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A CONSTRUCTION OF LIE ALGEBRAS FROM A CLASS OF TERNARY ALGEBRAS

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Abstract. A class of algebras with a ternary composition and alternating bilinear form is defined. The construction of a Lie algebra from a member of this class is given, and the Lie algebra is shown to be simple if the form is nondegenerate. A characterization of the Lie algebras so constructed in terms of their structure as modules for the three-dimensional simple Lie algebra is obtained in the case the base ring contains 1/2. Finally, some of the Lie algebras are identified; in particular, Lie algebras of type E_8 are obtained.

A construction of Lie algebras from Jordan algebras discovered independently by J. Tits [7] and M. Koecher [4] has been useful in the study of both kinds of algebras. In this paper, we give a similar construction of Lie algebras from a ternary algebra with a skew bilinear form satisfying certain axioms. These ternary algebras are a variation on the Freudenthal triple systems considered in [1]. Most of the results we obtain for our construction are parallel to those for the Tits-Koecher construction (see [3, Chapter VIII]).

In §1, we define the ternary algebras, derive some basic results about them, and give two examples of such algebras. In §2, the Lie algebras are constructed and shown to be simple if and only if the skew bilinear form is nondegenerate. In §3, we give a characterization, in the case the base ring contains 1/2, of the Lie algebras obtained by our construction in terms of their structure as modules for the three-dimensional simple Lie algebra. Finally, in §4, we identify some of the simple Lie algebras obtained by our construction from the examples of §1. In particular, we show that we can construct a Lie algebra of type E_8 from a 56-dimensional space which is a module for a Lie algebra of type E_7. A similar construction was given by H. Freudenthal in [2].

1. **A class of ternary algebras.** We shall be interested in a module $\mathfrak{M}$ over an arbitrary commutative associative ring Φ with 1 which possesses an alternating bilinear form $\langle\ ,\ \rangle$ and a ternary product $\langle\ ,\ ,\ \rangle$ which satisfy

(T1) $\langle x, y, z\rangle = \langle y, x, z\rangle + \langle x, y\rangle z$ for $x, y, z \in \mathfrak{M}$;

(T2) $\langle x, y, z\rangle = \langle x, z, y\rangle + \langle y, z\rangle x$ for $x, y, z \in \mathfrak{M}$;

(T3) $\langle\langle x, y, z\rangle, w\rangle = \langle\langle x, y, w\rangle, z\rangle + \langle x, y\rangle\langle z, w\rangle$ for $x, y, z, w \in \mathfrak{M}$;

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(T4) $\langle\langle x, y, z\rangle, v, w\rangle = \langle\langle x, v, w\rangle, y, z\rangle + \langle x, \langle y, v, w\rangle, z\rangle + \langle x, y, \langle z, w, v\rangle\rangle$ for $x, y, z, v, w \in \mathfrak{M}$.

We can define a four-linear form q on $\mathfrak{M}$ by

$$q(x, y, z, w) = \langle\langle x, y, z\rangle, w\rangle \quad \text{for } x, y, z, w \in \mathfrak{M}. \tag{1.1}$$

Axioms (T1)–(T3) then yield

$$\begin{aligned} q(x, y, z, w) &= q(y, x, z, w) + \langle x, y\rangle\langle z, w\rangle \\ &= q(x, z, y, w) + \langle y, z\rangle\langle x, w\rangle \\ &= q(x, y, w, z) + \langle x, y\rangle\langle z, w\rangle \quad \text{for } x, y, z, w \in \mathfrak{M}. \end{aligned} \tag{1.2}$$

An easy consequence of (1.2) is

$$q(x_{1\pi}, x_{2\pi}, x_{3\pi}, x_{4\pi}) = q(x_1, x_2, x_3, x_4) \quad \text{for } x_i \in \mathfrak{M} \text{ and } \pi \in K, \tag{1.3}$$

where K is the permutation group $\{1, (12)(34), (13)(24), (14)(23)\}$.

By (T4), we have

$$\begin{aligned} &\langle\langle\langle x_1, x_2, x_3\rangle, x_5, x_6\rangle, x_4\rangle \\ &\quad = \langle\langle\langle x_1, x_5, x_6\rangle, x_2, x_3\rangle, x_4\rangle + \langle\langle x_1, \langle x_2, x_5, x_6\rangle, x_3\rangle, x_4\rangle \\ &\qquad + \langle\langle x_1, x_2, \langle x_3, x_6, x_5\rangle\rangle, x_4\rangle \quad \text{for } x_i \in \mathfrak{M}. \end{aligned}$$

Using (T2), we see

$$\begin{aligned} &-\langle\langle\langle x_1, x_2, x_3\rangle, x_6, x_5\rangle, x_4\rangle + \langle\langle\langle x_1, x_5, x_6\rangle, x_2, x_3\rangle, x_4\rangle \\ &\quad + \langle\langle x_1, \langle x_2, x_5, x_6\rangle, x_3\rangle, x_4\rangle + \langle\langle x_1, x_2, \langle x_3, x_5, x_6\rangle\rangle, x_4\rangle \\ &\qquad = 2\langle x_5, x_6\rangle q(x_1, x_2, x_3, x_4). \end{aligned}$$

Using (1.3) this last identity can be rewritten as

$$\sum_{\pi \in K} \langle\langle x_1, x_2, x_3\rangle, \langle x_4, x_5, x_6\rangle\rangle^{\pi} = 2\langle x_5, x_6\rangle q(x_1, x_2, x_3, x_4) \quad \text{for } x_i \in \mathfrak{M}, \tag{1.4}$$

where K is considered to be a subgroup of the symmetric group S_6 and the superscript π means π is applied to each subscript i of the x_i's.

If $\langle\ ,\ \rangle$ is nondegenerate and Φ is a field, then (1.1) and (1.2) imply (T1)–(T3) and the argument used to establish (1.4) can be reversed to obtain (T4). Thus, we have shown

LEMMA 1. *If a vector space $\mathfrak{M}$ over a field Φ possesses a nondegenerate alternating form $\langle\ ,\ \rangle$ and four-linear form $q(\ ,\ ,\ ,\ )$ satisfying* (1.2) *and if $\langle\ ,\ ,\ \rangle$ defined by* (1.1) *satisfies* (1.4), *then $\langle\ ,\ \rangle$ and $\langle\ ,\ ,\ \rangle$ satisfy* (T1)–(T4).

We shall now give two examples of $\mathfrak{M}$, $\langle\ ,\ \rangle$ and $\langle\ ,\ ,\ \rangle$ satisfying (T1)–(T4).

