ALMOST LOCALLY TAME 2-MANIFOLDS
IN A 3-MANIFOLD\(^{(1)}\)

BY

HARVEY ROSEN

Abstract. Several conditions are given which together imply that a 2-manifold \(M\) in a 3-manifold is locally tame from one of its complementary domains, \(U\), at all except possibly one point. One of these conditions is that certain arbitrarily small simple closed curves on \(M\) can be collared from \(U\). Another condition is that there exists a certain sequence \(M_1, M_2, \ldots\) of 2-manifolds in \(U\) converging to \(M\) with the property that each unknotted, sufficiently small simple closed curve on each \(M_i\) is nullhomologous on \(M_i\). Moreover, if each of these simple closed curves bounds a disk on a member of the sequence, then it is shown that \(M\) is tame from \(U (M \neq S^2)\). As a result, if \(U\) is the complementary domain of a torus in \(S^3\) that is wild from \(U\) at just one point, then \(U\) is not homeomorphic to the complement of a tame knot in \(S^3\).

1. Introduction. We let \(M\) always denote a compact connected 2-manifold which lies in the interior of a connected triangulated (cf. [2], [20]) 3-manifold \(M^3\) with or without boundary and which separates \(M^3\), and we let \(U\) denote a component of \(M^3 - M\). In Theorem 7 of [7], Burgess gave a sufficient condition for \(M\) to be locally tame from \(U\) mod one point. In §3, we prove that if we add to his condition the requirement that \(M\) not be a 2-sphere, then \(M\) is tame from \(U\). This suggests that there is a weaker sufficient condition for \(M\) to be locally tame from \(U\) mod one point. In §5, we give such a condition.

2. Notations and definitions. We use the abbreviations \(\text{Bd}, \text{cl}, \text{diam}, \text{dist}, \text{Ext},\) and \(\text{Int}\) for “boundary,” “closure,” “diameter,” “distance,” “exterior,” and “interior,” respectively. \(N(K, \epsilon)\) denotes the \(\epsilon\)-neighborhood of a set \(K\), and \(d\) denotes the metric of \(M^3\). We let \(ab\) denote an arc with \(a\) and \(b\) as endpoints and let \(I = [0, 1]\).

Let \(K\) be a subset of \(M^3\). If there exists a homeomorphism \(g\) of \(M^3\) onto itself such that \(g(K)\) is a subpolyhedron of \(M^3\), then we say that \(K\) is tame. \(M\) is said to

\(^{(1)}\) This paper is part of the author’s dissertation, which was prepared under the direction of Professor O. G. Harrold at Florida State University. This research was supported by NASA grant LEWI 56-902 and NSF grants GP 5458 and 8417.

Received by the editors August 6, 1969.


Key words and phrases. Almost locally tame 2-manifolds, 2-manifolds in 3-manifolds, tameness from a complementary domain, wildness from a complementary domain, locally peripherally collared 2-manifolds, convergent sequence of 2-manifolds, locally spanned 2-manifolds, piercing disk, almost locally polyhedral tori, complements of tame knots.
be locally tame from \( U \) at a point \( p \) of \( M \) if there are a 3-cell \( C \) and a disk \( D \) such that
\[
p \in \text{Int } D \subseteq D \subseteq \text{Bd } C, \quad C \cap M = D, \quad \text{and} \quad C - D \subseteq U.
\]

We say that \( M \) is tame from \( U \) if \( M \) is locally tame from \( U \) at all its points. If \( M \) fails to be locally tame from \( U \) at a point \( p \), then \( M \) is said to be wild from \( U \) at \( p \).

The next definition is equivalent to the one given in [7]. We say that \( M \) can be locally peripherally collared from \( U \) at a point \( p \) of \( M \) if for each \( \varepsilon > 0 \) there exist a disk \( D \) and an annulus \( A \) such that
\[
p \in \text{Int } D \subseteq D \subseteq M, \quad A \cap M = \text{Bd } D \subseteq \text{Bd } A, \quad A - \text{Bd } D \subseteq U, \quad \text{and} \quad \text{diam } (A \cup D) < \varepsilon.
\]

If we substitute the word disk for annulus in the preceding definition, then we obtain what is meant by \( M \) can be locally spanned from \( U \) at \( p \). If \( M \) can be locally peripherally collared from \( U \) at all its points, then we say that \( M \) can be locally peripherally collared from \( U \).

