators:

$$(1.7) \qquad G^m(F; y) = \int_0^\infty F(x)K^m(x, y)\, dx, \qquad F \in L^1[0, \infty),$$

and

$$(1.8) \qquad H^m(G; y) = \int_0^\infty G(y)K^m(x, y)\, dy, \qquad G \in L^1[0, \infty).$$

With $m = \frac{1}{2}$ or $m = -\frac{1}{2}$, these operators become the cosine or sine transform. We will postpone treatment of the existence of G^m and H^m for arbitrary m until we have some estimates for $K^m(x, y)$.

Formally, at least, G^m and H^m are inverses of each other. Hence, it is natural to raise the mean convergence question: in what (Lebesgue integral) norms do the partial integrals for $H^m(G^m(F))$ converge to F? For the Fourier transform, mean convergence results follow from the boundedness of the Hilbert transform $f \to \int f(x)/(x-y)\, dx$ on the real line (or an any subinterval). The boundedness of this latter operator is a well-known result of M. Riesz [17] for L^p ($1 < p < \infty$), extended to $L^{p,\alpha}$ ($1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$), by Hardy and Littlewood [11].

Mean convergence results are also known for series expansions of a function by means of classical orthogonal polynomials. Pollard [16] investigated the problem for Legendre series and obtained positive results for L^p with $4/3 < p < 4$. Hirschman [12] used Jacobi expansions to get a projection theorem; his result can be viewed as a mean convergence theorem for Jacobi coefficients. Also in the area of mean convergence theorems is a result of Wing [22] dealing with partial Hankel integrals, operating on L^p with $1 < p \leq 2$.

Transplantation theorems are somewhat stronger results than mean convergence theorems. Very roughly, they amount to comparing $\|TF\|$ and $\|T'F\|$, where T and T' are given operators. Important applications arise when, for example, mean convergence results are known for one operator, giving analogous results for the second operator in the transplantation pair. D. L. Guy [10] found a theorem on transplanting between integral transforms induced by the kernels $(xy)^{1/2}J_a(xy)$ for $a \geq -\frac{1}{2}$ on $L^{p,\alpha}$ ($1 < p < \infty$, $1/p < \alpha < 1 - 1/p$). Askey and Wainger [3] obtained transplantation theorems arising from consideration of certain ultraspherical series. They also showed [2] that one can transplant between two sets of ultraspherical coefficients of a given function provided the parameter is, in each case, greater than zero. Askey [1] obtained a similar result for Jacobi coefficients provided all parameters are at least $-\frac{1}{2}$. Our principal transplantation results will be: If $m \leq \frac{1}{2}$ or $m = 1, 2, \ldots$; if $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$, then

$$\int_0^\infty |G^m(F; y)|^p y^{\alpha p}\, dy \leq A_{p,\alpha}^m \int_0^\infty |\hat{F}(y)|^p y^{\alpha p}\, dy$$

and

$$\int_0^\infty |H^m(F; y)|^p y^{\alpha p}\, dy \leq A_{p,\alpha}^m \int_0^\infty |\hat{F}(y)|^p y^{\alpha p}\, dy,$$

where $A_{p,\alpha}^m$ is independent of F and $\hat{F}$ stands for the Fourier cosine transform of F.

The significance of transplantation theorems is that they enable us to lift results known for one operator to another. A typical example of this procedure is the way in which a Fourier multiplier theorem is used to obtain multiplier theorems for other operators.

We state the definition of $L^{p,\alpha}[0, \infty)$:

DEFINITION. For $1 \leq p < \infty$, $f \in L^{p,\alpha}[0, \infty)$ if f is measurable and

$$\|f\|_{p,\alpha}^p = \int_0^\infty |f(x)|^p x^{\alpha p}\, dx < \infty.$$

In this paper, we will take $1 < p < \infty$. It is easy to see here that $L^{p,\alpha}$ is a Banach space with dual $L^{q,-\alpha}$ where $1/p + 1/q = 1$. If $-1/p < \alpha < 1 - 1/p$, we also have $-1/q < -\alpha < 1 - 1/q$.

We will be using the Fourier multiplier theorem in the following form:

THEOREM 1. *Let* $|\varphi(t)| \leq C$, $\int_0^\infty |d\varphi(t)| \leq C$, $1 < p < \infty$ *and* $-1/p < \alpha < 1 - 1/p$. *Then there is a constant* A, *depending only on* p *and* α, *such that, for* g, $\hat{g} \in L^1[0,\infty)$,

$$\left\| \int_0^\infty \cos ty\, \varphi(y)\hat{g}(y)\, dy \right\|_{p,\alpha} \leq AC \|g\|_{p,\alpha}$$

and

$$\left\| \int_0^\infty \sin ty\, \varphi(y)\hat{g}(y)\, dy \right\|_{p,\alpha} \leq AC \|g\|_{p,\alpha}.$$

REMARK. Theorem 1 remains true if $\varphi(y)$ is replaced by a bounded function of t multiplied by $\varphi(y)$ or by sums of such terms. (AC would obviously be replaced by some other constant.)

We will also use the following form of Guy's lemma [10, 8C]:

THEOREM 2. *If* $\mu \geq -\frac{1}{2}$, $1 < p < \infty$ *and* $-1/p < \alpha < 1 - 1/p$, *there is a constant* $A_{p,\alpha}^\mu$ *such that, for* $g \in L^1[0, \infty)$,

$$\left\| \int_0^\infty (ty)^{1/2} J_\mu(ty) g(y)\, dy \right\|_{p,\alpha} \leq A_{p,\alpha}^\mu \left\| \int_0^\infty \cos ty\, g(y)\, dy \right\|_{p,\alpha}.$$

The fact that a mapping $f \to Tf$ majorized by $\int_0^1 M|f(y)|\,dy + \int_1^\infty (M/y)|f(y)|\,dy$ is a bounded operator of $L^{p,\alpha}[0, \infty)$ into $L^{p,\alpha}[0, 1]$ will be used frequently. We will also use the boundedness (on $L^{p,\alpha}[0, \infty)$ into itself) of operators majorized by $\int_0^\infty M|f(y)|/(x+y)\,dy$. In both situations, $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$. These facts will be needed to deal with some error terms.

Our results will hold when the parameter $m \leq \frac{1}{2}$ or $m = 1, 2, \ldots$. To simplify the problem a bit, we use the fact [5, p. 140, 3.3(7)] that, for $m = 1, 2, \ldots$,

$$P_{-1/2+ix}^m(z) = \frac{\Gamma(\frac{1}{2}+m+ix)}{\Gamma(\frac{1}{2}-m+ix)} P_{-1/2+ix}^{-m}(z).$$

The quotient is a polynomial of degree $2m$ and, by induction on m, it equals $(-1)^m(\frac{1}{4}+x^2)\cdots([(2^m-1)/2]^2+x^2)$, which has constant sign $(-1)^m$. From (1.5)

$$K^m(x, y) = \frac{k_m(x)}{k_{-m}(x)} \frac{\Gamma(\frac{1}{2}+m+ix)}{\Gamma(\frac{1}{2}-m-ix)} K^{-m}(x, y)$$

$$= \left\{ \frac{\Gamma(\frac{1}{2}+m+ix)}{\Gamma(\frac{1}{2}-m+ix)} \Big/ \left| \frac{\Gamma(\frac{1}{2}+m+ix)}{\Gamma(\frac{1}{2}-m+ix)} \right| \right\} K^{-m}(x, y)$$

so that

(1.9) $K^m(x, y) = (-1)^m K^{-m}(x, y)$ for $m = 1, 2, \ldots$.

2. Asymptotic expansions.

2.1. To get our transplantation results, we will need some asymptotic expansions

for $K^m(x, y)$. We begin with the region where $x \geq 1$ and $y \geq 1$. From [5, p. 128, 3.2(26)], we have

$$P^m_{-1/2+ix}(\cosh y)$$

$$= \frac{1}{(2\pi)^{1/2}} (\sinh y)^{-1/2} \left\{ \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \cdot e^{-ixy} \cdot F\left(\frac{1}{2}+m, \frac{1}{2}-m; 1+ix; \frac{-1}{e^{2y}-1}\right) \right.$$

$$\left. + \frac{\Gamma(ix)}{\Gamma(\frac{1}{2}-m+ix)} e^{ixy} F\left(\frac{1}{2}+m, \frac{1}{2}-m; 1-ix; \frac{-1}{e^{2y}-1}\right) \right\};$$

this has the form $w + \bar{w}$. Hence, combined with (1.5) and (1.6), it gives

$$\text{(2.1.1)} \qquad K^m(x, y) = \left(\frac{2}{\pi}\right)^{1/2} \text{Re} \left\{ \left[\frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \middle/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \right| \right] \right.$$

$$\left. \cdot e^{-ixy} F\left(\frac{1}{2}+m, \frac{1}{2}-m; 1+ix; \frac{-1}{e^{2y}-1}\right) \right\}.$$

For the quotient we write

$$\text{(2.1.2)} \qquad \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \middle/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \right|$$

$$= \exp\left[i \arg \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \right] = \exp\left[i \operatorname{Im} \log \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \right].$$

Now using Barnes' expansion for $\log(z+a)$ and fixed n [5, p. 48, 1.18(12)] twice and subtracting, we have

$$\log \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} = -(\tfrac{1}{2}-m)\log(-ix) + \sum_{n=1}^{N} \frac{B_{n+1}(0) - B_{n+1}(\frac{1}{2}-m)}{n(n+1)} \left(\frac{1}{-ix}\right)^n + H_N(x)$$

where B_n is the nth degree Bernoulli polynomial and $H_N(x) = O(x^{-N-1})$. Also from Barnes' expansion,

$$t(z) = \log \Gamma\left(\frac{1}{z}+a\right) - \left(\frac{1}{z}+a-\frac{1}{2}\right) \log \frac{1}{z} + \frac{1}{z} - \frac{1}{2} \log 2\pi$$

$$- \frac{B_2(a)}{1 \cdot 2} z - \cdots - \frac{(-1)^{n+1} B_{n+1}(a)}{n(n+1)} z^n = O(z^{n+1})$$

for $|\arg z| < \pi$, and for real a, $t(z)$ is analytic off the real axis. For imaginary $z \neq 0$, let γ_z be the circle of radius $|z|/2$ centered at z. Then by Cauchy's formula for $t'(z)$, applied to γ_z, we obtain $|t'(z)| = O(|z|^n)$, $z \sim 0$, z imaginary. Replacing z by $1/-ix$ for x real therefore gives $H'_N(x) = O(x^{-N-2})$, $x \sim \infty$.

Let $N = 1$ in the preceding formula. Then [5, p. 36, 1.13(3)] $B_2(0) - B_2(\frac{1}{2}-m) = -m^2 + \frac{1}{4}$ so that

$$\log \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} = -(\tfrac{1}{2}-m)\log(-ix) - \frac{(m^2-\frac{1}{4})}{2} \frac{1}{-ix} + H_1(x)$$

and from (2.1.2) and the Taylor expansion for the exponential function

$$(2.1.3) \quad \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \bigg/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-m-ix)} \right|$$

$$= \exp\left[-i(2m-1)\pi/4\right]\{1 - (m^2-\tfrac{1}{4})/2ix + R(x)\}$$

where $R(x) = O(x^{-2})$, $R'(x) = O(x^{-3})$.

To obtain the needed expansion for the hypergeometric term, we make use of the fact that it is analytic in x and in y. Let

$$(2.1.4) \quad \zeta(w, z) = F\left(\frac{1}{2}+m, \frac{1}{2}-m; 1+iz; \frac{-1}{e^{2w}-1}\right) - 1 - \frac{(\frac{1}{2}+m)(\frac{1}{2}-m)}{(1+iz)\cdot 1}\left(\frac{-1}{e^{2w}-1}\right).$$

Then ζ is analytic in $\{(w, z) : \operatorname{Re} w > 0, \operatorname{Re} z > 0\}$. First fix $y \geq 1$ and let $x \geq 1$. Let γ_x be a circle centered at x of radius $\frac{1}{2}$. By analyticity

$$\frac{\partial}{\partial x}\zeta(y, x) = \frac{1}{2\pi i}\oint_{\gamma_x} \frac{\zeta(y, z)}{(z-x)^2}\, dz$$

so that

$$\left|\frac{\partial}{\partial x}\zeta(y, x)\right| \leq 2 \sup_{z \in \Omega_x} |\zeta(y, z)|,$$

where Ω_x = closed disc of radius $\frac{1}{2}$ about x. Similarly, we find that

$$\left|\frac{\partial}{\partial y}\zeta(y, x)\right| \leq 2 \sup_{w \in \Omega_y} |\zeta(w, x)|.$$

Thus once we can estimate $|\zeta|$, we will also have bounds for the first partial derivatives of ζ.