EXAMPLE 1. If Φ is a commutative associative ring with 1 containing $\frac{1}{2}$ with $\frac{1}{2}+\frac{1}{2}=1$ and $\mathfrak{M}$ is a Φ-module with an alternating bilinear form $\langle\ ,\ \rangle$, then $\langle\ ,\ \rangle$ and $\langle\ ,\ ,\ \rangle$ defined by $\langle x, y, z\rangle = \frac{1}{2}(\langle x, y\rangle z + \langle y, z\rangle x + \langle x, z\rangle y)$, $x, y, z \in \mathfrak{M}$, satisfy (T1)–(T4).

The verification of Example 1 is straightforward, and we omit it. A more complicated and more interesting example is

EXAMPLE 2. Let $\mathfrak{J}=\mathfrak{J}(N, 1)$ be a quadratic Jordan algebra with 1 over a field Φ constructed as in [5] from an admissible nondegenerate cubic form N with base-point 1. Recall $yU_x=T(x, y)x-x^{\#}\times y$ where $T(\ ,\ )$ and $x\to x^{\#}$ are respectively the associated nondegenerate bilinear form and quadratic mapping and $x\times y=(x+y)^{\#}-x^{\#}-y^{\#}$, $x, y\in\mathfrak{J}$. Let

$$\mathfrak{M} = \left\{\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix} \mid \alpha, \beta\in\Phi;\ x, y\in\mathfrak{J}\right\}.$$

For

$$x_i = \begin{pmatrix}\alpha_i & a_i\\ b_i & \beta_i\end{pmatrix}\in\mathfrak{M},$$

we define

$$\langle x_1, x_2\rangle = \alpha_1\beta_2-\alpha_2\beta_1-T(a_1, b_2)+T(a_2, b_1), \tag{1.5}$$

$$\langle x_1, x_2, x_3\rangle = \begin{pmatrix}\gamma & c\\ d & \delta\end{pmatrix} \tag{1.6}$$

where

$$\gamma = \alpha_1\beta_2\alpha_3+2\alpha_1\alpha_2\beta_3-\alpha_3T(a_1, b_2)-\alpha_2T(a_1, b_3)-\alpha_1T(a_2, b_3)+T(a_1, a_2\times a_3),$$
$$c = (\alpha_2\beta_3+T(b_2, a_3))a_1+(\alpha_1\beta_3+T(b_1, a_3))a_2 + (\alpha_1\beta_2+T(b_1, a_2))a_3$$
$$-\alpha_1b_2\times b_3-\alpha_2b_1\times b_3-\alpha_3b_1\times b_2-\{a_1b_2a_3\}-\{a_1b_3a_2\}-\{a_2b_1a_3\},$$
$$\delta = -\gamma^{\sigma},\qquad d = -c^{\sigma},\quad \text{where } \sigma = (\alpha\beta)(ab).$$

(Note γ^{σ} is the term obtained from γ by interchanging α and β as well as a and b.) If we define $q(\ ,\ ,\ ,\ )$ by (1.1), we shall show that the conditions of Lemma 1 are satisfied. Actually we shall show (T1)–(T3) and (1.4), which is clearly sufficient since $T(\ ,\ )$ nondegenerate implies $\langle\ ,\ \rangle$ is also.

We see $\gamma-\gamma^{(12)}=\langle x_1, x_2\rangle\alpha_3$ since $T(a_1, a_2\times a_3)$ is symmetric in all three variables. Also, $c-c^{(12)}=\langle x_1, x_2\rangle a_3$. Since $\langle x_1, x_2\rangle^{\sigma}=-\langle x_1, x_2\rangle$, we see $\delta-\delta^{(12)}=\langle x_1, x_2\rangle\beta_3$, $d-d^{(12)}=\langle x_1, x_2\rangle b_3$, and (T1) holds. A similar argument establishes (T2).

To show (T3), we shall show $q(x_1, x_2, x_3, x_4)=q(x_2, x_1, x_4, x_3)$, which with (T1) yields (T3). We note

$$\begin{aligned}
q(x_1, x_2, x_3, x_4) &= \gamma\beta_4+\gamma^{\sigma}\alpha_4-T(c, b_4)-T(c^{\sigma}, a_4)= [\gamma\beta_4-T(c, b_4)]^{(1+\sigma)}\\
&= [(\alpha_1\beta_2\alpha_3\beta_4)+(2\alpha_1\alpha_2\beta_3\beta_4)-(\alpha_3\beta_4T(a_1, b_2))\\
&\quad -(\alpha_2\beta_4T(a_1, b_3)+\alpha_1\beta_3T(a_2, b_4))-(\alpha_1\beta_4T(a_2, b_3)+\alpha_2\beta_3T(a_1, b_4))\\
&\quad +(\beta_4T(a_1, a_2\times a_3)+\alpha_3T(b_1\times b_2, b_4))-(T(a_3, b_2)T(a_1, b_4))\\
&\quad -(T(a_3, b_1)T(a_2, b_4))-(\alpha_1\beta_2T(a_3, b_4))-(T(a_2, b_1)T(a_3, b_4))\\
&\quad +(\alpha_1T(b_2\times b_3, b_4)+\alpha_2T(b_1\times b_3, b_4))+(T(\{a_1b_2a_3\}, b_4))\\
&\qquad +(T(\{a_1b_3a_2\}, b_4))+(T(\{a_2b_1a_3\}, b_4))]^{(1+\sigma)}\\
&= q(x_2, x_1, x_4, x_3)
\end{aligned}$$

as desired since each term in parenthesis above is invariant up to σ by (12)(34). Here we have used $T(a, b\times c)$ is symmetric, and

$$T(\{abc\}, d) = T(c, \{bad\}) = T(b, \{adc\})$$

which hold in $\mathfrak{J}$.

We shall now give a verification of (1.4). Letting

$$\langle x_4, x_5, x_6\rangle = \begin{pmatrix}\gamma' & c'\\ d' & \delta'\end{pmatrix},$$

we see that

$$\begin{aligned}\langle\langle x_1, x_2, x_3\rangle, \langle x_4, x_5, x_6\rangle\rangle &= \gamma\delta'-\gamma'\delta-T(c, d')+T(c', d) = [\gamma^\sigma\gamma'-T(c^\sigma, c')]^{(1-\sigma)}\\ &= [\gamma^\sigma\alpha_4(\beta_5\alpha_6+2\alpha_5\beta_6-T(a_5, b_6))+T(\gamma^\sigma a_4, -\alpha_5 b_6-\alpha_6 b_5+a_5\times a_6)\\ &\quad -T(c^\sigma\times b_4, -\alpha_5 b_6-\alpha_6 b_5+a_5\times a_6)-T(\alpha_4 c^\sigma, \beta_6 a_5+\beta_5 a_6-b_5\times b_6)\\ &\quad -T(c^\sigma, a_4)(\alpha_5\beta_6+T(b_5, a_6))-T(c^\sigma, -a_4L)]^{(1-\sigma)}\end{aligned}$$

where $uL=\{ub_5a_6\}+\{ub_6a_5\}$ so $T(uL, v)=T(u, vL^\sigma)$. Here we have used $\{a_5b_4a_6\}=T(b_4, a_5)a_6+T(b_4, a_6)a_5-(a_5\times a_6)\times b_4$ and the symmetry of $T(a, b\times c)$.