For a subset \( K \) of \( M^3 \) and a point \( p \) of \( K \), we say that \( K \) is locally polyhedral at \( p \) if there is a neighborhood \( W \) of \( p \) in \( M^3 \) such that \( K \cap \overline{W} \) is a polyhedron. In particular, whenever we say that \( K \) is locally polyhedral mod \( X \), we mean that \( K - X \) is locally polyhedral at all its points.

3. 2-manifolds that are tame. The following hypothesis is called \( H(k) \), \( k=1 \) or \( 2 \), if condition \( (k) \) below always holds:

Let \( M \) be a compact connected 2-manifold which separates a connected 3-manifold \( M^3 \), and let \( U \) be a component of \( M^3 - M \). Suppose \( M \) can be locally peripherally collared from \( U \). Furthermore, suppose there exists a sequence \( M_1, M_2, M_3, \ldots \) of polyhedral 2-manifolds converging to \( M \) such that for some \( y \in U \) each \( M_i \) separates \( y \) from \( M \) in \( M^3 \) and, for some \( \gamma > 0 \) and for each positive integer \( j \), every unknotted simple closed curve in \( M_j \) of diameter less than \( \gamma \) is either

1. the boundary of a disk in \( M_j \), or
2. homologous to 0 on \( M_j \).

Clearly, \( H(1) \) implies \( H(2) \). In [7, Theorem 7, p. 328], Burgess proved that if \( H(1) \) holds, then \( M \) is locally tame from \( U \) modulo one point. Our proofs of Theorem 1 and Theorem 2 rely heavily on Burgess' proofs of [7, Theorem 1, p. 322; Theorem 7, p. 328].

Now let \( M^3, M, M_1, \) and \( U \) be as in \( H(k) \), \( k=1 \) or \( 2 \). Choose two distinct points \( p_1, p_2 \in M \) and a number \( \delta > 0 \). There is a disk \( K \) in \( M \) which can be chosen in either of two ways: either (i) \( \text{Int } K \) contains both \( p_1 \) and \( p_2 \) or (ii) \( \text{Int } K \) contains at least \( p_1 \), and some polyhedral 3-cell \( C \) contains \( N(K, \delta) \). For convenience in stating the following two lemmas, we assume (i) and (ii) simultaneously hold. Since \( M \) can be locally peripherally collared from \( U \), there exist (for \( i=1,2 \)) disjoint disks \( D_i \) and disjoint annuli \( A_i \) such that
\[
p_i \in \text{Int } D_i \subseteq D_i \subseteq \text{Int } K, \quad A_i \cap M = \text{Bd } D_i \subseteq \text{Bd } A_i, \quad A_i - \text{Bd } D_i \subseteq U, \quad \text{and} \quad \text{diam } (A_i \cup D_i) < \delta.
\]
Then the following result is due to Burgess [7, p. 329].

**Lemma 1.** There exists a positive number $\sigma < \delta$ such that each simple closed curve in $N(K, \sigma)$ can be shrunk to a point in the component of $M - M_1$ that contains $K$ and such that for each arc $p_i x$ $(i = 1, 2)$ in $N(M, \sigma)$ with $\text{diam } p_i x \geq 3\delta$ there are a subarc $p_i b$ of $p_i x$, a point $c \in K - (D_1 \cup D_2)$ and an arc $bc$ for which

$$p_i b \subset N(K, \sigma), \quad \text{diam } bc < \sigma, \quad \text{and } \quad bc \cap (A_1 \cup D_1 \cup A_2 \cup D_2) = \emptyset.$$

We now prove a similar lemma.