Now

$$\zeta(w, z) = \sum_{k=2}^{\infty} \frac{(\frac{1}{2}+m)_k(\frac{1}{2}-m)_k}{(1+iz)_k k!}\left(\frac{-1}{e^{2w}-1}\right)^k \quad \text{on } \operatorname{Re} w \geq \log 2^{1/2}.$$

It is easy to see that

$$(\tfrac{1}{2}+m)_k(\tfrac{1}{2}-m)_k = (\tfrac{1}{2}+m)(3/2+m)(5/2+m)_{k-2}(\tfrac{1}{2}-m)(3/2-m)(5/2-m)_{k-2}$$

$$= a_m(5/2+m)_{k-2}(5/2-m)_{k-2}$$

and that

$$(1+iz)_k = (1+iz)(2+iz)(3+iz)_{k-2};$$

$z \in \Omega_x$ $(x \geq 1)$ implies $|1+iz| \geq \operatorname{Re} z \geq x/2$, $|2+iz| \geq x/2$ and, for $n \geq 2$, $|n+iz| \geq n - \operatorname{Im} z \geq n - \frac{1}{2}$. Hence, for $z \in \Omega_x$,

$$|(1+iz)_k| \geq \tfrac{1}{4}x^2(5/2)_{k-2}.$$

We also know that if $w \in \Omega_y$, $y \geqq 1$, then

$$|e^{2w} - 1| \geqq e^{2(y-1/2)} - 1 \geqq e^{2y}/5.$$

Consequently, $x \geqq 1$, $y \geqq 1$, $(w, z) \in \Omega_y \times \Omega_x$ implies

$$|\zeta(w, z)| \leqq \frac{|a_m|}{\frac{1}{4}x^2(\frac{1}{4}e^{2y})^2} \sum_{k=0}^{\infty} \frac{(5/2+m)_k(5/2-m)_k}{(5/2)_k k!} (5e^{-2y})^k$$

$$\leqq \frac{C_m}{x^2 e^{4y}} \sum_{k=0}^{\infty} \frac{(5/2+m)_k(5/2-m)_k}{(5/2)_k k!} \left(\frac{5}{e^2}\right)^k = \frac{C'_m}{x^2 e^{4y}}.$$

We conclude that

$$(2.1.5) \qquad \zeta(y, x), \frac{\partial}{\partial y} \zeta(y, x), \frac{\partial}{\partial x} \zeta(y, x) = O(x^{-2} e^{-4y}).$$

Combining (2.1.3), (2.1.4) and (2.1.5) with (2.1.1) gives

$$
\begin{aligned}
K^m(x, y) = \ & (2/\pi)^{1/2} \cos(xy + [2m-1]\pi/4) \\
& -(2/\pi)^{1/2} \frac{m^2 - \frac{1}{4}}{2x} \coth y \sin(xy + [2m-1]\pi/4) \\
& + \cos(xy + [2m-1]\pi/4) R_1(x, y) + \sin(xy + [2m-1]\pi/4) R_2(x, y) \\
& + \cos(xy + [2m-1]\pi/4) D_1(x, y) + \sin(xy + [2m-1]\pi/4) D_2(x, y)
\end{aligned}
$$

(2.1.6)

where $R_j(x, y) = S_j(x) + T_j(x)/(e^{2y} - 1)$ with

$$S_j(x), T_j(x) = O(x^{-2}), \qquad S'_j(x), T'_j(x) = O(x^{-3}),$$

and

$$D_j(x, y), \frac{\partial}{\partial y} D_j(x, y), \frac{\partial}{\partial x} D_j(x, y) = O(x^{-2} e^{-4y}),$$

for $x \geqq 1$, $y \geqq 1$.

2.2. To obtain an expansion for the region $x \leqq 1$, $y \geqq 1$, we again make use of [5, p. 128, 3.2(26)], so that we now consider

$$\frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \bigg/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \right|$$

for small x. For the moment, assume $m \neq \frac{1}{2}, \frac{3}{2}, \ldots$. With this restriction on m,

$$\log \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} = \log \frac{i}{x} + \log \frac{\Gamma(1 - ix)}{\Gamma(\frac{1}{2} - m - ix)}$$

and $\log \Gamma(1 - ix)/\Gamma(\frac{1}{2} - m - ix)$ is analytic for x close to zero. These facts yield the expansion

$$\log \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} = \log \frac{i}{x} - \log \Gamma(\frac{1}{2} - m) + g_1(x),$$

where $g_1(x) = O(x)$, $g_1'(x) = O(1)$. Making use of (2.1.2) shows that, for $m \neq \frac{1}{2}, \frac{3}{2}, \ldots$,

$$(2.2.1) \qquad \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \bigg/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \right| = \pm i + g(x),$$

where $g(x) = O(x)$ and $g'(x) = O(1)$; the " $\pm$ " sign depends on sgn $\Gamma(\frac{1}{2} - m)$.

Note that if we let $m = k + \frac{1}{2}$ for $k = 0, 1, 2, \ldots$, and call the corresponding quotient, $(\Gamma(-ix)/\Gamma(\frac{1}{2} - m - ix))/|\Gamma(-ix),/\Gamma(\frac{1}{2} - m - ix)|$, $\rho_k(x)$, then $\rho_{k+1}(x) = \rho_k(x)$ $\cdot (k - 1 - ix)/|k - 1 - ix|$. Since $\rho_0(x) = 1$, we obtain, for $m = \frac{1}{2}, \frac{3}{2}, \ldots$,

$$(2.2.2) \qquad \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \bigg/ \left| \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - m - ix)} \right| = 1 + g(x)$$

where $g(x) = O(x)$, $g'(x) = O(1)$.

We write the relevant hypergeometric term as

$$F\left(\frac{1}{2} + m, \frac{1}{2} - m; 1 + ix; \frac{-1}{e^{2y} - 1}\right) = 1 + s(x, y),$$

and from the series for F, we have $|s(x, y)| \leq A e^{-2y}$ (A = absolute constant). The analyticity of F in x for $|x| < 1$ shows, by Cauchy's formula, that for small x,

$$|\partial s(x, y)/\partial x| \leq A e^{-2y}.$$

(We use Cauchy's formula for the derivative on a *fixed* circle of, say radius $1 - \delta$ (some $\delta > 0$) about the origin.) The function $s(x, y)$ is also analytic on Re $y > 0$ and by the same methods we used for ζ as defined in (2.1.4)

$$|\partial s(x, y)/\partial y| \leq A e^{-2y}.$$

Now according to [5, p. 128, 3.2(26)], we combine (2.2.1) with these facts on the hypergeometric function so that

$$(2.2.3) \qquad \begin{aligned} K^m(x, y) = & \pm (2/\pi)^{1/2} \sin xy + \cos xy \, h_1(x) + \sin xy \, h_2(x) \\ & + \cos xy \, s_1(x, y) + \sin xy \, s_2(x, y) \end{aligned}$$

for $m \neq \frac{1}{2}, \frac{3}{2}, \ldots$, where $h_j(x) = O(x)$, $h_j'(x) = O(1)$ and

$$s_j(x, y), \, \partial s_j(x, y)/\partial x, \, \partial s_j(x, y)/\partial y = O(e^{-2y}) \quad \text{for } x \leq 1, y \geq 1.$$

When $m = \frac{1}{2}, \frac{3}{2}, \ldots$, one gets by (2.2.2)

$$(2.2.4) \qquad \begin{aligned} K^m(x, y) = & (2/\pi)^{1/2} \cos xy + \cos xy \, h_1(x) + \sin xy \, h_2(x) \\ & + \cos xy \, s_1(x, y) + \sin xy \, s_2(x, y), \end{aligned}$$

where h_j, s_j have the same properties as in (2.2.3), for $x \leq 1, y \geq 1$.

2.3. We will now do the expansion of $K^m(x, y)$ for $x \leq 1$, $y \leq 1$. We exclude the case $m = 1, 2, \ldots$. (Recall that by (1.9), this restriction involves no loss of generality.) Then by [5, p. 122, 3.2(3)],

$$(\sinh y)^{1/2} P^m_{-1/2+ix}(\cosh y) = \frac{1}{\Gamma(1-m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}}$$

(2.3.1)

$$\cdot F(\tfrac{1}{2} - ix, \tfrac{1}{2} + ix; 1 - m; \tfrac{1}{2} - \tfrac{1}{2} \cosh y).$$

Setting the hypergeometric function equal to $1 + t(x, y)$ and using the power series shows that $|t(x, y)| \leq Ay^2$. Also, the formula for differentiating a hypergeometric series (with respect to its last argument) is well known; applying it yields $|\partial t(x, y)/\partial y| \leq Ay$. Moreover, $t(x, y)$ is an entire function of x; by Cauchy's formula we therefore conclude that $|\partial t(x, y)/\partial x| \leq Ay^2$.

Now $k_m(x) = \exp \operatorname{Re} \log \Gamma(\tfrac{1}{2} - m - ix)/\Gamma(-ix)$ from (1.5). If $m \neq \tfrac{1}{2}, \tfrac{3}{2}, \ldots$, we have, by power series,

(2.3.2) $$k_m(x) = x \cdot |\Gamma(\tfrac{1}{2} - m)|\{1 + r(x)\}$$

with $r(x) = O(x^2)$, $r'(x) = O(x)$.

For $m = \tfrac{1}{2}, \tfrac{3}{2}, \ldots$, let $m = n + \tfrac{1}{2}$ and $\sigma_n(x) = k_m(x)$. Then $\sigma_{n+1}(x) = \sigma_n(x)/|n + 1 + ix|$. We have $\sigma_0(x) = 1$ so that if $m = \tfrac{1}{2}, \tfrac{3}{2}, \ldots$,

(2.3.3) $$k_m(x) = 1/(m - \tfrac{1}{2})! + r_1(x)$$

where $r_1(x) = O(x)$, $r_1'(x) = O(1)$.

Combining our information on the hypergeometric term with (2.3.1), letting

$$t_1(x, y) = \frac{1}{\Gamma(1-m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} t(x, y), \qquad s(x) = |\Gamma(\tfrac{1}{2} - m)| x r(x)$$

leads to

$$K^m(x, y) = \frac{|\Gamma(\tfrac{1}{2} - m)|}{\Gamma(1-m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} x$$

(2.3.4)

$$+ |\Gamma(\tfrac{1}{2} - m)| x t_1(x, y) + \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} \frac{1}{\Gamma(1-m)} s(x) + s(x) t_1(x, y)$$

for $m \neq \tfrac{1}{2}, \tfrac{3}{2}, \ldots$, where $t_1(x, y)$, $\partial t_1(x, y)/\partial x = O(y^{5/2-m})$, $\partial t_1(x, y)/\partial y = O(y^{3/2-m})$ $s(x) = O(x^3)$ and $s'(x) = O(x^2)$, for $x \leq 1$, $y \leq 1$.

If, on the other hand, $m = \tfrac{1}{2}, \tfrac{3}{2}, \ldots$ then (2.3.1) gives rise to the formula

$$K^m(x, y) = \frac{1}{\Gamma(1-m)(m-\tfrac{1}{2})!} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}}$$

(2.3.5)

$$+ \frac{1}{(m-\tfrac{1}{2})!} t_1(x, y) + \frac{1}{\Gamma(1-m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} s_1(x) + r_1(x) t_1(x, y)$$

where $t_1(x, y)$ has the same properties as in (2.3.4) and $r_1(x) = O(x)$, $r_1'(x) = O(1)$, for $x \leq 1$, $y \leq 1$.

2.4. To develop the kind of expansion we will need for the region $x \geq 1$, $y \leq 1$ requires more work than was required to get expansions for the previous regions. We will use an integral representation for the Legendre function, comparing it to a similar representation for the Bessel function. Both representations are valid only if $m < \frac{1}{2}$ and so from here on, we will be making this restriction. To obtain bounds for the remainder that occurs in the expansion for $P^m_{-1/2+ix}(\cosh y)$, we have adapted methods developed by G. Szegö and appearing in [20] and [21].