We note that

$$T(\alpha_4 c, \beta_6 a_5+\beta_5 a_6-b_5\times b_6)^{(1-\sigma)} = T(\beta_4 c, -\alpha_6 b_5-\alpha_5 b_6+a_5\times a_6)^{(1-\sigma)}$$

and

$$\begin{aligned}&\sum_{\pi\in K}(\gamma^\sigma a_4-c^\sigma\times b_4-\beta_4 c)^\pi\\ &\quad=\sum_{\pi\in K}(T(b_1, b_2\times b_3)a_4+T(b_1, a_2)b_3\times b_4-(a_2\times(b_1\times b_3))\times b_4\\ &\qquad +T(a_3, b_2)b_1+b_4-(a_3\times(b_1\times b_2))\times b_4+T(a_1, b_3)b_2\times b_4\\ &\qquad -(a_1\times(b_2\times b_3))\times b_4)^\pi\\ &\quad=\sum_{\pi\in K}(T(b_1, b_2\times b_3)a_4+T(b_3, a_4)b_1\times b_2-(a_4\times(b_3\times b_1))\times b_2\\ &\qquad +T(a_4, b_1)b_2\times b_3-(a_4\times(b_2\times b_1))\times b_3+T(a_4, b_2)b_3\times b_1\\ &\qquad -(a_4\times(b_3\times b_2))\times b_1)^\pi = 0\end{aligned}$$

by the linearization of $N(b)a+T(a, b)b^{\#}=(a\times b^{\#})\times b$ which holds in $\mathfrak{J}$.

Also, $T(b_1, a_4L)^\sigma=T(a_1L, b_4)=T(b_1, a_4L)^{(14)}$, so

$$[(\beta_2\alpha_3+T(a_2, b_3))T(b_1, a_4L)]^\sigma = [(\beta_2\alpha_3+T(a_2, b_3))T(b_1, a_4L)]^{(14)(23)}.$$

Moreover,

$$\begin{aligned}T(\{b_1a_2b_3\}, a_4L)^\sigma &= T(\{a_1b_2a_3\}L, b_4)\\ &= T(\{a_1Lb_2a_3\}, b_4)-T(\{a_1b_2L^\sigma a_3\}, b_4)+T(\{a_1b_2a_3L\}, b_4),\end{aligned}$$

so

$$[T(\{b_1a_2b_3\}, a_4L)^{1+(13)(24)}]^\sigma = T(\{b_1a_2b_3\}, a_4L)^{(14)(23)+(12)(34)}.$$

These and similar expressions yield

$$\sum_{\pi\in K} T(c^\sigma, a_4L)^{\pi(1-\sigma)}$$
$$= -\sum_{\pi\in K} (\beta_1T(a_2\times a_3, a_4L)+\beta_1T(a_2\times a_4, a_3L)+\beta_1T(a_3\times a_4, a_2L))^{\pi(1-\sigma)}$$
$$= -2(T(b_5, a_6)+T(b_6, a_5))\sum_{\pi\in K} \beta_1(T(a_2, a_3\times a_4))^{\pi(1-\sigma)}.$$

The last equality follows from the linearization of $T(u^\#, \{uab\})=2T(a, b)N(u)$ which holds in $\mathfrak{J}$.

Finally, we note that $\sum_{\pi\in K} (\gamma^\sigma\alpha_4-T(b_1, b_2\times b_3)\alpha_4)^\pi$ and

$$\sum_{\pi\in K} (T(c^\sigma, a_4)+\beta_1T(a_2\times a_3, a_4)+\beta_2T(a_1\times a_3, a_4)+\beta_3T(a_1\times a_2, a_4))^\pi$$

are invariant under σ and their difference is

$$2(\gamma^\sigma\alpha_4-T(c^\sigma, a_4)-\beta_1T(a_2\times a_3, a_4)-\beta_2T(a_1+a_3, a_4)$$
$$-\beta_3T(a_1\times a_2, a_4)-\beta_4T(a_1, a_2\times a_3))^{(1+\sigma)}.$$

Hence

$$\sum_{\pi\in K} \langle\langle x_1, x_2, x_3\rangle, \langle x_4, x_5, x_6\rangle\rangle^\pi$$
$$= \sum_{\pi\in K} (\gamma^\sigma\alpha_4-T(b_1, b_2\times b_3)\alpha_4)^\pi\langle x_5, x_6\rangle$$
$$+\sum_{\pi\in K} (T(b_1, b_2\times b_3)\alpha_4(\beta_5\alpha_6+2\alpha_5\beta_6-T(a_5, b_6)))^{\pi(1-\sigma)}$$
$$-\sum_{\pi\in K} (T(c^\sigma, a_4)+\beta_1T(a_2\times a_3, a_4)+\beta_2T(a_1\times a_3, a_4)+\beta_3T(a_1\times a_2, a_4))^\pi\langle x_5, x_6\rangle$$
$$+\sum_{\pi\in K} (3\beta_1T(a_2\times a_3, a_4)(\alpha_5\beta_6+T(b_5, a_6)))^{\pi(1-\sigma)}$$
$$-\sum_{\pi\in K} (2\beta_1T(a_2\times a_3, a_4)(T(b_5, a_6)+T(b_6, a_5)))^{\pi(1-\sigma)}$$
$$= 2(\gamma^\sigma\alpha_4-T(c^\sigma, a_4))^{(1+\sigma)}\langle x_5, x_6\rangle$$
$$= 2q(x_1, x_2, x_3, x_4)\langle x_5, x_6\rangle$$

establishing (1.4).

2. **Construction of the Lie algebras.** Starting with a module $\mathfrak{M}$ over a commutative associative ring Φ with 1 which possesses an alternating bilinear form $\langle\ ,\ \rangle$ and a ternary product $\langle\ ,\ ,\ \rangle$ which satisfy (T1)–(T4), we shall construct some Lie algebras.

First, we construct $\mathfrak{N}=\mathfrak{M}\oplus\Phi u$ and the associative subalgebra $\mathfrak{A}(\mathfrak{M})$ of $\mathrm{Hom}_\Phi(\mathfrak{N}, \mathfrak{N})$ consisting of $A\in\mathrm{Hom}_\Phi(\mathfrak{N}, \mathfrak{N})$ such that $uA\in\Phi u$ and $\mathfrak{M}A\subseteq\mathfrak{M}$. We let $\mathfrak{A}(\mathfrak{M})^-$ denote the Lie algebra structure on $\mathfrak{A}(\mathfrak{M})$ where $[AB]=AB-BA$.