**Lemma 2.** There exist positive numbers $\sigma < \sigma' < \delta$ such that each simple closed curve in $N(K, \sigma')$ can be shrunk to a point in $C - M_1$ and such that, for each arc $p_i x$ in $N(M, \sigma)$ with $\text{diam } p_i x \geq 3\delta$, there are a subarc $p_i b$ of $p_i x$, a point $c \in K - D_1$, and an arc $bc$ for which

$$p_i b \subset N(K, \sigma), \quad \text{diam } bc < \sigma', \quad \text{and } \quad bc \cap (A_1 \cup D_1) = \emptyset.$$

**Proof.** We may assume, by choosing a smaller number $\delta$ if necessary, that there is a disk $D_1'$ such that

$$D_1' \subset \text{Int } D_1' \subset \text{Int } K \quad \text{and} \quad \text{dist } (\text{cl } (M - D_1'), A_1 \cup D_1) \geq 3\delta.$$

Choose a positive number $\alpha < \delta/2$ such that $N(K, \alpha) \subset C$ and $M_1 \cap N(K, \alpha) = \emptyset$. Since $K$ is an absolute neighborhood retract, there exists a neighborhood $V$ of $K$ in $N(K, \alpha)$ and a retraction $r : V \to K$. By [22, Lemma 1, p. 5], there is a neighborhood $W$ of $K$ in $V$ and a homotopy $h : W \times I \to V$ such that $h_0 = \text{the identity map}$ and $h_1 = r$. Choose a positive number $\sigma' < \alpha$ such that $N(K, \sigma') \subset W$. It follows that each simple closed curve in $N(K, \sigma')$ can be shrunk to a point in $V$. Since $V \subset C - M_1$, each simple closed curve in $N(K, \sigma')$ can be shrunk to a point in $C - M_1$. Since $C$ is uniformly locally arcwise connected, there exists a positive number $\sigma < \sigma'$ such that if $y$ and $z$ are two points in $C$ for which $d(y, z) < \sigma$, then there is an arc $yz$ for which $\text{diam } yz < \sigma'$.

Now let $p_i x$ be an arc in $N(M, \sigma)$ with $\text{diam } p_i x \geq 3\delta$. Let $b$ be the first point of $p_i x$, as one goes from $p_1$ to $x$, such that $b \in \text{Fr } [N(p_1, 3\delta/2)]$ where $\text{Fr}$ denotes frontier. Then $p_i b - b \subset N(p_1, 3\delta/2)$. Since $d(\text{cl } (M - D_1'), A_1 \cup D_1) \geq 3\delta$ and $\sigma < \delta/2, N(\text{cl } (M - D_1'), \sigma) \cap \text{cl } N(p_1, 3\delta/2) = \emptyset$. Therefore

$$p_i b \subset N(M, \sigma) \cap \text{cl } N(p_1, 3\delta/2) \subset N(K, \sigma).$$

Since $b \in N(K, \sigma)$, there is a point $c \in K$ such that $d(b, c) < \sigma$. Assume $c \in D_1$. Then

$$d(p_1, b) \leq d(p_1, c) + d(c, b) < \delta + \sigma < \delta + \delta/2 = 3\delta/2.$$

Therefore $b \in N(p_1, 3\delta/2)$, a contradiction to the way $b$ was chosen. Hence $c \notin D_1$. This shows that $c \in K - D_1$.

Since $d(b, c) < \sigma$, there is an arc $bc$ for which $\text{diam } bc < \sigma'$. Let $y \in bc$. By the triangle inequality,

$$d(p_1, y) \geq d(p_1, b) - d(y, b) > 3\delta/2 - \sigma' > 3\delta/2 - \delta/2 = \delta.$$
Therefore since \( p_1 \in A_1 \cup D_1 \) and diam \((A_1 \cup D_1) < \delta\), \( y \notin A_1 \cup D_1 \). This shows that \( be \cap (A_1 \cup D_1) = \emptyset \).