According to [5, p. 156, 3.7(8)]

$$P^m_{-1/2+ix}(\cosh y) = \frac{(\frac{1}{2}\pi)^{-1/2}}{\Gamma(\frac{1}{2}-m)} (\sinh y)^m \int_0^y (\cosh y - \cosh v)^{-m-1/2} \cos xv \, dv$$

which we write as

$$
(2.4.1(1)) \quad
\begin{aligned}
&P^m_{-1/2+ix}(\cosh y) \\
&\quad = \frac{2^m \pi^{-1/2}}{\Gamma(\frac{1}{2}-m)} (\sinh y)^m \int_{-y}^y (2\cosh y - 2\cosh v)^{-m-1/2} e^{ixv} \, dv.
\end{aligned}
$$

The expression to which we will compare this is [6, p. 81, 7.12(7)]

$$(2.4.1(2)) \qquad \Gamma(\mu+\tfrac{1}{2}) J_\mu(z) = \pi^{-1/2}(\tfrac{1}{2}z)^\mu \int_{-1}^1 (1-t^2)^{\mu-1/2} e^{izt} \, dt,$$

which is valid for $\mu > -\frac{1}{2}$.

Our aim is to expand

$$\left(\frac{2\cosh y - 2\cosh v}{y^2 - v^2} \right)^{-m-1/2}$$

in a series of powers of $(y^2 - v^2)$, which gives an expansion of $(2\cosh y - 2\cosh v)^{-m-1/2}$ in powers of $(y^2 - v^2)$. A change of variables will then allow us to compare (2.4.1(1)) with the kind of representation we have in (2.4.1(2)).

Let $\tau = y^2 - v^2$ and set

$$r(y, \tau) = \frac{2\cosh y - 2\cosh v}{y^2 - v^2}, \qquad \tau \neq 0,$$

$$= \frac{\sinh y}{y}, \qquad \tau = 0,$$

for y and v complex variables. Since the hyperbolic cosine is an even function, r is entire in y and τ. Its only zeroes occur where either $v - y$ or y is a nonzero multiple of $2\pi i$. Hence if $|y| \leq \pi$, then $r(y, \tau) = 0$ implies $v = y + 2n\pi i$, for some nonzero integer n, and so $v^2 = y^2 + 4n\pi i - 4n^2\pi^2$ or $\operatorname{Re} \tau = 4n^2\pi^2 \geq 4\pi^2$. It follows that

$r(y, \tau) \neq 0$ whenever $|y| \leqq \pi$ and $|\tau| \leqq 7\pi^2/2$. Within this region $r(y, \tau)$ has analytic powers so for such points

(2.4.1(3)) $$r(y, \tau)^{-m-1/2} = \left(\frac{\sinh y}{y}\right)^{-m-1/2} \sum_{k=0}^{\infty} \varphi_k(y) \tau^k$$

where

(2.4.1(4)) $$2\pi i \varphi_k(y) = \left(\frac{\sinh y}{y}\right)^{m+1/2} \oint_{|\tau| = 3\pi^2} \frac{r(y, \tau)^{-m-1/2}}{\tau^{k+1}} \, d\tau;$$

we also have $\varphi_0(y) = 1$.

From here on, we take $-\pi \leqq y \leqq \pi$. Then if $v \in [-y, y]$, we have

$$0 \leqq \tau = y^2 - v^2 \leqq \pi^2;$$

hence the series in (2.4.1(3)) converges uniformly on

$$\{\tau : v = (y^2 - v^2)^{1/2} \in [-y, y]\}.$$

We obtain the representation

$$\int_{-y}^{y} (2 \cosh y - 2 \cosh v)^{-m-1/2} e^{ixv} \, dv$$
$$= \left(\frac{\sinh y}{y}\right)^{-m-1/2} \cdot \sum_{k=0}^{\infty} \varphi_k(y) \int_{-y}^{y} (y^2 - v^2)^{k-m-1/2} e^{ixv} \, dv.$$

A change of variables gives

$$\int_{-y}^{y} (y^2 - v^2)^{k-m-1/2} e^{ixv} \, dv = y^{2k-2m} \int_{-1}^{1} (1-t^2)^{k-m-1/2} e^{ixyt} \, dt$$
$$= \pi^{1/2} 2^{k-m} \Gamma(k-m+\tfrac{1}{2})(y/x)^{k-m} J_{k-m}(xy),$$

by (2.4.1(2)) since $k - m > -\tfrac{1}{2}$. Thus, in consequence of (2.4.1(1)), the formula

$$P^m_{-1/2+ix}(\cosh y)$$

(2.4.1(5))
$$= \left(\frac{y}{\sinh y}\right)^{1/2} x^m \cdot \sum_{k=0}^{\infty} 2^k \frac{\Gamma(k-m+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}-m)} \varphi_k(y) \left(\frac{y}{x}\right)^k J_{k-m}(xy),$$

valid for $|y| \leqq \pi$, arises. Moreover, if we let the error in this expansion after p terms be $R_p(x, y) = R_p$, then

(2.4.1(6))
$$R_p = \frac{2^m \pi^{-1/2}}{\Gamma(\tfrac{1}{2}-m)} (\sinh y)^m \left(\frac{\sinh y}{y}\right)^{-m-1/2}$$
$$\cdot \sum_{k=p}^{\infty}{}' \varphi_k(y) \int_{-y}^{y} (y^2 - v^2)^{k-m-1/2} e^{ixv} \, dv.$$

We also want to differentiate this formula with respect to x. (We will later differentiate a shorter expansion with respect to y.) Clearly,

$$\int_{-y}^{y} (2 \cosh y - 2 \cosh v)^{-m-1/2} e^{ixv} \, dv$$

is differentiable in x, and

$$\frac{\partial}{\partial x} \int_{-y}^{y} (2 \cosh y - 2 \cosh v)^{-m-1/2} e^{ixv} \, dv$$

$$= \int_{-y}^{y} (2 \cosh y - 2 \cosh v)^{-m-1/2} iv e^{ixv} \, dv$$

$$= \left(\frac{\sinh y}{y}\right)^{-m-1/2} \sum_{k=0}^{\infty} \varphi_k(y) \int_{-y}^{y} \{(y^2 - v^2)^{k-m-1/2} iv\} e^{ixv} \, dv$$

$$= \left(\frac{\sinh y}{y}\right)^{-m-1/2} \sum_{k=0}^{\infty} -\varphi_k(y) \frac{x}{2(k-m+\frac{1}{2})} \int_{-y}^{y} (y^2 - v^2)^{k+1-m-1/2} e^{ixv} \, dv$$

by parts. With the same change of variables just used, and an application of (2.4.1(2)), this expression becomes

$$\left(\frac{\sinh y}{y}\right)^{-m-1/2} \sum_{k=0}^{\infty} \varphi_k(y)\{-\pi^{1/2} 2^{k-m} \Gamma(k-m+\tfrac{1}{2}) \cdot y(y/x)^{k-m} J_{k+1-m}(xy)\}.$$

Hence, for $|y| \le \pi$, we get by (2.4.1(1)),

(2.4.1(7))
$$\frac{\partial}{\partial x} P_{-1/2+ix}^{m}(\cosh y)$$
$$= -\left(\frac{y}{\sinh y}\right)^{1/2} x^m y \cdot \sum_{k=0}^{\infty} 2^k \frac{\Gamma(k-m+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}-m)} \varphi_k(y) \left(\frac{y}{x}\right)^k J_{k+1-m}(xy).$$

Letting the error after p terms of this expression be denoted by $\tilde{R}_p(x, y) = \tilde{R}_p$, we have

(2.4.1(8))
$$\tilde{R}_p = \frac{2^m \pi^{-1/2}}{\Gamma(\tfrac{1}{2}-m)} (\sinh y)^m \left(\frac{\sinh y}{y}\right)^{-m-1/2}$$
$$\cdot \sum_{k=p}^{\infty} \varphi_k(y) \int_{-y}^{y} (y^2 - v^2)^{k-m-1/2} iv e^{ixv} \, dv.$$

2.4.2. To consider the accuracy of our expansions, we need to estimate $|\varphi_k(y)|$; we use (2.4.1(4)). We recall that r is entire, thus continuous in (y, τ) and that if $|y| \le \pi$ and $|\pi| \le 7/2\pi^2$ then $r(y, \tau) \ne 0$. It follows that there is a constant, M, such that, for y and τ as indicated,

(2.4.2(1))
$$\left| \left(\frac{\sinh y}{y}\right)^{m+1/2} r(y, \tau)^{-m-1/2} \right| \le M.$$

Therefore by (2.4.1(4)),

(2.4.2(2)) $$|\varphi_k(y)| \leqq M(3\pi^2)^{-k}$$

for $-\pi \leqq y \leqq \pi$.

Now let $0 < y < \pi$; the error terms for $-\pi < y < 0$ will obviously be of the same order of growth. We wish to bound R_p and $\tilde{R}_p$; to this end, let

$$(2.4.2(3)) \qquad S_p = \sum_{k=p}^{\infty} \varphi_k(y) \int_{-y}^{y} (y^2 - v^2)^{k-m-1/2} e^{ixv}\, dv$$

and

$$(2.4.2(4)) \qquad \tilde{S}_p = \sum_{k=p}^{\infty} \varphi_k(y) \int_{-y}^{y} iv(y^2 - v^2)^{k-m-1/2} e^{ixv}\, dv.$$

Note that $R_p = O(y^m)S_p$ and $\tilde{R}_p = O(y^m)\tilde{S}_p$. Two cases arise: $xy \leqq 1$ and $xy \geqq 1$. First take $xy \leqq 1$ and let $p \geqq 1$. Then $p - m - \frac{1}{2} > \frac{1}{2} - m > 0$ so that

$$|S_p| \leqq 2M \sum_{k=p}^{\infty} (3\pi^2)^{-k} y^{2k-2m} = O(y^{2p-2m}).$$

Moreover, integration by parts shows that we can write

$$\tilde{S}_p = -\sum_{k=p}^{\infty} \varphi_k(y) \frac{x}{2(k-m+\frac{1}{2})} \int_{-y}^{y} (y^2 - v^2)^{k+1-m-1/2} e^{ixv}\, dv.$$

Since $p - m + \frac{1}{2} \geqq p \geqq 1$,

$$|\tilde{S}_p| \leqq Mx \sum_{k=p}^{\infty} (3\pi^2)^{-k} y^{2k+2-2m} = O(xy^{2p+2-2m}).$$

Therefore, if $xy \leqq 1$,

(2.4.2(5)) $$R_p(x, y) = O(y^{2p-m})$$

and

$$(2.4.2(6)) \qquad \partial R_p(x, y)/\partial x = \tilde{R}_p(x, y) = O(xy^{2p+2-m}).$$

Finding bounds when $xy \geqq 1$ is a bit harder; however a device due to Szegö [20] is very helpful. It consists of repeatedly integrating by parts, noting that

$$\frac{\partial^n}{\partial v^n}(y^2 - v^2)^{k-m-1/2} = \frac{\partial^n}{\partial v^n}\{(y-v)^{k-m-1/2}(y+v)^{k-m-1/2}\}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \frac{\partial^j}{\partial v^j}(y-v)^{k-m-1/2} \frac{\partial^{n-j}}{\partial v^{n-j}}(y+v)^{k-m-1/2},$$

which gives rise to the sequence of estimates

$$\left| \frac{\partial^n}{\partial v^n} (y^2 - v^2)^{k-m-1/2} \right| \leq \sum_{j=0}^{n} \binom{n}{j} \left| \frac{\partial^j}{\partial v^j} (y-v)^{k-m-1/2} \right| \left| \frac{\partial^{n-j}}{\partial v^{n-j}} (y+v)^{k-m-1/2} \right|$$

$$\leq (k-m-\tfrac{1}{2})^n \sum_{j=0}^{n} \binom{n}{j} (y-v)^{k-m-1/2-j}(y+v)^{k-m-1/2-(n-j)}$$

$$= (k-m-\tfrac{1}{2})^n (y^2-v^2)^{k-m-1/2} \sum_{j=0}^{n} \binom{n}{j}\left(\frac{1}{y-v}\right)^j\left(\frac{1}{y+v}\right)^{n-j}$$

$$= (k-m-\tfrac{1}{2})^n (y^2-v^2)^{k-m-1/2-n} 2^n y^n$$

and consequently, for $k-m-\tfrac{1}{2}-n\geq 0$ and $-y\leq v\leq y$,

$$(2.4.2(7)) \qquad \left| \frac{\partial^n}{\partial v^n}(y^2-v^2)^{k-m-1/2} \right| \leq C(n)(k-m-\tfrac{1}{2})^n y^{2k-2m-1-n}.$$