If we define $U \in \mathfrak{A}(\mathfrak{M})^-$ by $uU=2u$ and $xU=x$ for $x \in \mathfrak{M}$, then it is clear that U is in the center of $\mathfrak{A}(\mathfrak{M})^-$. We may also define $\rho(A) \in \Phi$ for $A \in \mathfrak{A}(\mathfrak{M})^-$ by

$$uA = \rho(A)u. \tag{2.1}$$

If $A \in \mathfrak{A}(\mathfrak{M})^-$, we set

$$A' = A - \rho(A)U \tag{2.2}$$

and note that $[AB]'=[AB]=[A'B']$ for $A, B \in \mathfrak{A}(\mathfrak{M})^-$, so $A \to A'$ is an automorphism of $\mathfrak{A}(\mathfrak{M})^-$ of order two.

We next define $R(x, y) \in \mathfrak{A}(\mathfrak{M})^-$ for $x, y \in \mathfrak{M}$ by

$$\begin{aligned} uR(x, y) &= \langle x, y\rangle u, \\ zR(x, y) &= \langle z, x, y\rangle \quad \text{for } z \in \mathfrak{M}. \end{aligned} \tag{2.3}$$

Let $\mathfrak{R}^*(\mathfrak{M})$ consist of those $R \in \mathfrak{A}(\mathfrak{M})^-$ such that

$$[R(x, y)R] = R(xR, y) + R(x, yR') \quad \text{for } x, y \in \mathfrak{M}. \tag{2.4}$$

One checks immediately that $\mathfrak{R}^*(\mathfrak{M})$ is a Lie subalgebra of $\mathfrak{A}(\mathfrak{M})^-$ containing U and hence invariant under $A \to A$.

It is clear from (T1) that

$$R(x, y) - R(y, x) = \langle x, y\rangle U, \qquad x, y \in \mathfrak{M}, \tag{2.5}$$

and hence $R'(x, y) = R(y, x)$. Since $q(x_1, x_3, x_4, x_2) = q(x_2, x_4, x_3, x_1)$ by (1.3), we see by (T4) that

$$[R(x_1, x_2)R(x_3, x_4)] = R(x_1R(x_3, x_4), x_2) + R(x_1, x_2R(x_4, x_3)) \quad \text{for } x_i \in \mathfrak{M},\ i = 1, 2, 3, 4. \tag{2.6}$$

Hence, $R(x, y) \in \mathfrak{R}^*(\mathfrak{M})$ for $x, y \in \mathfrak{M}$. Indeed $\{R(x, y) \mid x, y \in \mathfrak{M}\} \cup \{U\}$ spans an ideal $\mathfrak{R}(\mathfrak{M})$ of $\mathfrak{R}^*(\mathfrak{M})$. We note that if $\langle\ ,\ \rangle$ represents 1, then (2.5) implies that $\mathfrak{R}(\mathfrak{M})$ is spanned by $\{R(x, y) \mid x, y \in \mathfrak{M}\}$.

Applying (2.4) to u we get

$$\langle xR, y\rangle + \langle x, yR'\rangle = 0 \quad \text{for } x, y \in \mathfrak{M},\ R \in \mathfrak{R}^*(\mathfrak{M}). \tag{2.7}$$

Now let $\mathfrak{R}'$ be any Lie subalgebra of $\mathfrak{R}^*(\mathfrak{M})$ containing $\mathfrak{R}(\mathfrak{M})$ and let $\tilde{\mathfrak{N}}$ denote a second copy of $\mathfrak{N}$. Form $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}') = \mathfrak{N} \oplus \tilde{\mathfrak{N}} \oplus \mathfrak{R}' = \mathfrak{M} \oplus \tilde{\mathfrak{M}} \oplus \Phi u \oplus \Phi\tilde{u} \oplus \mathfrak{R}'$. We may define a Lie product on $\mathfrak{S} = \mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ by

$$\begin{aligned} &[x_1 + \tilde{y}_1 + \alpha_1 u + \beta_1\tilde{u} + R_1,\ x_2 + \tilde{y}_2 + \alpha_2 u + \beta_2 u + R_2] \\ &\quad = (x_1R_2 - x_2R_1 + \alpha_1 y_2 - \alpha_2 y_1) + (y_1R_2' - y_2R_1' + \beta_2 x_1 - \beta_1 x_2)^{\sim} \\ &\qquad + (\langle x_1, x_2\rangle + \alpha_1\rho(R_2) - \alpha_2\rho(R_1))u \\ &\qquad + (\langle y_1, y_2\rangle - \beta_1\rho(R_2) + \beta_2\rho(R_1))\tilde{u} \\ &\qquad + (R(x_1, y_2) - R(x_2, y_1) + (\alpha_1\beta_2 - \alpha_2\beta_1)U + [R_1, R_2]) \\ &\qquad\qquad \text{for } x_i, y_i \in \mathfrak{M},\ \alpha_i, \beta_i \in \Phi,\ R_i \in \mathfrak{R}', \end{aligned} \tag{2.8}$$

where $[R_1R_2]$ is the Lie product in $\mathfrak{R}'$. Clearly $[SS]=0$ for $S\in\mathfrak{S}$, and we need only show the Jacobi identity.

If $S_i=x_i+\tilde{y}_i+\alpha_i u+\beta_i\tilde{u}+R_i$, $x_i, y_i\in\mathfrak{M}$, $\alpha_i,\beta_i\in\Phi$, $R_i\in\mathfrak{R}'$, $i=1,2,3$, then