**Theorem 1.** If \( H(1) \) holds and if \( M \) is not a 2-sphere, then \( M \) is tame from \( U \).

**Proof.** By [19, Theorem 2, p. 166], there exists in an arbitrary neighborhood of \( M \) in \( M^3 \) a polyhedral subset \( L \) that is homeomorphic to \( M \times I \) and there exists a finite disjoint collection \( H_1, H_2, \ldots, H_q \) of arbitrarily small polyhedral cubes with handles in \( M^3 \) such that

- each \( H_i \) meets \( L \) precisely in a disk in \((\text{Bd } H_i) \cap \text{Bd } L\),
- \( M \subseteq \text{Int} \left( L \cup H_1 \cup H_2 \cup \cdots \cup H_q \right) \), and \( y \notin L \cup H_1 \cup H_2 \cup \cdots \cup H_q \), where \( y \) is some point of \( U \) which each \( M_i \) separates from \( M \) in \( M^3 \).

By [7, Theorem 7, p. 328], there exists a point \( p_1 \in M \) such that \( M \) is locally tame from \( U \) at each point of \( M - p_1 \). Let \( \varepsilon \) and \( \delta \) be positive numbers such that

\[
\delta < \gamma, \quad 7\delta < \varepsilon,
\]

and

\[
N(M, \delta) \subseteq \text{Int} \left( L \cup H_1 \cup H_2 \cup \cdots \cup H_q \right)
\]

and such that there exist a disk \( K \) and a polyhedral 3-cell \( C \) for which

\[
p_1 \in \text{Int } K \subseteq K \subseteq M, \\
N(K, \delta) \subseteq C \subseteq \text{Int} \left( L \cup H_1 \cup H_2 \cup \cdots \cup H_q \right),
\]

and

\[
C \cap \bigcup_{i=1}^{q} [(\text{Bd } H_i) \cap \text{Bd } L] = \emptyset.
\]

For this last condition to hold, it might be necessary to make a slight adjustment of \( \text{Bd } L \) near \( p_1 \).

Since \( M \) can be locally peripherally collared from \( U \), there exist a disk \( D_1 \) and an annulus \( A_1 \) such that

- \( p_1 \in \text{Int } D_1 \subseteq D_1 \subseteq \text{Int } K \), \( A_1 \cap M = \text{Bd } D_1 \subseteq \text{Bd } A_1 \),
- \( A_1 - \text{Bd } D_1 \subseteq U \), and \( \text{diam } (A_1 \cup D_1) \) < \( \delta \).

Therefore \( A_1 \cup D_1 \subseteq C \). Let \( \sigma \) and \( \sigma' \) be as in Lemma 2. Let \( J = (\text{Bd } A_1) - \text{Bd } D_1 \), \( a_1 \in J \), and \( A'_1 \) be an annulus in \( A_1 \) such that

\[
A'_1 \cap M = \text{Bd } D_1, \quad A'_1 \cap M_1 = \emptyset, \quad \text{and } A'_1 \subseteq N(K, \sigma).
\]

By [3, Theorem 7, p. 478], we may assume \( A_1 \) is locally polyhedral mod \( \text{Bd } D_1 \).

Without loss of generality, we assume

\[
M_1 \subseteq N(M, \delta), \\
M_1 \text{ separates } J \text{ from } M \text{ in } M^3,
\]
there is a 2-manifold $M_0$ which separates $M_1$ from $M_2$ in $M^3$ such that for each point $w$ in $U$ such that $\dist(w, M) \geq \sigma$, there is an arc $w_Y$ such that $w_Y \cap M_0 = \emptyset$, $M_0 \cap A_1 = \emptyset$, $M_2 \subset N(M, \sigma)$, $M_2$ separates $M_0 \cup M_1 \cup (A_1 - A_1^*)$ from $M$ in $M^3$, and $M_2$ and $A_1$ are in relative general position.