Now let $l=0, 1, \dots$ be chosen so that $l\leq -m+\tfrac{1}{2}<l+1$. Put $n=p+l$ in (2.4.2(7)) to get

$$|S_{p+1}| = \left| \sum_{k=p+1}^{\infty} \varphi_k(y) \int_{-y}^{y} \frac{\partial^{p+l}}{\partial v^{p+l}}(y^2-v^2)^{k-m-1/2} \frac{e^{ixv}}{(ix)^{p+l}} dv \right|$$

$$\leq C'(p) x^{-p-l} \sum_{k=p+1}^{\infty} (k-m-\tfrac{1}{2})^{p+l} y^{2k-2m-(p+l)},$$

by (2.4.2(2)),

$$= C'(p) x^{-p-l} y^{-p-l-2m} \sum_{k=p+1}^{\infty} (k-m-\tfrac{1}{2})^{p+l} y^{2k}$$

$$\leq C''(p) x^{-p-l} y^{p-l-2m+2}$$

for $y\sim 0$. Now $l+m+\tfrac{1}{2}>0$ and $y^{-1}<x$ so that $y^{-(l+m+1/2)}\leq x^{l+m+1/2}$ or $y^{-l-m}\leq x^{l+m+1/2}y^{1/2}$, leading to

$$|S_{p+1}| \leq C''(p) x^{m+1/2-p} y^{-m+5/2+p}.$$

Moreover, the leading term of the series for S_p can be evaluated in terms of Bessel functions by (2.4.1(2)) giving

$$\varphi_p(y) \int_{-y}^{y} (y^2-v^2)^{p-m-1/2} e^{ixv} dv = \varphi_p(y) C(p)\left(\frac{y}{x}\right)^{p-m} J_{p-m}(xy)$$

and by well-known estimates on $J_\mu(t)$ for $t\sim\infty$, the leading term is bounded in absolute value by

$$(2.4.2(8)) \qquad C(p) x^{m-1/2-p} y^{-m-1/2+p}.$$

Combining this with the estimate for S_{p+1} shows that $S_p = O(x^{m+1/2-p} y^{-m-1/2+p})$. Replacement of p by $p+1$ in this gives $S_{p+1} = O(x^{m-1/2-p} y^{-m+1/2+p})$ which when used in conjunction with (2.4.2(8)) leads to $S_p = O(x^{m-1/2-p} y^{-m-1/2+p})$ and so

$$(2.4.2(9)) \qquad R_p = O(x^{m-1/2-p} y^{-1/2+p}), \qquad xy \geq 1, y\sim 0.$$

To estimate $\tilde{S}_p$, first note that for $k - m - \frac{1}{2} \geq n$, by (2.4.2(7)), we have for $-y \leq v \leq y$

$$\left| \frac{\partial^n}{\partial v^n} v(y^2 - v^2)^{k-m-1/2} \right| \leq \left| v \frac{\partial^n}{\partial v^n} (y^2 - v^2)^{k-m-1/2} \right| + n \left| \frac{\partial^{n-1}}{\partial v^{n-1}} (y^2 - v^2)^{k-m-1/2} \right|$$

$$\leq C(n)(k - m - \tfrac{1}{2})^n y^{2k - 2m - n}.$$

Thus, with $n = p + l$, we obtain, by the same procedure as before, for $y \sim 0$,

$$|\tilde{S}_{p+1}| \leq C''(p)x^{-p-l}y^{p-l-2m+3} \leq C''(p)x^{m+1/2-p}y^{-m+7/2+p}$$

by choice of l. Consideration of the leading term of $\tilde{S}_p$ yields, by a similar sequence of steps as for S_p, $\tilde{S}_p = O(x^{m-1/2-p}y^{-m+1/2+p})$ and

$$(2.4.2(10)) \qquad \tilde{R}_p = O(\dot{x}^{m-1/2-p}y^{1/2+p}), \qquad xy \geq 1, y \sim 0.$$

2.4.3. We note an expansion for $k_m(x) = \exp \operatorname{Re} \log(\Gamma(\frac{1}{2} - m - ix)/\Gamma(-ix))$, for $x \geq 1$. We use [5, p. 48, 1.18(12)]; the differentiation property of this expansion follows in the same way as for the quotient of Γ-functions we looked at on the region $x \geq 1, y \geq 1$. We thus obtain

$$\log \frac{\Gamma(\frac{1}{2} - m - ix)}{\Gamma(-ix)} = (\tfrac{1}{2} - m) \log(-ix) + \frac{B_2(\frac{1}{2} - m) - B_2(0)}{(-ix)1 \cdot 2} + l_1(x)$$

with

$$l_1(x) = O(x^{-2}), l_1'(x) = O(x^{-3}).$$

The Bernoulli polynomials have real coefficients; therefore taking the real part and exponentiating gives

$$(2.4.3) \qquad k_m(x) = x^{1/2-m}\{1 + \tilde{k}_m(x)\}$$

where $\tilde{k}_m(x) = O(x^{-2})$ and $\tilde{k}_m'(x) = O(x^{-3})$.

We can now put the pieces together and expand $K^m(x, y)$. Using only two terms of (2.4.1(5)) we obtain

$$(2.4.4) \qquad \begin{aligned} (\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y) \\ = y^{1/2}x^m J_{-m}(xy) + 2(\tfrac{1}{2} - m)\varphi_1(y)y^{3/2}x^{m-1}J_{1-m}(xy) + E_m(x, y), \end{aligned}$$

where

$$(2.4.5) \qquad \begin{aligned} E_m(x, y) &= O(y^{9/2-m}), & xy &\leq 1; \\ &= O(x^{m-5/2}y^2), & xy &\geq 1, \end{aligned}$$

by (2.4.2(5)) and (2.4.2(9)).

Multiplying $k_m(x)$ and $(\sinh y)^{1/2}P_{-1/2+ix}(\cosh y)$, using these last two formulas and (2.4.5) gives

$$(2.4.6) \quad K^m(x, y) = (xy)^{1/2}J_{-m}(xy) + 2(\tfrac{1}{2} - m)\varphi_1(y)y^{3/2}x^{-1/2}J_{1-m}(xy) + F_m(x, y)$$

and $F_m(x, y)$ is a sum of four terms, each of which we will examine. To do this, we use the estimates for the Bessel function: $J_\mu(t) = O(t^\mu)$, $t \sim 0$; $J_\mu(t) = O(t^{-1/2})$, $t \sim \infty$.

Note also that $\varphi_1(y) = O(1)$. We tabulate below estimates for the terms adding up to F_m:

Term	$O(\ldots)$ $xy \leqq 1$	$xy \geqq 1$
$x^{1/2-m}E_m(x, y)$	$x^{1/2-m}y^{9/2-m}$	$x^{-2}y^2$
$(xy)^{1/2}\check{k}_m(x)J_{-m}(xy)$	$x^{-3/2-m}y^{1/2-m}$	x^{-2}
$2(\frac{1}{2}-m)\varphi_1(y)y^{3/2}x^{-1/2}\check{k}_m(x)J_{1-m}(xy)$	$x^{-3/2-m}y^{5/2-m}$	$x^{-3}y$
$x^{1/2-m}\check{k}_m(x)E_m(x, y)$	$x^{-3/2-m}y^{9/2-m}$	$x^{-4}y^2$

Each of these terms, *excepting the second*, is of smaller magnitude in x, and no larger in y than the second term in (2.4.6). Consequently, F_m can be written as

$$(2.4.7) \quad F_m(x, y) = (xy)^{1/2}\check{k}_m(x)J_{-m}(xy) + \begin{cases} O(x^{-m-3/2}y^{-m+5/2}), & xy \leqq 1; \\ O(x^{-2}y^2), & xy \geqq 1. \end{cases}$$

The next problem is that of estimating $\partial F_m(x, y)/\partial x$. Evidently

$$\frac{\partial}{\partial x} K^m(x, y) = (\tfrac{1}{2}-m)x^{-1/2-m}\{1+\check{k}_m(x)\}(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$$
$$+ x^{1/2-m}\check{k}'_m(x)(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y)$$
$$+ x^{1/2-m}\{1+\check{k}_m(x)\}\cdot\frac{\partial}{\partial x}(\sinh y)^{1/2}P^m_{-1/2+ix}(\cosh y).$$

If we now substitute from (2.4.4) and (2.4.5) and use (2.4.1(7)) with two terms plus remainder (using (2.4.2(6)) and (2.4.2(10))), this expression can be written as

$$\frac{\partial}{\partial x} K^m(x, y) = (\tfrac{1}{2}-m)x^{-1/2}y^{1/2}J_{-m}(xy) - x^{1/2}y^{3/2}J_{1-m}(xy)$$
$$+ 2(\tfrac{1}{2}-m)\varphi_1(y)\{x^{-3/2}y^{3/2}J_{1-m}(xy) - x^{-1/2}y^{5/2}J_{2-m}(xy)\}$$
$$+ \frac{\partial}{\partial x}\{\check{k}_m(x)(xy)^{1/2}J_{-m}(xy)\}$$
$$+ \begin{cases} O(x^{-m-5/2}y^{-m+5/2}), & xy \leqq 1, \\ O(x^{-2}y^3), & xy \geqq 1. \end{cases}$$

Differentiation of the first two terms of (2.4.6), using [6, p. 11, 7.2.8(51)], and comparison with this expression, shows

$$(2.4.8) \quad \frac{\partial}{\partial x} F_m(x, y) = \frac{\partial}{\partial x}\{\check{k}_m(x)(xy)^{1/2}J_{-m}(xy)\}$$
$$+ \begin{cases} O(x^{-m-5/2}y^{-m+5/2}), & xy \leqq 1, \\ O(x^{-2}y^3), & xy \geqq 1. \end{cases}$$

Finally, we come to differentiation with respect to y. For our purpose, we only need to use one term plus remainder for K^m. Using (2.4.6) and (2.4.7) we get

(2.4.9) $$K^m(x, y) = (1 + \check{k}_m(x))(xy)^{1/2}J_{-m}(xy) + W_m(x, y)$$

with

$$W_m(x, y) = O(x^{-m+1/2}y^{-m+5/2}), \qquad xy \leqq 1,$$
$$= O(x^{-1}y), \qquad\qquad xy \geqq 1.$$

Next, we differentiate $K^m(x, y)$, making use of the recurrence relation [5, p. 161, 3.8(9)] to arrive at

$$\frac{\partial}{\partial y} K^m(x, y) = (\tfrac{1}{2} - m) \coth y \, K^m(x, y) - \{(m - \tfrac{1}{2})^2 + x^2\} \frac{k_m(x)}{k_{m-1}(x)} K^{m-1}(x, y).$$

Noting that $k_m(x)/k_{m-1}(x) = \{(m - \tfrac{1}{2})^2 + x^2\}^{-1/2}$, we write this as

(2.4.10) $$\frac{\partial}{\partial y} K_m(x, y) = (\tfrac{1}{2} - m) \coth y \, K^m(x, y) - ((m - \tfrac{1}{2})^2 + x^2)^{1/2} K^{m-1}(x, y).$$

Moreover, since

$$\frac{\partial}{\partial y} (xy)^{1/2}J_{-m}(xy) = (\tfrac{1}{2} - m)x^{1/2}y^{-1/2}J_{-m}(xy) - x^{3/2}y^{1/2}J_{1-m}(xy),$$

termwise differentiation of (2.4.9) yields

$$\frac{\partial}{\partial y} K^m(x, y) = (\tfrac{1}{2} - m)(1 + \check{k}_m(x))x^{1/2}y^{-1/2}J_{-m}(xy)$$
$$- (1 + \check{k}_m(x))x^{3/2}y^{1/2}J_{1-m}(xy) + \frac{\partial}{\partial y} W_m(x, y).$$

Substitution of (2.4.9), with m and with $m-1$, into (2.4.10) shows that

$$\frac{\partial}{\partial y} K^m(x, y) = (\tfrac{1}{2} - m) \coth y \, \{y(1 + \check{k}_m(x))x^{1/2}y^{-1/2}J_{-m}(xy)\}$$
$$+ (\tfrac{1}{2} - m) \coth y \, W_m(x, y)$$
$$- ((m - \tfrac{1}{2})^2 + x^2)^{1/2}\{x^{-1}(1 + \check{k}_{m-1}(x))x^{3/2}y^{1/2}J_{1-m}(xy)\}$$
$$- ((m - \tfrac{1}{2})^2 + x^2)^{1/2}W_{m-1}(x, y).$$

Now

$$y \coth y - 1 = O(y);$$
$$(((m - \tfrac{1}{2})^2 + x^2)^{1/2}/x)(1 + k_{m-1}(x)) - (1 + k_m(x)) = O(x^{-2}).$$

Consequently, by subtracting the two previous expressions for $\partial K^m(x, y)/\partial y$, we arrive at

(2.4.11) $$\frac{\partial}{\partial y} W_m(x, y) = O(x^{1/2-m}y^{1/2-m}), \qquad xy \leqq 1,$$
$$= O(1), \qquad\qquad xy \geqq 1.$$

3. **Transplantation theorems.** We now have enough information to prove our transplantation theorems. The first one will be enough to give a mean convergence theorem. It follows:

THEOREM 3. *Let $G \in L^1[0, \infty)$. Then if $m \leq \frac{1}{2}$ or $m = 1, 2, \ldots,$*

$$(3.1) \qquad H^m(G; x) = \int_0^\infty G(y) K^m(x, y) \, dy = H^m(x)$$

exists as a Lebesgue integral; moreover, if $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$, there is a constant $A_{p,\alpha}^m$ such that

$$\|H^m\|_{p,\alpha} \leq A_{p,\alpha}^m \|H^{1/2}\|_{p,\alpha}.$$

By (1.9) it suffices to prove this theorem for $m < \frac{1}{2}$. By (2.1.6), (2.2.3), (2.3.4) and (2.4.4), $K^m(x, y)$ is a bounded function of y for each x. Since $G \in L^1$, the existence of (3.1), for $m < \frac{1}{2}$, follows. Thus (3.1) exists.