$$
\begin{aligned}
[[S_1S_2]S_3] = {} & \{(x_1R_2R_3-x_2R_1R_3-x_3[R_1R_2])\\
&+(\alpha_1y_2R_3-\alpha_2\rho(R_1)y_3-\alpha_3y_1R_2')+(-\alpha_2y_1R_3+\alpha_1\rho(R_2)y_3+\alpha_3y_2R_1')\\
&+(-\langle x_3x_1y_2\rangle+\langle x_3x_2y_1\rangle+\langle x_1,x_2\rangle y_3)\\
&\qquad+(-\alpha_1\beta_2x_3+\alpha_3\beta_1x_2)+(\alpha_2\beta_1x_3-\alpha_3\beta_2x_1)\}\\
&+\{(y_1R_2'R_3'-y_2R_1'R_3'-y_3[R_1R_2]')\\
&+(\beta_2x_1R_3'+\beta_1\rho(R_2)x_3-\beta_3x_2R_1)+(-\beta_1x_2R_3'-\beta_2\rho(R_1)x_3+\beta_3x_1R_2)\\
&+(\langle y_3y_1x_2\rangle-\langle y_3y_2x_1\rangle-\langle y_1,y_2\rangle x_3)\\
&\qquad+(\alpha_1\beta_2y_3-\alpha_2\beta_3y_1)+(-\alpha_2\beta_1y_3+\alpha_1\beta_3y_2)\}^{\sim}\\
&+\{(\langle x_1R_2,x_3\rangle-\langle x_2R_1,x_3\rangle+\langle x_1,x_2\rangle\rho(R_3))\\
&+(\alpha_1\langle y_2,x_3\rangle-\alpha_3\langle y_1,x_2\rangle)+(\alpha_3\langle y_2,x_1\rangle-\alpha_2\langle y_1,x_3\rangle)\\
&\qquad+(\alpha_1\rho(R_2)\rho(R_3)-\alpha_2\rho(R_1)\rho(R_3))+(2\alpha_2\alpha_3\beta_1-2\alpha_3\alpha_1\beta_2)\}u\\
&+\{(\langle y_1R_2',y_3\rangle-\langle y_2R_1',y_3\rangle-\langle y_1,y_2\rangle\rho(R_3))\\
&+(\beta_2\langle x_1,y_3\rangle-\beta_3\langle x_2,y_1\rangle)+(\beta_3\langle x_1,y_2\rangle-\beta_1\langle x_2,y_3\rangle)\\
&\qquad+(\beta_1\rho(R_2)\rho(R_3)-\beta_2\rho(R_1)\rho(R_3))+(2\beta_3\alpha_1\beta_2-2\beta_1\alpha_2\beta_3)\}\tilde{u}\\
&+\{(R(x_1R_2,y_3)+R(x_3,y_2R_1')-[R(x_2,y_1)R_3])\\
&+([R(x_1,y_2)R_3]-R(x_2R_1,y_3)-R(x_3,y_1R_2'))\\
&+(\alpha_1R(y_2,y_3)-\alpha_2R(y_1,y_3)-\alpha_3\langle y_1,y_2\rangle U)\\
&+(\beta_1R(x_3,x_2)-\beta_2R(x_3,x_1)+\beta_3\langle x_1,x_2\rangle U)\\
&+(\alpha_1\beta_3\rho(R_2)-\alpha_2\beta_2\rho(R_1))U\\
&\qquad+(\alpha_3\beta_1\rho(R_2)-\alpha_2\beta_3\rho(R_1))U+([[R_1R_2]R_3])\}.
\end{aligned}
$$

If the subscripts of each term in parenthesis above are permuted cyclically and the resulting three terms summed, the summand will be zero. Hence, the Jacobi identity holds in $\mathfrak{S}$, and $\mathfrak{S}$ is a Lie algebra.

We shall next give a condition for simplicity of $\mathfrak{S}$.

THEOREM 1. *If $\mathfrak{M}$ is a vector space over a field Φ with an alternating bilinear form $\langle\ ,\ \rangle$ and a ternary product $\langle\ ,\ ,\ \rangle$ satisfying* (T1)–(T4) *and if $\mathfrak{S}=\mathfrak{S}(\mathfrak{M},\mathfrak{R}(\mathfrak{M}))$ is constructed as above then $\mathfrak{S}$ is a simple Lie algebra if and only if $\langle\ ,\ \rangle$ is nondegenerate.*

The theorem will follow from the next two lemmas, but first we shall define an *ideal* of $\mathfrak{M}$ to be a subspace $\mathfrak{I}$ with $\langle x_1,x_2,x_3\rangle\in\mathfrak{I}$ for $x_i\in\mathfrak{I}$, $x_j,x_k\in\mathfrak{M}$, i,j,k are not equal. $\mathfrak{M}$ is *simple*, if $\mathfrak{M}$ and $\{0\}$ are the only ideals in $\mathfrak{M}$ and $\langle\mathfrak{M}\mathfrak{M}\mathfrak{M}\rangle\neq 0$.

LEMMA 2. *Let* $\mathfrak{M}$ *be as in Theorem* 1 *and let* Rad $(\mathfrak{M})=\{x\in\mathfrak{M} \mid \langle x, y\rangle=0$ *for all* $y\in\mathfrak{M}\}$, *then* Rad $(\mathfrak{M})$ *is an ideal of* $\mathfrak{M}$ *containing every ideal* $\mathfrak{J}$ *of* $\mathfrak{M}$ *with* $\mathfrak{J}\neq\mathfrak{M}$.

Proof. If $x\in\mathfrak{J}\neq\mathfrak{M}$ an ideal of $\mathfrak{M}$, then $\langle x, y\rangle z=\langle x, y, z\rangle-\langle y, x, z\rangle$ for all $y, z\in\mathfrak{M}$ implies $x\in$ Rad $(\mathfrak{M})$. On the other hand, we see $\langle\langle x_1, x_2, x_3\rangle, x_4\rangle=0$ for $x_i\in$ Rad $(\mathfrak{M})$, $x_j, x_k, x_l\in\mathfrak{M}$, i, j, k, l not equal, by (1.3). Hence, Rad $(\mathfrak{M})$ is an ideal of $\mathfrak{M}$.

LEMMA 3. *Let* $\mathfrak{M}$ *be as in Theorem* 1 *and let* $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ *be the Lie algebra constructed as above. If* $\mathfrak{K}$ *is an ideal of* $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$, *then* $\mathfrak{K}\cap\mathfrak{M}$ *is an ideal of* $\mathfrak{M}$. *Also, if* $\mathfrak{J}$ *is an ideal of* $\mathfrak{M}$, $\mathfrak{J}\neq\mathfrak{M}$, *then* $\mathfrak{J}+\tilde{\mathfrak{J}}+\mathfrak{R}(\mathfrak{J}, \mathfrak{M})$ *is an ideal of* $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$ *where* $\mathfrak{R}(\mathfrak{J}, \mathfrak{M})$ *is the subspace of* $\mathfrak{R}(\mathfrak{M})$ *spanned by* $\{R(x, y) \mid x\in\mathfrak{J}, y\in\mathfrak{M}\}$.

Proof. Since $\langle x_1, x_2, x_3\rangle=[x_1[x_2[x_3\tilde{u}]]]$, the first statement is clear. The second follows immediately from (2.8) and the fact $\mathfrak{J}\subseteq$ Rad $(\mathfrak{M})$.