It follows that each component of $A_1 \cap M_2$ is a simple closed curve in $A_1$'. Since each simple closed curve in $A_1$ is unknotted and has diameter less than $\gamma$, then according to $H(1)$, each simple closed curve in $A_1 \cap M_2$ is the boundary of a disk in $M_2$. Furthermore, since $M_2$ separates $J$ from $\bdy D_1$ in $M^3$, some component of $A_1 \cap M_2$ separates $J$ from $\bdy D_1$ in $A_1$. Therefore there exists a disk $D' \subset M_2$ such that

\[
\bdy D' \subset A_1^*,
\]

\[
\bdy D' \text{ is not the boundary of a disk in } A_1, \quad \text{and each component of } A_1 \cap \text{Int } D' \text{ is the boundary of a disk in } A_1'.
\]

Thus, by replacing certain subdisks of $D'$ by disks near $A_1'$, we can adjust $D'$ to a polyhedral disk $D$ such that

\[
\bdy D = \bdy D', \quad \text{Int } D \subset U - (A_1 \cup M_0 \cup M_1), \quad \text{and } D \subset D' \cup N(K, \sigma).
\]

The last inclusion follows from the fact that $A_1 \subset N(K, \sigma)$.

Let $A'$ be the annulus in $A_1'$ such that $\bdy A' = (\bdy D_1) \cup \bdy D_1$ and let $S = A' \cup D \cup D_1$. $S$ is a 2-sphere which is locally polyhedral mod $D_1$, and $S - D_1 \subset U$.

Now suppose $M$ is orientable. Therefore $L \cup H_1 \cup H_2 \cup \cdots \cup H_q$ can be considered to be already imbedded in $S^3$. It then follows from the next lemma (Lemma 3) that $M$ is tame from $U$.

Next, suppose $M$ is nonorientable and has genus $p$. For convenience, we identify $L$ with a polyhedron $B^p \times I \subset M^3$, where $B^p$ is homeomorphic to $M$. Then

\[
L \cup H_1 \cup H_2 \cup \cdots \cup H_q = (B^p \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q.
\]

The following argument, which shows that $S$ separates $M^3$, is analogous to the proof of [12, Lemma 15, p. 414].

Let $J_p$ be a median of a Moebius band $N$ in $B^p$. Since each $H_i \cap (B^p \times I)$ is a disk which does not intersect $C$ and since $D_1 \subset C$, it may be assumed that

\[
N \cap S = \emptyset \quad \text{and} \quad (J_p \times I) \cap \left( D_1 \cup \bigcup_{i=1}^{q} \left[ \bdy H_i \cap \bdy L \right] \right) = \emptyset.
\]

By relative general position, it may be assumed that each component of $S \cap (J_p \times I)$ is a polyhedral simple closed curve. Let $F$ be such a component which is the boundary of a disk $E$ in $S$ such that

\[
(\text{Int } E) \cap (J_p \times I) = \emptyset \quad \text{and} \quad E \cap D_1 = \emptyset.
\]
$F$ must be nullhomotopic in $J_p \times I$; otherwise, there is an annulus $A$ in $J_p \times I$ such that $\text{Bd } A = F \cup J_p$. Then $A \cup E$ is a disk such that $(A \cup E) \cap N = J_p$, a situation which contradicts [12, Lemma 7, p. 409]. Thus $F$ must be the boundary of a disk $E'$ in $J_p \times I$. It follows from the proof of [12, Lemma 15, p. 414] that the 2-sphere $E \cup E'$ is the boundary of a 3-cell $G$ in $(B^p \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q$. By deforming $E$ across $G$ onto $E'$ and then off $J_p \times I$, we can eliminate the component $F$ of $S \cap (J_p \times I)$. Therefore we may assume a priori that $S \cap (J_p \times I) = \emptyset$. By cutting $B^p$ along $J_p$, we obtain a 2-manifold $B^p \smash{\times}^{-1}$ of genus $p - 1$ with one contour such that

$$S \subset \text{Int } [(B^p \smash{\times}^{-1} I) \cup H_1 \cup H_2 \cup \cdots \cup H_q].$$

By induction, we obtain a 2-manifold $B^0$ of genus zero with $p$ contours such that

$$S \subset \text{Int } [(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q].$$

Since $B^0$ is orientable, $(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q$ can be considered to be already imbedded in $S^3$. The closure of each component of $S^3 - S$ is a crumpled cube (by definition). Since $\text{Bd } B^0 \neq \emptyset$, $\text{Bd } [(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q]$ is connected and therefore contained in a single component of $S^3 - S$. Then the closure of the other component is a crumpled cube $Q$ in

$$\text{Int } [(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q]$$

and hence in $\text{Int } (L \cup H_1 \cup H_2 \cup \cdots \cup H_q)$. Consequently $S$ separates $M^3$.