To prove the required norm inequality, we recall, from elementary Fourier analysis, that for $G \in L^1$,

$$(2/\pi)^{1/2} \int_0^\infty H^{1/2}(u) e^{-\varepsilon u} \cos uy \, du \xrightarrow{L^1} G(y) \quad \text{as } \varepsilon \downarrow 0.$$

Since $K^m(x, y) \in L^\infty(dy)$, we have

$$H^m(x) = \lim_{\varepsilon \downarrow 0} \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty K^m(x, y) \int_0^\infty H^{1/2}(u) e^{-\varepsilon u} \cos uy \, du \, dy.$$

Hence, by Fatou's lemma,

$$\|H^m\|_{p,\alpha} \leq \lim_{\varepsilon \downarrow 0} \left(\frac{2}{\pi}\right)^{1/2} \left\| \int_0^\infty K^m(x, y) \int_0^\infty H^{1/2}(u) e^{-\varepsilon u} \cos uy \, du \, dy \right\|_{p,\alpha}.$$

Setting $H(u) = H^{1/2}(u) e^{-\varepsilon u}$ and noting that $|H(u)| \leq |H^{1/2}(u)|$, we see that Theorem 3 is implied by the following:

THEOREM 4. *Let $m < \frac{1}{2}$, with p and α as in Theorem 3. Then there is a constant $A_{p,\alpha}^m$ such that for $H \in L^{p,\alpha} \cap L^1$, $\hat{H} \in L^1$,*

$$\left\| \int_0^\infty K^m(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

(Here, as in the rest of this paper, "^" stands for the Fourier cosine transform.) Theorem 4 (as well as Theorem 7) is analogous to results of Askey and Wainger (see [3]; there all parameters must be at least zero).

For the proof of Theorem 4, we will write

$$\int_0^\infty K^m(x, y) \hat{H}(y) \, dy = \int_0^1 K^m(x, y) \hat{H}(y) \, dy + \int_1^\infty K^m(x, y) \hat{H}(y) \, dy.$$

The expansions we obtained in §2 for $K^m(x, y)$ were in terms of $\sin xy$, $\cos xy$ and

$J_{-m}(xy)$. In view of Theorem 1, we will try to show that the coefficients of $\sin xy$ and $\cos xy$ form acceptable Fourier multipliers. For expansions involving $J_{-m}(xy)$, we will use Guy's transplantation result (Theorem 2) and then Theorem 1.

We begin with $\int_1^\infty K^m(x, y)\hat{H}(y)\, dy$. Let

(3.2.1) $$\alpha_m = \cos (2m-1)\pi/4, \qquad \beta_m = \sin (2m-1)\pi/4.$$

Then

(3.2.2)
$$\cos (xy+(2m-1)\pi/4) = \alpha_m \cos xy - \beta_m \sin xy,$$
$$\sin (xy+(2m-1)\pi/4) = \beta_m \cos xy + \alpha_m \sin xy.$$

Let $x \geq 1$. The sum of the two leading terms of (2.1.6) is

$$-\left(\frac{2}{\pi}\right)^{1/2}\left(\beta_m + \alpha_m \frac{m^2 - \frac{1}{4}}{2x}\coth y\right)\sin xy + \left(\frac{2}{\pi}\right)^{1/2}\left(\alpha_m - \beta_m \frac{m^2-\frac{1}{4}}{2x}\right)\cos xy.$$

Now $d(\coth y)/dy = -\operatorname{csch}^2 y \in L^1[1, \infty)$ and hence, by Theorem 1 and the remark following it, the functions equal to the above coefficients of $\sin xy$ or of $\cos xy$ for $y \geq 1$, and equal to zero for $0 \leq y \leq 1$, form satisfactory multipliers. The next two terms of (2.1.6) have for coefficients of $\sin xy$ or of $\cos xy$ terms of the type $\psi_1(x) + \psi_2(x)/(e^{2y}-1)$, with ψ_1, ψ_2 bounded. The same results again apply. We may now conclude that

$$\left\{\int_1^\infty \left|\int_1^\infty \tilde{K}^m(x, y)\hat{H}(y)\, dy\right|^p x^{\alpha p}\, dx\right\}^{1/p} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}$$

where $\tilde{K}^m$ denotes the sum of the first four terms in the expansion (2.1.6).

We must now deal with the last two error terms. The situation is now somewhat different, inasmuch as instead of actual quotients for coefficients, we have terms which are merely bounded by quotients of the right type. We look at

$$\int_1^\infty \cos \left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y)\hat{H}(y)\, dy$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\int_1^\infty \cos \left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y)\int_0^\infty H(u)\cos uy\, du\, dy$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty H(u)\int_1^\infty \cos \left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y)\cos uy\, dy\, du.$$

(Fubini's theorem applies since $D_1 \in L^1(dy)$.) Then we have

$$\left|\int_1^\infty \cos \left(xy + [2m-1]\frac{\pi}{4}\right)D_1(x, y)\cos uy\, dy\right| \leq \int_1^\infty |D_1(x, y)|\, dy = O\left(\frac{1}{x}\right).$$

Also,

$$\int_1^\infty \cos\left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y) \cos uy \, dy$$

$$= \cos\left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y) \frac{\sin uy}{u} \Big|_{y=1}^{y=\infty}$$

$$- \frac{1}{u} \int_1^\infty \left\{ -x \sin\left(xy + [2m-1]\frac{\pi}{4}\right) \cdot O(x^{-2}e^{-4y}) + O(x^{-2}e^{-4y}) \right\} \sin uy \, dy$$

$$= O\left(\frac{1}{u}\right).$$

Now if $x \geq u$, $O(1/x) = O(1/(x+u))$, and if $x \leq u$, $O(1/u) = O(1/(x+u))$. Hence, we obtain

$$\left\| \int_1^\infty \cos\left(xy + [2m-1]\frac{\pi}{4}\right) D_1(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha}$$

$$\leq A_{p,\alpha}^m \left\| \int_0^\infty \frac{|H(u)|}{u+x} \, du \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

Similarly,

$$\left\| \int_1^\infty \sin\left(xy + [2m-1]\frac{\pi}{4}\right) D_2(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

This completes treatment of the case $x \geq 1$, giving

$$(3.3) \qquad \left\| \int_1^\infty K^m(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha; \{x:x \geq 1\}} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

(The actual value of $A_{p,\alpha}^m$ has already changed several times; it will again.)

Now we will let $x \leq 1$; the relevant expansion is (2.2.3). The leading terms are $(\pm(2/\pi)^{1/2} + h_2(x)) \sin xy + h_1(x) \cos xy$, and $\pm(2/\pi)^{1/2} + h_2(x)$, $h_1(x)$ are bounded, so they are permissible Fourier multipliers.

For the first remaining error term, we consider

$$\int_1^\infty \cos xy \, s_1(x, y) \hat{H}(y) \, dy = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty H(u) \int_1^\infty \cos xy \, s_1(x, y) \cos uy \, dy \, du$$

by Fubini's theorem, applicable since $s_1 \in L^1(dy)$. Now if $u \leq 1$, we note that

$$\left| \int_1^\infty \cos xy \, s_1(x, y) \cos uy \, dy \right| \leq M < \infty.$$

If, on the other hand, $u \geq 1$, then we integrate by parts:

$$\int_1^\infty \cos xy \, s_1(x, y) \cos uy \, dy = \cos xy \, s_1(x, y) \frac{\sin uy}{u} \Big|_{y=1}^{y=\infty}$$

$$- \frac{1}{u} \int_1^\infty \left\{ -x \sin xy \, s_1(x, y) + \cos xy \frac{\partial}{\partial y} s_1(x, y) \right\} \sin uy \, dy = O\left(\frac{1}{u}\right)$$

since $x \leq 1$. We obtain

$$\int_1^\infty \cos xy \, s_1(x, y) \hat{H}(y) \, dy = \int_0^1 H(u) \cdot O(1) \, du + \int_1^\infty H(u) O\left(\frac{1}{u}\right) du.$$

Similarly,

$$\int_1^\infty \sin xy \, s_2(x, y) \hat{H}(y) \, dy = \int_0^1 H(u) \cdot O(1) \, du + \int_1^\infty H(u) O\left(\frac{1}{u}\right) du.$$

The remark following Theorem 2 completes the case $x \leq 1$, $y \geq 1$. Combining these two facts with our remarks on the relevant multipliers and with (3.3), we conclude that the mapping

$$H \to \int_1^\infty K^m(x, y) \hat{H}(y) \, dy$$

is a bounded transformation of $L^1 \cap L^{p,\alpha}[0, \infty) \subset L^{p,\alpha}$.

We now deal with $\int_0^1 K^m(x, y) \hat{H}(y) \, dy$. We address ourselves first to the case where $x \leq 1$ and use (2.3.4); evidently an appeal to the Fourier multiplier theorem would not help. We have, instead,

$$\int_0^1 K^m(x, y) \hat{H}(y) \, dy = \left(\frac{2}{\pi}\right)^{1/2} \int_0^1 K^m(x, y) \int_0^\infty H(u) \cos uy \, du \, dy$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty H(u) \int_0^1 K^m(x, y) \cos uy \, dy \, du$$

since $K^m \in L^1([0, 1]; dy)$. From (2.3.4), $K^m(x, y) \leq M$. Hence,

$$\int_0^1 H(u) \int_0^1 K^m(x, y) \cos uy \, dy \, du = \int_0^1 H(u) \cdot O(1) \, du.$$

Also, from (2.3.4), $\int_0^1 |\partial K^m(x, y)/\partial y| \, dy < M$, so that, by integrating by parts, $\int_0^1 K^m(x, y) \cos uy \, dy = O(1/u)$ for $u \geq 1$ and

$$\int_1^\infty H(u) \int_0^1 K^m(x, y) \cos uy \, dy \, du = \int_1^\infty H(u) \cdot O\left(\frac{1}{u}\right) du,$$

so we obtain, as before,

$$(3.4) \qquad \left\{\int_0^1 \left|\int_0^1 K^m(x, y) \hat{H}(y) \, dy\right|^p x^{\alpha p} \, dx\right\}^{1/p} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

We proceed next to the remaining situation: $x \geq 1$, $y \leq 1$. Our principal tool will be Guy's transplantation theorem for the Hankel transform (Theorem 2). It will be sufficient to use the short expansion (see (2.4.9) and (2.4.11)):

$$(3.5) \qquad K^m(x, y) = (1 + \check{k}_m(x))(xy)^{1/2} J_{-m}(xy) + W_m(x, y),$$

where

$$\tilde{k}_m(x) = O(x^{-2});$$
$$W_m(x, y) = O(x^{-m+1/2}y^{-m+5/2}), \qquad xy \leq 1,$$
$$= O(x^{-1}y), \qquad xy \geq 1;$$
$$\partial W_m(x, y)/\partial y = O(x^{-m+1/2}y^{-m+1/2}), \qquad xy \leq 1,$$
$$= O(1), \qquad xy \geq 1.$$