To prove Theorem 1, we first assume $\langle\ ,\ \rangle$ is nondegenerate. If $\mathfrak{K}\neq 0$ is an ideal of $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$, then (2.8) shows $\mathfrak{K}\cap\mathfrak{M}\neq 0$. By Lemmas 2 and 3, we have $\mathfrak{K}\cap\mathfrak{M}=\mathfrak{M}$. But $\mathfrak{M}$ generates $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$ by (2.8) so $\mathfrak{K}=\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$. If $\langle\ ,\ \rangle$ is degenerate, then $\mathfrak{J}=$ Rad $(\mathfrak{M})$ is a nonzero ideal of $\mathfrak{M}$. Hence $\mathfrak{K}=\mathfrak{J}+\tilde{\mathfrak{J}}+\mathfrak{R}(\mathfrak{J}, \mathfrak{M})$ is a nonzero ideal of $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$. But $\mathfrak{K}\neq\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$, since $u\notin\mathfrak{K}$.

3. **A characterization of the Lie algebras.** In this section, we shall obtain a characterization of the Lie algebras $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ constructed as in §2 from a module $\mathfrak{M}$ over a commutative associative ring Φ with 1 containing $\frac{1}{2}$ with $\frac{1}{2}+\frac{1}{2}=1$ where $\mathfrak{M}$ possesses an alternating bilinear form $\langle\ ,\ \rangle$ and ternary product $\langle\ ,\ ,\ \rangle$ satisfying (T1)–(T4). Let $\mathfrak{S}=\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ be such a Lie algebra, and let $e=u$, $f=\tilde{u}$, and $h=U$. We have by (2.8) that

$$(3.1) \qquad [ef] = h, \qquad [eh] = 2e, \qquad [fh] = -2f.$$

Hence, the subalgebra $\mathfrak{A}=\Phi e+\Phi f+\Phi h$ of $\mathfrak{S}$ has a faithful representation $v\to va$, $v\in V$, $a\in\mathfrak{A}$, on $V=\Phi v_1\oplus\Phi v_2$ given by

$$(3.2) \quad v_1e = 0, \quad v_2e = -v_1; \qquad v_1f = v_2, \quad v_2f = 0; \qquad v_1h = v_1, \quad v_2h = -v_2.$$

If $x\in\mathfrak{M}$, then the $\mathfrak{A}$-submodule of $\mathfrak{S}$ under the adjoint action of $\mathfrak{A}$ generated by x is $\Phi x+\Phi\tilde{x}$ which is a homomorphic image of V.

We note that if $D\in\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ is a derivation (i.e., $\langle x, y, z\rangle D=\langle xD, y, z\rangle+\langle x, yD, z\rangle+\langle x, y, zD\rangle$), then D can be extended uniquely to an element $D\in\mathfrak{R}^*(\mathfrak{M})$ with $\rho(D)=0$. Conversely, $D\in\mathfrak{R}^*(\mathfrak{M})$ with $\rho(D)=0$ restricts to a derivation of $\mathfrak{M}$. We shall identify $\mathfrak{D}=\{D\in\mathfrak{R}^*(\mathfrak{M}) \mid \rho(D)=0\}$ with the derivations of $\mathfrak{M}$. We have an ideal $\mathfrak{D}_i$ of $\mathfrak{D}$ consisting of elements of the form $\sum_i R(x_i, y_i)$ with $\sum_i\langle x_i, y_i\rangle=0$. Such elements are called *inner derivations* of $\mathfrak{M}$. Since $\frac{1}{2}\in\Phi$, we see that $\mathfrak{R}^*(\mathfrak{M})=\Phi U \oplus \mathfrak{D}$. Hence $\mathfrak{R}'=\Phi U \oplus \mathfrak{D}'$ where $\mathfrak{D}'=\mathfrak{D} \cap \mathfrak{R}'$. We may now write

$$\mathfrak{S} = \sum_{x\in\mathfrak{M}} (\Phi x+\Phi\tilde{x})+\mathfrak{A}+\mathfrak{D}'. \tag{3.3}$$

It is clear that $\mathfrak{D}'$ is the centralizer of $\mathfrak{A}$ in $\mathfrak{S}$. Also, if $D \in \mathfrak{D}'$, then $[\mathfrak{M}D]\subseteq\mathfrak{M}$ and $[\mathfrak{M}, D]\subseteq\mathfrak{D}'$ only if $D=0$. Hence, $\mathfrak{D}'$ contains no nonzero ideals of $\mathfrak{S}$. We have shown half of

THEOREM 2. *A Lie algebra $\mathfrak{S}$ over a commutative associative ring $\Phi \ni \frac{1}{2}$ is isomorphic to a Lie algebra $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ constructed as in §2 if and only if $\mathfrak{S}$ satisfies*:

(i) *$\mathfrak{S}$ contains a subalgebra $\mathfrak{A}=\Phi e+\Phi f+\Phi h$ having a representation on $V=\Phi v_1 \oplus \Phi v_2$ given by* (3.2),

(ii) *$\mathfrak{S}$ as an $\mathfrak{A}$ module under the adjoint action is a sum of $\mathfrak{A}$, submodules which are homomorphic images of V, and the centralizer $\mathfrak{D}'$ of $\mathfrak{A}$ in $\mathfrak{S}$*,

(iii) *$\mathfrak{D}'$ contains no nonzero ideals of $\mathfrak{S}$*.

Proof. Let $\mathfrak{S}$ satisfy (i)–(iii). Set $\mathfrak{S}_i=\{x \in \mathfrak{S} \mid [xh]=ix,\ i=0, \pm 1, \pm 2\}$. Clearly $\mathfrak{S}=\mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_{-1}+\mathfrak{S}_{-2}+\mathfrak{S}_0$ and $\mathfrak{S}_0=\Phi h \oplus \mathfrak{D}'$. Also, we see $\mathfrak{S}_i \cap \mathfrak{S}_j=0$ for $i\neq j$ unless $i-j=\pm 3$ and $3=0$ in Φ. Letting $\mathfrak{M}$ (respectively $\tilde{\mathfrak{M}}$) be the set of images of Φv_1 (respectively Φv_2) under the homomorphisms of V onto submodules of $\mathfrak{S}$, we see $\mathfrak{M}\subseteq\mathfrak{S}_1$ and $\tilde{\mathfrak{M}}\subseteq\mathfrak{S}_{-1}$. It is clear that $x \to \tilde{x}=[xf]$ is a bijection of $\mathfrak{M}$ with $\tilde{\mathfrak{M}}$. Also, $\Phi e\subseteq\mathfrak{S}_2$, $\Phi f\subseteq\mathfrak{S}_{-2}$, and

$$\mathfrak{S} = \mathfrak{M} \oplus \Phi e \oplus \tilde{\mathfrak{M}} \oplus \Phi f \oplus \Phi h \oplus \mathfrak{D}'. \tag{3.4}$$

We have $[\mathfrak{M}\mathfrak{M}]\subseteq\mathfrak{S}_2\subseteq\Phi e+\tilde{\mathfrak{M}}$. If $[xy]=\alpha e+\tilde{z}$ with $x, y, z \in \mathfrak{M}$, $\alpha \in \Phi$, we see $-z=[[xy]e]=[[xe]y]+[x[ye]]=0$. Hence $[\mathfrak{M}\mathfrak{M}]\subseteq \Phi e$, and we may define a skew bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{M}$ by $\langle x, y\rangle e=[x, y]$, $x, y \in \mathfrak{M}$.