Assume $S$ separates $y$ from $M - D_1$ in $M^3$. Then since $y \notin \text{Int } Q$, $M - D_1 \subset \text{Int } Q$. Since $D_1 \subset S \subset Q$, $M \subset Q$. Therefore since $Q$ is a crumpled cube, it along with $M$ can be imbedded in $S^3$. This is a contradiction to the fact that a closed nonorientable 2-manifold like $M$ cannot be imbedded in $S^3$ [8, Theorem 22, p. 182]. Therefore $S$ does not separate $y$ from $M - D_1$ in $M^3$.

We now show that $\text{diam } D \leq 68$. Assume otherwise. Using the methods of [23, p. 66], we construct an arc $p_2a_1$ such that $p_2a_1 \subset (p_1 \cup a_1) \subset U - A_1$. There exists an arc $a_1y$ such that $a_1y - a_1 \subset U - (A_1 \cup M_1 \cup p_2a_1)$. Since $y$ and $M - D_1$ are in the same component of $M^3 - S$, there is an arc $yp_2$ such that

$$p_2 \in M - D_1 \quad \text{and} \quad yp_2 \subset U - (A_1 \cup D \cup p_2a_1 \cup a_1y).$$

Let $p_2p_1$ be an arc such that

$$p_2p_1 \subset (p_2 \cup p_1) \subset M^3 - (M \cup U),$$

and let $J^*$ denote the simple closed curve $p_1a_1 \cup a_1y \cup yp_2 \cup p_2p_1$. By construction $J^* \cap D = p_1a_1 \cap D, J^* \cap D \neq \emptyset$ because

$$J^* \cap (S - D) = p_1^*, \quad J^* \text{ pierces } S \text{ at } p_1, \quad \text{and } \quad S \text{ separates } M^3.$$

Therefore $p_1a_1 \cap D \neq \emptyset$. Let $a$ be the first point of $p_1a_1$ (as one goes from $p_1$ to $a_1$) such that $a \in D$, and let $p_1a$ denote the subarc of $p_1a_1$. Since it is assumed that $\text{diam } D \geq 68$, there is a point $x \in D$ such that $d(a, x) \geq 38$. Let $ax$ be an arc in $D,$
and let \( p_1 x = p_1 a \cup ax \). Then \( \operatorname{diam} p_1 x \geq 3 \delta \). Assume \( p_1 a \notin N(M, \sigma) \). Then there exists a point \( w \in \operatorname{Int} p_1 a \) such that \( \operatorname{dist} (w, M) \geq \sigma \). Therefore there is an arc \( wy \) such that
\[
\omega - (w \cup y) \subset U - (A' \cup D \cup p_1 a \cup yp_2).
\]
Let \( p_1 w \) be the subarc of \( p_1 a \). Then \( J^* = p_1 w \cup wy \cup yp_2 \cup p_2 p_1 \) is a simple closed curve which intersects and pierces \( S \) precisely at \( p_1 \), a contradiction. Hence \( p_1 a \in N(M, \sigma) \). Since \( ax \subset D \subset N(M, \sigma) \), \( p_1 x \subset N(M, \sigma) \). Therefore by Lemma 2, there exist a subarc \( p_1 b \) of \( p_1 x \), a point \( c \in K - D_1 \), and an arc \( bc \) such that
\[
\begin{align*}
p_1 b & \subset N(K, \sigma), \quad \operatorname{diam} bc < \sigma', \quad \text{and} \quad bc \cap (A_1 \cup D_1 \cup \operatorname{Int} p_1 b) = \emptyset.
\end{align*}
\]
Let \( cp_1 \) be an arc in \( N(K, \sigma) \) such that
\[
\omega - (c \cup p_1) \subset M^3 - (M \cup U \cup bc),
\]
and let \( J' \) denote the simple closed curve \( p_1 b \cup bc \cup cp_1 \). By construction,
\[
J' \subset N(K, \sigma), \quad J' \cap (A_1 \cup D_1) = p_1, \quad \text{and} \quad J' \text{ pierces } A_1 \cup D_1 \text{ at } p_1.
\]
Therefore, since \( J' \cup A_1 \cup D_1 \subset C \), \( J' \) links \( J \) in \( C \). But since \( J' \subset N(K, \sigma) \), we can apply Lemma 2 to shrink \( J' \) to a point in the component of \( C - M_1 \) that contains \( K \). Since \( M_1 \) separates \( J \) from \( K \) in \( M^3 \), \( C \cap M_1 \) separates \( J \) from \( K \) in \( C \). Therefore \( J' \) does not link \( J \) in \( C \), a contradiction. Hence we must have that \( \operatorname{diam} D < 6 \delta \).