First,

$$\left\| \int_0^1 (1+\tilde{k}_m(x))(xy)^{1/2}J_{-m}(x, y) \cdot \hat{H}(y)\, dy \right\|_{p,\alpha;\{x:x\geq 1\}}$$
$$\leq C \left\| \int_0^1 (xy)^{1/2}J_{-m}(xy)\hat{H}(y)\, dy \right\|_{p,\alpha}$$
$$\leq A_{p,\alpha}^m \left\| \int_0^1 \hat{H}(y)\cos uy\, dy \right\|_{p,\alpha}$$

by Guy's result, with the function g replaced by the restriction of $\hat{H}$ to $[0, 1]$. The mean convergence result (for partial integrals) for the cosine transform implies that the last norm is bounded by a multiple of $\|H\|_{p,\alpha}$; hence

$$(3.6) \qquad \left\| \int_0^1 (1+\tilde{k}_m(x))(xy)^{1/2}J_{-m}(xy)\hat{H}(y)\, dy \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

To dispose of the error term, we note that if $xy \leq 1$, then

$$x^{-m+1/2}y^{-m+5/2} = x^{-m+1/2}y^{-m+1/2}y^2 = O(yx^{-1}).$$

Therefore $W_m(x, y) = O(yx^{-1})$, $x \geq 1$, $y \leq 1$ and so

$$\int_0^1 W_m(x, y)\cos uy\, dy = O\left(\frac{1}{x}\right).$$

Moreover, since $m < \frac{1}{2}$, $W_m(x, y) = O(1)$, and $\partial W_m(x, y)/\partial y = O(1)$, integration by parts gives

$$\int_0^1 W_m(x, y)\cos uy\, dy = O\left(\frac{1}{u}\right).$$

These two integral estimates show that

$$\int_0^1 W_m(x, y)\hat{H}(y)\, dy = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty H(u) \int_0^1 W_m(x, y)\cos uy\, dy\, du$$
$$= \int_0^\infty H(u)O\left(\frac{1}{u+x}\right) du.$$

(Fubini's theorem applies because $W_m(x, y) = O(1)$.) Using this in conjunction with (3.6) and (3.4) we see that the mapping

$$H \to \int_0^1 K^m(x, y)\hat{H}(y)\, dy$$

is a bounded transformation of $L^1 \cap L^{p,\alpha}[0, \infty) \subset L^{k,\alpha}$, as desired.

COROLLARY 5. *Let p, α, m be as in Theorem 3. Then if $H \in L^1 \cap L^{p,\alpha}$, we can define a function $S_1^m H$ by*

$$(S_1^m H)(x) = \underset{N \to \infty}{\text{l.i.m.}}^{p,\alpha} \int_0^N K^m(x, y) \hat{H}(y) \, dy;$$

moreover

$$\|S_1^m H\|_{p,\alpha} \leq A_{p,\alpha}^m \|H\|_{p,\alpha}.$$

Proof. Let $0 \leq N_1 < N_2 < \infty$ and put the $G(y)$ of Theorem 3 equal to zero off $[N_1, N_2]$, and equal to $\hat{H}(y)$ on $[N_1, N_2]$. Then $G \in L^1[0, \infty)$ so

$$\left\| \int_{N_1}^{N_2} K^m(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha} \leq A_{p,\alpha}^m \left\| \int_{N_1}^{N_2} \cos xy \hat{H}(y) \, dy \right\|.$$

The corollary is known to be true for $m = \frac{1}{2}$; thus $\left\| \int_{N_1}^{N_2} K^m(x, y) \hat{H}(y) \, dy \right\|_{p,\alpha} \to 0$ as $N_1, N_2 \to \infty$ and $S_1^m H$ exists in $L^{p,\alpha}$. Taking $N_1 = 0$ then gives the desired norm inequality.

Our second transplantation theorem (which, in turn, implies a projection theorem) is as follows:

THEOREM 6. *Let $F \in L^1[0, \infty)$. Then if $m \leq \frac{1}{2}$ or $m = 1, 2, \ldots$,*

$$(3.7) \qquad G^m(F; y) = \int_0^\infty F(x) K^m(x, y) \, dx = G^m(y)$$

exists as a Lebesgue integral; moreover if $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$ there is a constant $A_{p,\alpha}^m$ such that

$$\|G^m\|_{p,\alpha} \leq A_{p,\alpha}^m \|G^{1/2}\|_{p,\alpha}.$$

It is enough to prove Theorem 6 for $m < \frac{1}{2}$. By (2.1.6), (2.2.3), (2.3.4) and (2.4.9), $K^m(x, y)$ is bounded in x for each y. Since $F \in L^1$, $G^m(F; y)$ exists.

As with our first transplantation result, we will prove the following and it will imply Theorem 6:

THEOREM 7. *Let $m < \frac{1}{2}$, with p and α as in Theorem 6. Then there is a constant $A_{p,\alpha}^m$ such that for $G \in L^{p,\alpha} \cap L^1$, $\hat{G} \in L^1$,*

$$\left\| \int_0^\infty K^m(x, y) \hat{G}(x) \, dx \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

Again we write

$$\int_0^\infty K^m(x, y) \hat{G}(x) \, dx = \int_0^1 K^m(x, y) \hat{G}(x) \, dx + \int_1^\infty K^m(x, y) \hat{G}(x) \, dx$$

and we will begin by treating $\int_0^1 K^m(x, y) \hat{G}(x) \, dx$. As a start, we let $y \leq 1$ and use

(2.3.4). Since our expansion is not in terms of sines and cosines, we will not use multiplier theorems. Instead, we have

$$\int_0^1 K^m(x, y)\hat{G}(x)\, dx = \left(\frac{2}{\pi}\right)^{1/2} \int_0^1 K^m(x, y) \int_0^\infty G(u) \cos ux\, du\, dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty G(u) \int_0^1 K^m(x, y) \cos ux\, dx\, du;$$

the order of integration is interchangeable because $|K^m(x, y)| \leq C$. Now for $u \leq 1$, we note that $\int_0^1 K^m(x, y) \cos ux\, dx = O(1)$. For $u \geq 1$, we use the fact that $|K^m(x, y)| \leq C$ and $|\partial K^m(x, y)/\partial x| \leq C$, integrate by parts, and get

$$\int_0^1 K^m(x, y) \cos ux\, dx = O(1/u).$$

We conclude that

$$(3.8) \qquad \left\{ \int_0^1 \left| \int_0^1 K^m(x, y)\hat{G}(x)\, dx \right|^p y^{\alpha p}\, dy \right\}^{1/p} \leq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

Next, we let $y \geq 1$. The estimates we need are in (2.2.3). We write this as

$$K^m(x, y) = (\pm (2/\pi)^{1/2} + h_2(x)) \sin xy$$

$$+ h_1(x) \cos xy + s_1(x, y) \cos xy + s_2(x, y) \sin xy.$$

The coefficients $\pm (2/\pi)^{1/2} + h_2(x)$ and $h_1(x)$ satisfy the requirements for multipliers given in Theorem 1.

To deal with the error terms, we note that $s_j(x, y), \partial s_j(x, y)/\partial x = O(e^{-2y})$. Now

$$\int_0^1 \cos xy\, s_1(x, y)\hat{G}(x)\, dx = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty G(u) \int_0^1 s_1(x, y) \cos xy \cos ux\, dx\, du$$

since $\int_0^1 |s_1(x, y)|\, dx = O(1)$. Moreover, $|s_1(x, y)| < Me^{-2y}$ so that

$$\left| \int_0^1 s_1(x, y) \cos xy \cos ux\, dx \right| \leq Me^{-2y}.$$

Also, integrating by parts and using the condition on $\partial s_1(x, y)/\partial x$ gives $\left| \int_0^1 s_1(x, y) \cos xy \cos ux\, dx \right| \leq (M/u)e^{-2y}$. Now letting $\omega(u) = M$ on $0 \leq u \leq 1$, $\omega(u) = M/u$ on $u \geq 1$, and $1/p + 1/q = 1$, $\omega \in L^{q,-\alpha}[0, \infty)$, the dual of $L^{p,\alpha}$. Hence,

$$\left| \int_0^1 \cos xy\, s_1(x, y)\hat{G}(x)\, dx \right| \leq e^{-2y}\|G\|_{p,\alpha}\|\omega\|_{q,-\alpha}.$$

Clearly, $e^{-2y} \in L^{p,\alpha}[1, \infty)$ and so

$$\left\| \int_0^1 \cos xy\, s_1(x, y)\hat{G}(x)\, dx \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

The same argument works for the term $\sin xy \, s_2(x, y)$. Recalling (3.8), we conclude that

$$(3.9) \qquad \left\| \int_0^1 K^m(x, y)\hat{G}(x)\,dx \right\|_{p,\alpha} \leqq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

We will now deal with $\int_1^\infty K^m(x, y)\hat{G}(x)\,dx$, starting with $y \geqq 1$. According to (2.1.6), (3.2.1) and (3.2.2)

$$K^m(x, y) = -\left[\left(\frac{2}{\pi}\right)^{1/2}\beta_m + \left(\frac{2}{\pi}\right)^{1/2}\alpha_m \frac{m^2 - \frac{1}{4}}{2x}\coth y + \beta_m R_1(x, y) - \alpha_m R_2(x, y)\right]\sin xy$$

$$+ \left[\left(\frac{2}{\pi}\right)^{1/2}\alpha_m - \left(\frac{2}{\pi}\right)^{1/2}\beta_m \frac{m^2 - \frac{1}{4}}{2x}\coth y + \alpha_m R_1(x, y) + \beta_m R_2(x, y)\right]\cos xy$$

$$+ D_1(x, y)\cos(xy + [2m-1]\pi/4) + D_2(x, y)\sin(xy + [2m-1]\pi/4).$$

The coefficients of $\sin xy$ and of $\cos xy$ that appear in this expression satisfy the sufficient conditions given in and right after Theorem 1 for Fourier multipliers. The last two terms we treat differently.

$$\int_1^\infty \cos\left(xy + [2m-1]\frac{\pi}{4}\right)D_1(x, y)\hat{G}(x)\,dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty G(u)\int_1^\infty \cos\left(xy + [2m-1]\frac{\pi}{4}\right)D_1(x, y)\cos ux\,dx\,du,$$

since $\int_1^\infty |D_1(x, y)|\,dx = O(1)$. We also have

$$\left|\int_1^\infty \cos\left(xy + [2m-1]\frac{\pi}{4}\right)D_1(x, y)\cos ux\,dx\right| \leqq Me^{-4y}, \qquad u \leqq 1,$$

$$\leqq (M/u)e^{-4y}, \qquad u \geqq 1.$$

The same argument we used for the remainder terms when $x \leqq 1$, $y \geqq 1$ works here as well as for the term $\sin(xy + [2m-1]\pi/4)D_2(x, y)$. We have

$$(3.10) \qquad \left\{\int_1^\infty \left|\int_1^\infty K^m(x, y)\hat{G}(x)\,dx\right|^p y^{\alpha p}\,dy\right\}^{1/p} \leqq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

Now we assume $x \geqq 1$, $y \leqq 1$. The expansion we want is (2.4.6) with estimates (2.4.7) and (2.4.8):

$$K^m(x, y) = (1 + \check{k}_m(x))(xy)^{1/2}J_{-m}(xy) + 2(\tfrac{1}{2} - m)\varphi_1(y)y^{3/2}x^{-1/2}J_{1-m}(xy) + E(x, y)$$

with

$$\check{k}_m(x) = O(x^{-2}), \qquad \check{k}_m'(x) = O(x^{-3});$$

$$E(x, y) = O(x^{-m-3/2}y^{-m+5/2}), \qquad xy \leqq 1,$$

$$= O(x^{-2}y^2), \qquad xy \geqq 1;$$

and

$$\frac{\partial}{\partial x}E(x, y) = O(x^{-m-5/2}y^{-m+5/2}), \qquad xy \leqq 1,$$

$$= O(x^{-2}y^3), \qquad xy \geqq 1.$$

We replace the "g" of Guy's result (see Theorem 2) by the restriction to $[1, \infty)$ of $(1+\check{k}_m(x))\hat{G}(x)$ to get

$$\left\| \int_1^\infty (1+\check{k}_m(x))\hat{G}(x)(xy)^{1/2}J_{-m}(xy)\,dx \right\|_{p,\alpha}$$

$$\leq A_{p,\alpha}^m \left\| \int_1^\infty (1+\check{k}_m(x))\hat{G}(x)\cos xy\,dx \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|G\|_{p,\alpha};$$

the second inequality arises from the fact that $1+\check{k}_m(x)$ satisfies the hypothesis we have given for Fourier multipliers. For the second term, $\varphi_1(y)=O(1)$ and

$$\left\| \int_1^\infty 2(\tfrac{1}{2}-m)\varphi_1(y)y^{3/2}x^{-1/2}\hat{G}(x)J_{1-m}(xy)\,dx \right\|_{p,\alpha}$$

$$\leq C \left\| \int_1^\infty x^{-1}\hat{G}(x)(xy)^{1/2}J_{1-m}(xy)\,dx \right\|_{p,\alpha} \leq A_{p,\alpha}^m \left\| \int_1^\infty x^{-1}\hat{G}(x)\cos xy\,dx \right\|_{p,\alpha}$$

$$\leq A_{p,\alpha}^m \|G\|_{p,\alpha};$$

the second inequality comes from Guy's result, with his "g" equal to $x^{-1}\hat{G}(x)$ restricted to $[1, \infty)$. The third comes from treating x^{-1} on $[1, \infty)$ as a Fourier multiplier.