One sees that $[\mathfrak{M}[\mathfrak{M}\tilde{\mathfrak{M}}]]\subseteq\mathfrak{S}_1\subseteq\Phi f+\mathfrak{M}$. If $[x[y\tilde{z}]]=\alpha f+w$ with $\alpha \in \Phi$, $w \in \mathfrak{M}$, then $-\alpha h=[[x[y\tilde{z}]]e]=[x[y[\tilde{z}e]]]=-[x[yz]]=-\langle y, z\rangle[x, e]=0$. Hence

$$[\mathfrak{M}[\mathfrak{M}\tilde{\mathfrak{M}}]] \subseteq \mathfrak{M},$$

and we may define a ternary product $\langle x, y, z\rangle=[x[y\tilde{z}]] \in \mathfrak{M}$ for $x, y, z \in \mathfrak{M}$.

Since $\langle x, y, z\rangle=[[xy]\tilde{z}]+\langle y, x, z\rangle=\langle x, y\rangle z+\langle y, x, z\rangle$ for $x, y, z \in \mathfrak{M}$, we see (T1) holds for $\mathfrak{M}$. A similar calculation shows (T2). To show (T3), we calculate

$$\begin{aligned}\langle\langle x, y, z\rangle, w\rangle e &= [[x[y\tilde{z}]]w] = [[xw][y\tilde{z}]]+[[w[y\tilde{z}]]x] \\ &= \langle x, w\rangle\langle y, z\rangle e+\langle\langle w, y, z\rangle, x\rangle e; \qquad x, y, z, w \in \mathfrak{M}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle\langle x, y, z\rangle, w\rangle &= \langle\langle z, x, y\rangle, w\rangle+\langle y, z\rangle\langle x, w\rangle+\langle x, z\rangle\langle y, w\rangle \quad \text{(by (T2) and (T1))} \\ &= \langle\langle w, x, y\rangle, z\rangle+\langle z, w\rangle\langle x, y\rangle+\langle y, z\rangle\langle x, w\rangle+\langle x, z\rangle\langle y, w\rangle \\ &= \langle\langle x, y, w\rangle, z\rangle+\langle z, w\rangle\langle x, y\rangle \quad \text{(by (T1) and (T2))}\end{aligned}$$

for $x, y, z, w \in \mathfrak{M}$.

If $x, y, z, v, w \in \mathfrak{M}$ and $L=[v, \tilde{w}]$, then

$$\begin{aligned}\langle\langle x, y, z\rangle, v, w\rangle &= [\langle x, y, z\rangle L] = \langle xL, y, z\rangle+\langle x, yL, z\rangle+[x[y[\tilde{z}L]]]\\ &= \langle\langle x, v, w\rangle y, z\rangle+\langle x, \langle y, v, w\rangle, z\rangle+\langle x, y, \langle z, w, v\rangle\rangle,\end{aligned}$$

since

$$\begin{aligned}[\tilde{z}L] &= [[zf]L] = [[zL]f]+[z[fL]]\\ &= \langle z, v, w\rangle^{\sim}+[z[f[v\tilde{w}]]]] = \langle z, v, w\rangle^{\sim}-\langle v, w\rangle\tilde{z} = \langle z, w, v\rangle^{\sim}.\end{aligned}$$

Here we have used $[f[v\tilde{w}]]=-[\tilde{v}\tilde{w}]\in\Phi f$ and $[[[\tilde{v}, \tilde{w}]e]e]=2[[\tilde{v}e][\tilde{w}e]]=2[vw]=2\langle v, w\rangle e$, so $[\tilde{v}\tilde{w}]=\langle v, w\rangle f$. Thus, we have established (T4) and

$$[\tilde{v}\tilde{w}] = \langle v, w\rangle f, \qquad v, w\in\mathfrak{M}, \tag{3.5}$$

$$[\tilde{z}[v\tilde{w}]] = \langle z, w, v\rangle^{\sim}, \qquad z, v, w\in\mathfrak{M}. \tag{3.6}$$

If $d\in\mathfrak{D}'$ and $x\in\mathfrak{M}$, then $[xd]\in\mathfrak{S}_1\subseteq\mathfrak{M}+\Phi f$. If $[xd]=y+\alpha f$, $y\in\mathfrak{M}$, $\alpha\in\Phi$, then $-\alpha h=[[xd]e]=0$. Hence, $[xd]\in\mathfrak{M}$, and we may define $D_d\in\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ by $xD_d=[xd]$ for $x\in\mathfrak{M}$. We see that

$$[\tilde{x}d] = (xD_d)^{\sim} \quad \text{for } x\in\mathfrak{M},\ d\in\mathfrak{D}'. \tag{3.7}$$

Hence, $\langle x, y, z\rangle D_d=\langle xD_d, y, z\rangle+\langle x, yD_d, z\rangle+\langle x, y, zD_d\rangle$ for $x, y, z\in\mathfrak{M}$, $d\in\mathfrak{D}'$, and D_d is a derivation.

We now may define a linear map $\varphi\colon\mathfrak{S}\to\mathfrak{S}(\mathfrak{M}, \mathfrak{R}^*(\mathfrak{M}))$ by

$$\varphi\colon x+\tilde{y}+\alpha e+\beta f+\gamma h+d\to x+\tilde{y}+\alpha u+\beta\tilde{u}+\gamma U+D_d \tag{3.8}$$

where $x, y\in\mathfrak{M}$, $\alpha, \beta, \gamma\in\Phi$, and $d\in\mathfrak{D}'$. To check that φ is a Lie homomorphism, we first note that the structure of $\mathfrak{S}$ as an $\mathfrak{A}$-module yields $[sa]^\varphi=[s^\varphi a^\varphi]$ for $s\in\mathfrak{S}$, $a\in\mathfrak{A}$. Thus, we need only check

$$[x, y]^\varphi = \langle x, y\rangle u, \qquad x, y\in\mathfrak{M}, \tag{3.9}$$

$$[x, d]^\varphi = xD_d, \qquad x\in\mathfrak{M},\ d\in\mathfrak{D}', \tag{3.10}$$

$$[\tilde{x}, \tilde{y}]^\varphi = \langle x, y\rangle\tilde{u}, \qquad x, y\in\mathfrak{M}, \tag{3.11}$$

$$[\tilde{x}, d]^\varphi = (xD_d)^{\sim}, \qquad x\in\mathfrak{M},\ d\in\mathfrak{D}', \tag{3.12}$$

$$[cd]^\varphi = [D_cD_d], \qquad c, d\in\mathfrak{D}', \tag{3.13}$$

$$[x\tilde{y}]^\varphi = R(x, y), \qquad x, y\in\mathfrak{M}. \tag{3.14}$$

We note that (3.9) and (3.10) follow by definition, that (3.11) and (3.12) follow from (3.5) and (3.7) respectively, and that (3.13) is obvious. Since $[e[x\tilde{y}]]=[xy]=\langle x, y\rangle e$, $[f[x\tilde{y}]]=-[\tilde{x}, \tilde{y}]=-\langle x, y\rangle f$ and $[h[x\tilde{y}]]=0$ for $x, y\in\mathfrak{M}$, we have $d=[x\tilde{y}]-\frac{1}{2}\langle x, y\rangle h\in\mathfrak{D}'$. Now $z[x\tilde{y}]^\varphi=z(\frac{1}{2}\langle x, y\rangle U+D_d)=\langle z, x, y\rangle$ for $z\in\mathfrak{M}$, and $u[x\tilde{y}]^\varphi=\langle x, y\rangle u$ imply $[x\tilde{y}]^\varphi=R(x, y)$ to establish (3.14). Thus, φ is a homomorphism.

Since the kernel of φ is contained in $\mathfrak{D}'$, condition (iii) implies that φ is an isomorphism. Since $\mathfrak{R}(\mathfrak{M}) \subseteq \mathfrak{R}' \equiv (\Phi h + \mathfrak{D}')^{\varphi}$ by (3.14), we have $\mathfrak{S}$ isomorphic to $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}')$ as desired.

4. **Identification of the Lie algebras.** We wish to identify the simple Lie algebras $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$ constructed as in §2 from the ternary algebras of Example 1 with $\langle \, , \rangle$ nondegenerate and of Example 2 with $\mathfrak{J}$ an exceptional simple Jordan algebra of dimension 27. We shall do this for Φ a field of characteristic zero. Since $\langle \, , \rangle$ remains nondegenerate upon extension of the base field, we may assume in both cases that Φ is algebraically closed.

Example 1. We first consider the derivation algebra of $\mathfrak{M}$ which we have identified with $\mathfrak{D} = \{D \in \mathfrak{R}^*(\mathfrak{M}) \mid \rho(D) = 0\}$. By (2.2) and (2.7), we have $\mathfrak{D} \subseteq \mathfrak{L}$, the Lie algebra of linear transformations of $\mathfrak{M}$ which are skew relative to $\langle \, , \rangle$. An immediate calculation shows however, $D \in \mathfrak{L}$ is a derivation of $\mathfrak{M}$. Thus, if dim $\mathfrak{M} = 2l$, we have that $\mathfrak{D}$ is a Lie algebra of type C_l and dim $\mathfrak{D} = l(2l+1)$. Since $\mathfrak{D}$ is simple, we see that the inner derivation algebra $\mathfrak{D}_i = \{\sum_i R(x_i, y_i) \mid \sum_i \langle x_i, y_i \rangle = 0\} = \mathfrak{D}$ and $\mathfrak{R}^*(\mathfrak{M}) = \mathfrak{R}(\mathfrak{M})$.

Now $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M})) = \mathfrak{M} \oplus \tilde{\mathfrak{M}} \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus \mathfrak{D}$, so dim $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M})) = 4l + 3 + l(2l+1) = (l+1)(2(l+1)+1)$. By the classification theory of simple Lie algebras, we see that $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$ is of type C_{l+1}.

Example 2. Again we look first at the derivation algebra $\mathfrak{D}$. As before, $D \in \mathfrak{D}$ is skew relative to $\langle \, , \rangle$. Thus,

$$0 = \langle\langle x, y, z\rangle D, w\rangle + \langle\langle x, y, z\rangle, wD\rangle = \langle\langle xD, y, z\rangle, w\rangle + \langle\langle x, yD, z\rangle, w\rangle + \langle\langle x, y, zD\rangle, w\rangle + \langle\langle x, y, z\rangle, wD\rangle \quad \text{for } x, y, z, w \in \mathfrak{M},$$

and D is skew relative to the four-linear form $q(x, y, z, w) = \langle\langle x, y, z\rangle, w\rangle$. Conversely, if D is skew relative to q and $\langle \, , \rangle$, it is clear that D is a derivation of $\mathfrak{M}$. If Q is the quartic form $Q(x) = q(x, x, x, x)$, $x \in \mathfrak{M}$, and if $Q(x, y, z, w)$ is its linearization, then we see by (1.3) and (1.2) that

$$Q(x_1, x_2, x_3, x_4) = \sum_{\pi \in S_4} q(x_{1\pi}, x_{2\pi}, x_{3\pi}, x_{4\pi}) = 24q(x_1, x_2, x_3, x_4) + A$$

where S_4 is the symmetric group on $\{1, 2, 3, 4\}$ and A is a sum of terms of the form $\langle x_i, x_j\rangle\langle x_k, x_l\rangle$. Hence, D is skew relative to $\langle \, , \rangle$ and q if and only if D is skew relative to $\langle \, , \rangle$ and Q. Thus,

$$\mathfrak{D} = \{D \in \mathrm{Hom}_{\Phi}(\mathfrak{M}, \mathfrak{M}) \mid Q(xD, x, x, x) = 0 \text{ and } \langle xD, y\rangle + \langle x, yD\rangle = 0, \; x, y \in \mathfrak{M}\}. \tag{4.1}$$

Calculating Q, we find

$$Q(x) = 24(\alpha N(b) + \beta N(a) - T(a^{\#}, b^{\#}) + \tfrac{1}{4}(\alpha\beta - T(a, b))^2) \quad \text{for } x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \quad \alpha, \beta \in \Phi, \quad a, b \in \mathfrak{J}. \tag{4.2}$$

Thus, $\mathfrak{D}$ is a Lie algebra of type E_7 (see [6]), and dim $\mathfrak{D}=133$. Since $\mathfrak{D}$ is simple, $\mathfrak{D}_i=\mathfrak{D}$ and $\mathfrak{R}^*(\mathfrak{M})=\mathfrak{R}(\mathfrak{M})$. Hence dim $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))=2(56)+3+133=248$. Thus, by the classification of simple Lie algebras, we see that $\mathfrak{S}(\mathfrak{M}, \mathfrak{R}(\mathfrak{M}))$ is of type E_8.

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