We have to dispose of the error term $E(x, y)$. Writing

$$\int_1^\infty E(x, y)\hat{G}(x)\,dx = \left\{ \int_1^{1/y} + \int_{1/y}^\infty \right\} E(x, y)\hat{G}(x)\,dx,$$

we will treat $\int_1^{1/y} E(x, y)\hat{G}(x)\,dx$ first. Evidently $E(x, y)$ is bounded in x and y so we can interchange the order of integration and

$$\int_1^{1/y} E(x, y)\hat{G}(x) = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty G(u) \int_1^{1/y} E(x, y)\cos ux\,dx\,du.$$

We have

$$\left| \int_1^{1/y} E(x, y)\cos ux\,dx \right| \leq \int_1^{1/y} O(y^{-m+5/2}x^{-m-3/2})\,dx \leq M.$$

Also, by parts,

$$\left| \int_1^{1/y} E(x, y)\cos ux\,dx \right| \leq \frac{|E(x, y)|}{u}\Big|_{x=1}^{x=1/y} + \frac{1}{u}\int_1^{1/y} \left| \frac{\partial}{\partial x} E(x, y) \right| \cdot |\sin ux|\,dx$$

$$\leq \frac{M}{u}\int_1^{1/y} O(x^{-m-5/2}y^{-m+5/2})\,dx \leq \frac{M}{u}.$$

Once again, the remark following Theorem 2 implies that

$$\left\| \int_1^{1/y} E(x, y)\hat{G}(x)\,dx \right\|_{p,\alpha} \leq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

For the integral over $[1/y, \infty)$, we note that $\int_{1/y}^{\infty} |E(x, y)|\, dx = O(1)$ and so, by Fubini's theorem,

$$\int_{1/y}^{\infty} E(x, y)\hat{G}(x)\, dx = \left(\frac{2}{\pi}\right)^{1/2} \int_{0}^{\infty} G(u) \int_{1/y}^{\infty} E(x, y) \cos ux\, dx\, du.$$

First of all, since $E(x, y) = O(x^{-2}y^2)$ on the range in question,

$$\left| \int_{1/y}^{\infty} E(x, y) \cos ux\, dx \right| = O\left\{ y^2 \int_{1/y}^{\infty} x^{-2}\, dx \right\} = O(y^3) \leqq M.$$

Also, by parts, and since $\partial E(x, y)/\partial x = O(x^{-2}y^3)$, we get $\left| \int_{1/y}^{\infty} E(x, y) \cos ux\, dx \right|$ $\leqq M/u$ and the argument we just used works again.

Combining our results for this region ($x \geqq 1$, $y \leqq 1$) with (3.9) and (3.10), we now obtain the boundedness of the transformation

$$G \to \int_{0}^{\infty} K^m(x, y)\hat{G}(x)\, dx$$

of $L^1 \cap L^{p,\alpha}[0, \infty) \subset L^{p,\alpha}$.

COROLLARY 8. *Let p, α, m be as in Theorem 6. Then if $G \in L^1 \cap L^{p,\alpha}$, we can define a function $S_2^m G$ by*

$$(S_2^m G)(x) = \underset{N \to \infty}{\text{l.i.m.}} \int_{0}^{N} K^m(x, y)\hat{G}(x)\, dx;$$

moreover

$$\|S_2^m G\|_{p,\alpha} \leqq A_{p,\alpha}^m \|G\|_{p,\alpha}.$$

We omit the proof; it parallels that of Corollary 5.

4. **Applications.** Typical consequences of transplantation theorems are results about multipliers. To get these results, however, we need to be able to map from $G^{1/2}$ to G^m *and back*, and from $H^{1/2}$ to H^m and back. At the least we have to obtain sharp duals for Corollaries 5 and 8. Preliminary to this, we will obtain two mean convergence results, stated in one lemma. (The second part might be considered as a projection theorem.)

LEMMA 9. *Let* $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $m \leqq \frac{1}{2}$ *or* $m = 1, 2, \ldots$. *Then if* $f \in L^1 \cap L^{p,\alpha}$,

$$\underset{N \to \infty}{\text{l.i.m.}^{p,\alpha}} \int_{0}^{N} K^m(u, y)G^m(f; y)\, dy = f(u)$$

and

$$\underset{N \to \infty}{\text{l.i.m.}^{p,\alpha}} \int_{0}^{N} K^m(y, u)H^m(f; y)\, dy = f(u).$$

Proof. We will only prove the first statement; the proof of the second proceeds along similar lines. For $f \in L^1 \cap L^{p,\alpha}$, $|G^m(f; y)| \leqq \sup_{x,y} |K^m(x, y)| \cdot \|f\|_1 < M \|f\|_1$,

from the asymptotic expansions. Hence, $G^m(f; y)$ multiplied by the characteristic function of $[0, N]$ is in $L^1[0, \infty)$, so by Theorem 3,

$$\left\| \int_0^N K^m(u, y) G^m(f; y)\, dy \right\|_{p,\alpha} \leqq A_{p,\alpha}^m \left\| \int_0^N \cos uy\, G^m(f; y)\, dy \right\|_{p,\alpha}.$$

Take q so that $1/p + 1/q = 1$; the dual of $L^{p,\alpha}$ is $L^{q,-\alpha}$. Since the last norm equals $\sup_{k < \infty} \| \int_0^N \cos uy\, G^m(f; y)\, dy \|_{p,\alpha;\, \{u:u \leqq k\}}$, it is bounded by

$$\sup_{\|h\|_{q,-\alpha} \leqq 1;\, h \in L^1} \left| \int_0^\infty h(u) \int_0^N \cos uy\, G^m(f; y)\, dy\, du \right|.$$

For $h \in L^1$, we may use Fubini's theorem and

$$\left| \int_0^\infty h(u) \int_0^N \cos uy\, G^m(f; y)\, dy\, du \right| = \left(\frac{\pi}{2}\right)^{1/2} \left| \int_0^N G^m(f; y) \hat{h}(y)\, dy \right|$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \left| \int_0^N \int_0^\infty f(x) K^m(x, y) \hat{h}(y)\, dx\, dy \right|$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \left| \int_0^\infty f(x) \int_0^N K^m(x, y) \hat{h}(y)\, dy\, dx \right|,$$

since $f \in L^1[0, \infty)$ and $\hat{h} \in L^1[0, N]$.

We apply Theorem 3 to $\hat{h}$ multiplied by the characteristic function of $[0, N]$, since $-1/q < -\alpha < 1 - 1/q$, and

$$\left\| \int_0^N K^m(x, y) \hat{h}(y)\, dy \right\|_{q,-\alpha} \leqq A_{q,-\alpha}^m \left\| \int_0^N \cos xy\, \hat{h}(y)\, dy \right\|_{q,-\alpha}$$

$$\leqq A_{q,-\alpha}^m \|h\|_{q,-\alpha} \leqq A_{q,-\alpha};$$

the second inequality arises from mean convergence results for the cosine transform. Hölder's inequality then implies

$$(4.1) \quad \left\| \int_0^N K^m(u, y) G^m(f; y)\, dy \right\|_{p,\alpha} \leqq A_{p,\alpha}^m (\pi/2)^{1/2} A_{q,-\alpha}^m \|f\|_{p,\alpha} = B_{p,\alpha}^m \|f\|_{p,\alpha}.$$

This inequality shows that to get mean convergence for $f \in L^1 \cap L^{p,\alpha}$ it is enough to let f belong to a dense (in (p, α) norm) subspace of this intersection. We will let $f \in C_c^\infty(0, \infty) = \{f :$ support of f is a compact subset of $(0, \infty)$ and f has infinitely many derivatives$\}$. Then by [19]

$$f(u) = \lim_{N \to \infty} \int_0^N K^m(u, y) G^m(f; y)\, dy$$

for $u \geqq 0$. Let $(r_N f)(u) = \int_0^N K^m(u, y) G^m(f; y)\, dy$. We will show $r_N f \xrightarrow{L^{p,\alpha}} f$. To do this we have to show that $r_N f$ is bounded by a fixed function in $L^{p,\alpha}$ (and Lebesgue's Dominated Convergence Theorem completes the argument).

First, G^m is bounded (by $\sup_{x,y} |K^m(x, y)| \cdot \|f\|_1$). Hence, $G^m \in L^1[0, 1]$. If $y \geq 1$, we integrate by parts twice to get

$$G^m(f; y) = \int_{\text{support } f} f(x) K^m(x, y) \, dx$$

$$= \int_{\text{support } f} f''(x) \int_\delta^x \int_\delta^{x_1} K^m(x_2, y) \, dx_2 \, dx_1 \, dx,$$

where $\delta > 0$ depends on f. Then, from (2.1.6) and (2.2.3),

$$G^m(f; y) = O(y^{-2}) \in L^1[1, \infty).$$

Consequently, $\|G^m\|_1 < \infty$. Hence, on $0 \leq u \leq 1$,

$$|(r_N f)(u)| \leq \sup_{x,y} |K^m(x, y)| \cdot \|G^m\|_1 < \infty.$$

On $u \geq 1$, we integrate by parts and we have

$$(r_N f)(u) = G^m(f; N) \int_0^N K^m(u, t) \, dt - \int_0^N (G^m)'(f; y) \int_0^y K^m(u, t) \, dt \, dy.$$

It is easy to see from (2.1.6), (2.2.3) and (2.4.9) that

$$\left| \int_0^N K^m(u, t) \, dt \right| \leq \frac{C_1}{u} \quad \text{and} \quad \left| \int_0^y K^m(u, t) \, dt \right| \leq \frac{C_2}{u}.$$

Also $G^m(f)$ is bounded so

$$\left| G^m(f; N) \int_0^N K^m(u, t) \, dt \right| \leq \frac{C_3}{u}.$$

(C_1, C_2 and C_3 are independent of N.) We show that $(G^m)' \in L^1[0, \infty)$; this will give $|(r_N f)(u)| \leq C/u$ on $u \geq 1$. We may write

$$(G^m)'(f; y) = \int_{\text{supp } f} x f(x) \frac{1}{x} \frac{\partial}{\partial y} K^m(x, y) \, dx.$$

Clearly $(G^m)' \in L^1[0, 1]$. So let $y \geq 1$. Integrating by parts twice gives

$$(G^m)'(f; y) = \int_{\text{supp } f} \frac{d^2}{dx^2} x f(x) \cdot I^m(x, y) \, dx$$

where

$$I^m(x, y) = \int_\delta^x \int_\delta^{x_1} \frac{1}{t} \frac{\partial}{\partial y} K^m(t, y) \, dt \, dx_1$$

($\delta = \min \{x : f(x) \neq 0\}$). From (2.1.6) and (2.2.3) we see that $I^m(x, y) = O(1/y^2)$ for $y \geq 1$ and $x \in \text{supp} f$. Thus $(G^m)'(f; y) = O(1/y^2)$, $y \geq 1$, so $(G^m)' \in L^1[0, \infty)$. We are now able to conclude that

$$(r_N f)(u) \leq M \|G^m\|_1, \qquad u \leq 1,$$
$$\leq C/u, \qquad u \geq 1.$$

The function on the right is in $L^{p,\alpha}$; call it $s(u)$. Then

$$|(r_N f)(u) - f(u)|^p u^{\alpha p} \leq \{|s(u)| + |f(u)|\}^p u^{\alpha p} \leq 2^p \{|s(u)|^p + |f(u)|^p\} u^{\alpha p}$$

which belongs to $L^1[0, \infty)$. Hence, we conclude from Lebesgue's Dominated Convergence Theorem that $\lim_{N \to \infty} \|\int_0^N K^m(u, y) G^m(f; y) \, dy - f(u)\|_{p,\alpha} = 0$ for $f \in C_c^\infty(0, \infty)$ and hence for $f \in L^1 \cap L^{p,\alpha}$ by (4.1).

We may now prove a theorem which dualizes Theorems 4 and 7. Actually, we will only prove the first part.

THEOREM 10. *Let* $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $m \leq \frac{1}{2}$ *or* $m = 1, 2, \ldots$. *Then there is a constant* $B_{p,\alpha}^m$ *such that, for* $f \in L^1[0, \infty)$ *and* $0 \leq a < b < \infty$,

$$\left\| \int_a^b \cos ux \, G^m(f; u) \, du \right\|_{p,\alpha} \leq B_{p,\alpha}^m \left\| \int_a^b K^m(x, u) G^m(f; u) \, du \right\|_{p,\alpha}$$

and

$$\left\| \int_a^b \cos ux \, H^m(f; u) \, du \right\|_{p,\alpha} \leq B_{p,\alpha}^m \left\| \int_a^b K^m(u, x) H^m(f; u) \, du \right\|_{p,\alpha}.$$

Proof of the first inequality. Let $g(u) = G^m(f; u)$ on $[a, b]$, $g(u) = 0$ off $[a, b]$. Then $g \in L^{p,\alpha} \cap L^1$ and, by the lemma,

$$r_k(u) = \int_0^k K^m(y, u) H^m(g; y) \, dy \to g(u)$$

in $L^{p,\alpha}[a, b]$ and hence in $L^1[a, b]$. So

$$\left\| \int_a^b \cos ux \, G^m(f; u) \, du \right\|_{p,\alpha} = \left\| \lim_{k \to \infty} \int_a^b r_k(u) \cos ux \, du \right\|_{p,\alpha}$$

$$\leq \liminf_{k \to \infty} \left\| \int_a^b r_k(u) \cos ux \, du \right\|_{p,\alpha}$$

by Fatou's lemma. Moreover,

$$\int_a^b r_k(u) \cos ux \, du = \int_a^b \left\{ \int_0^k K^m(y, u) H^m(g; y) \, dy \right\} \cos ux \, du$$

$$= \int_0^k H^m(g; y) \int_a^b K^m(y, u) \cos ux \, du \, dy.$$

So, with $1/p + 1/q = 1$,

$$\left\| \int_a^b r_k(u) \cos ux \, du \right\|_{p,\alpha}$$

$$= \left\| \int_0^k H^m(g; y) \int_a^b K^m(y, u) \cos ux \, du \, dy \right\|_{p,\alpha}$$

$$= \sup_{\|h\|_{q, -\alpha} \leq 1; \, h \in L^1} \left| \int_0^\infty h(x) \int_0^k H^m(g; y) \int_a^b K^m(y, u) \cos ux \, du \, dy \, dx \right|$$

$$= \sup_{\|h\|_{q, -\alpha} \leq 1; \, h \in L^1} \left(\frac{\pi}{2} \right)^{1/2} \left| \int_0^k H^m(g; y) \int_a^b K^m(y, u) \hat{h}(u) \, du \, dy \right|.$$

By Corollary 5, $\|\int_a^b K^m(y, u)\hat{h}(u)\, du\|_{q, -\alpha} \leqq A_{q, -\alpha}\|h\|_{q, -\alpha}$. Thus

$$\left\| \int_a^b r_k(u) \cos ux\, du \right\|_{p,\alpha} \leqq B_{p,\alpha}^m \|H^m(g)\|_{p,\alpha}$$

by Hölder's inequality so that

$$\left\| \int_a^b \cos ux\, G^m(f; u)\, du \right\|_{p,\alpha} \leqq B_{p,\alpha}^m \|H^m(g)\|_{p,\alpha} = B_{p,\alpha}^m \left\| \int_a^b K^m(x, u)G^m(f; u)\, du \right\|_{p,\alpha},$$

as desired.

We include our multiplier results in one theorem; once again, we will only prove the first result.

THEOREM 11. *Let* $|\varphi(t)| \leqq C$, $\int_0^t |s\, d\varphi(s)| \leqq Ct$, $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $m \leqq \frac{1}{2}$ *or* $m = 1, 2, \ldots$. *Then for* $f \in L^1 \cap L^{p,\alpha}[0, \infty)$, *the functions*

$$\underset{N \to \infty}{\text{l.i.m.}}^{p,\alpha} \int_0^N K^m(x, y)\varphi(y)G^m(f; y)\, dy = (T_{1,\varphi}f)(x)$$

and

$$\underset{N \to \infty}{\text{l.i.m.}}^{p,\alpha} \int_0^N K^m(y, x)\varphi(y)H^m(f; y)\, dy = (T_{2,\varphi}f)(x)$$

exist, and for some constant $A_{p,\alpha}^m$

$$\|T_{1,\varphi}f\|_{p,\alpha} \leqq A_{p,\alpha}^m C\|f\|_{p,\alpha}$$

and

$$\|T_{2,\varphi}f\|_{p,\alpha} \leqq A_{p,\alpha}^m C\|f\|_{p,\alpha}.$$

We will need a lemma, the dual of the Fourier multiplier theorem.

LEMMA 12. *Let* $|\varphi(t)| \leqq C$, $\int_0^t |s\varphi(s)| \leqq Ct$, $1 < p < \infty$ *and* $-1/p < \alpha < 1 - 1/p$. *Then for a constant* A, *we have for all* $g \in L^1[0, \infty)$,

$$\left\| \int_0^\infty \cos ty\, \varphi(y)g(y)\, dy \right\|_{p,\alpha} \leqq AC \left\| \int_0^\infty \cos ty\, g(y)\, dy \right\|_{p,\alpha}.$$

Proof of Theorem 11. We write $(T_N f)(x) = \int_0^N K^m(x, y)\varphi(y)G^m(f; y)\, dy$. We will show $\|T_{N_1}f - T_{N_2}f\|_{p,\alpha} \to 0$ as $N_1, N_2 \to \infty$ (which implies $T_{1\,\varphi}$ exists). For $0 \leqq N_1 < N_2 < \infty$,

$$\|T_{N_1}f - T_{N_2}f\|_{p,\alpha} = \left\| \int_{N_1}^{N_2} K^m(x, y)\varphi(y)G^m(f; y)\, dy \right\|_{p,\alpha}$$

and since the characteristic function of $[N_1, N_2]$ times $\varphi(y)G^m(f; y)$ is in $L^1[0, \infty)$ Theorem 3 yields

$$\left\| \int_{N_1}^{N_2} K^m(x, y)\varphi(y)G^m(f; y)\, dy \right\|_{p,\alpha} \leqq A_{p,\alpha}^m \left\| \int_{N_1}^{N_2} \cos xy\, \varphi(y)G^m(f; y)\, dy \right\|_{p,\alpha}.$$

Letting $g(y) = G^m(f; y)$ on $[N_1, N_2]$, $g(y) = 0$ elsewhere, gives, by Lemma 12,

$$\left\| \int_{N_1}^{N_2} \cos xy\, \varphi(y) G^m(f; y)\, dy \right\|_{p,\alpha} = \left\| \int_0^\infty \cos xy\, \varphi(y) g(y)\, dy \right\|_{p,\alpha}$$

$$\leq AC \left\| \int_0^\infty \cos xy\, g(y)\, dy \right\|_{p,\alpha}$$

$$= AC \left\| \int_{N_1}^{N_2} \cos xy\, G^m(f; y)\, dy \right\|_{p,\alpha}$$

$$\leq AC \cdot B_{p,\alpha}^m \left\| \int_{N1}^{N_2} K^m(x, y) G^m(f; y)\, dy \right\|_{p,\alpha}$$

by Theorem 10. By Lemma 9, this last norm approaches zero as $N_1, N_2 \to \infty$, for each $f \in L^1 \cap L^{p,\alpha}$. Hence, $T_{1,\varphi} f$ exists in $L^{p,\alpha}$. With $N_1 = 0$, the preceding inequalities show that

$$\left\| \int_0^{N_2} K^m(x, y)\varphi(y) G^m(f; y)\, dy \right\|_{p,\alpha} \leq ACB_{p,\alpha}^m \left\| \int_0^{N_2} K^m(x, y) G^m(f; y)\, dy \right\|_{p,\alpha}$$

$$\leq ACB_{p,\alpha}^m \|f\|_{p,\alpha},$$

also by Lemma 9. Letting $N_2 \to \infty$ implies that $\|T_{1,\varphi} f\|_{p,\alpha} \leq ACB_{p,\alpha}^m \|f\|_{p,\alpha}$.

Returning briefly to the first mean convergence problem, let us note that its solution depends on an analysis of the integral transform

$$(T_N f)(u) = \int_0^\infty f(x) \int_0^N K^m(x, u) K^m(x, y)\, dy\, dx.$$

By [5, p. 169, 3.12(1)] and [5, p. 161, 3.8(9)], it is easy to see that, for $m < 1$,

$$\int_0^N K^m(x, y) K^m(u, y)\, dy = \frac{-1}{x^2 - u^2} \{ K^m(x, N)((\tfrac{1}{2} - m)^2 + u^2)^{1/2} K^{m-1}(u, N)$$

$$- K^m(u, N)((\tfrac{1}{2} - m)^2 + x^2)^{1/2} K^{m-1}(x, N) \},$$

and that for $m > 1$ the kernel does not exist as a Lebesgue integral. Using the Hilbert transform in conjunction with expansions (2.1.6) and (2.2.3) and the transform induced by the kernel $1/(u+x)$ gives the mean convergence result for $m < 1$.

In the case of the other mean convergence result, we do not now have access to a closed formula for the integral kernel which occurs. If one compares recursion formulas for Bessel functions with those for Legendre functions, however, one can probably make a plausible argument for the failure of the second kind of mean convergence result in Lemma 9 for Legendre functions with $m > \tfrac{1}{2}$, $m \neq 1, 2, \ldots$. Such an argument would lean on the expansions for $x \geq 1$, $y \leq 1$, and would use the failure of (p, α) mean convergence for the Hankel transform with $m > \tfrac{1}{2}$.

Boas [2] has noted the following theorem for the Fourier cosine transform:

THEOREM. *If $g \in L^1$, $\hat{g}(t) \downarrow 0$ as $t \to \infty$ and $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, then $g \in L^{p,\alpha}[0, \infty)$ if and only if $\hat{g} \in L^{p,1-\alpha-2/p}$.*

The roles played by g and $\hat{g}$ can be switched here keeping g in L^1. Moreover, $-1/p < 1 - \alpha - 2/p < 1 - 1/p$ if and only if $-1/p < \alpha < 1 - 1/p$. One can also show, by Lemma 9, that if $\chi_{[0,N]}(x) = $ characteristic function of $[0, N]$, then for $g \in L^1 \cap L^{p,\gamma}$ (usual hypotheses on p and γ) we have for $N \geq N(g)$

$$\|g\|_{p,\gamma} \leq 2A_{p,\gamma}^m \|[\chi_{[0,N]}G^m(g)]^\wedge\|_{p,\gamma}.$$

One can then show the following: For $g \in L^1$, $g(t) \downarrow 0$ as $t \to \infty$, $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$, $G^m g \in L^{p,\alpha}$ if and only if $g \in L^{p,1-\alpha-2/p}$. Recalling the definitions (1.6) and (1.7) of k_m and K^m, we can restate this as

THEOREM 13. *Let* $\int_0^\infty |f(x)|\, dx/k_m(x) < \infty$ *and suppose* $f(x)/k_m(x) \downarrow 0$ *as* $x \to \infty$. *Put* $r(w) = \int_0^\infty f(x)P_{-1/2+ix}^m(w)\, dx$. *Then if* $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $m < \frac{1}{2}$ *or* $m = 1, 2, \ldots,$

$$\int_0^1 |f(x)|^p x^{-\alpha p - 2}\, dx + \int_1^\infty |f(x)|^p x^{mp - \alpha p - p/2 - 2}\, dx \; < \; \infty$$

if and only if

$$\int_1^\infty |r(w)|^p (w^2 - 1)^{p/4 - 1/2} \, [\log(w + (w^2 - 1)^{1/2})]^{\alpha p}\, dw \; < \; \infty.$$

We note that by using H^m in place of G^m, the same conclusions obtain if the hypotheses are replaced by

"$\int_0^\infty |g(\cosh y)|\, (\sinh y)^{1/2}\, dy < \infty$ and $g(\cosh y)(\sinh y)^{1/2} \downarrow 0$ as $y \to \infty$."

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