Therefore
\[
\operatorname{diam} (D_1 \cup A' \cup D) \leq \operatorname{diam} (D_1 \cup A') + \operatorname{diam} D < 8 + 6 \delta = 7 \delta < \varepsilon.
\]

Since the disks \( D_1 \) and \( A' \cup D \) are those in the definition of local spanning, we have shown that \( M \) can be locally spanned from \( U \) at \( p_1 \). Since \( M \) is locally spanned from \( U \) at all other points, it follows from [6, Theorem 10, p. 88] and from the proof of [6, Theorem 16, pp. 95–96] that \( M \) is locally tame from \( U \).

The proof of the following lemma finishes the proof of Theorem 1.

**Lemma 3.** If \( H(1) \) holds for \( M^3 = S^3 \) and if \( M \) is not a 2-sphere, then \( M \) is tame from \( U \).

**Proof.** We let \( D_3, A_3, D', A', S \), and \( p_1 a_1 \) be those sets constructed in the proof of the nonorientable case of Theorem 1. In the proof of that case of Theorem 1, the nonorientability of \( M \) was used just to show that \( p_1 a_1 \cap D \neq \emptyset \). Therefore we only need to show again that \( p_1 a_1 \cap D \neq \emptyset \) for the case when \( M \) is an orientable manifold in \( S^3 \).

On the contrary, assume \( p_1 a_1 \cap D = \emptyset \). If we assume that \( S \) does not separate \( y \) from \( M - D_1 \) in \( S^3 \), then we can construct the simple closed curve \( J^* \) exactly as done in the proof of Theorem 1. But now \( J^* \) intersects and pierces \( S \) precisely at \( p_1 \), a contradiction. Therefore \( S \) must separate \( y \) from \( M - D_1 \) in \( S^3 \). Let \( Q_0 \) be that component of \( S^3 - S \) which contains \( y \). Then \( Q_0 \subset U \).
It follows from [4, Theorem 5, p. 302] and from [17, p. 666] or [18, Theorem 2, p. 541] that we may assume $M$ is tame from $S^3 - \text{cl } U$. By [2, Theorem 9, p. 157], we may further assume that $M$ is locally polyhedral mod $p_1$; and consequently, we may assume that $S$ was constructed to be locally polyhedral mod $p_1$. It now follows from [9, Theorem 1, p. 250] that $Q_0$ is an open 3-cell.

Since $S$ could have been constructed in an arbitrary neighborhood of $M$ in $S^3$, it is clear that $S$ is just one member of a sequence $S_0 (= S), S_1, S_2, \ldots$ of 2-spheres such that, for each nonnegative integer $i$,

1. $S_i - [S_i \cap (A_i \cup D_i)]$ is a polyhedral disk in $U$,
2. $S_i \subset N(M, 1/i)$,
3. $S_i$ separates $y$ from $M - D_i$ in $S^3$.
4. if $Q_i$ is that component of $S^3 - S_i$ containing $y$, then $Q_i$ is an open 3-cell in $U$, and
5. $Q_i \subset Q_{i+1}$.

Now (1), (2), and (3) imply that $S_0, S_1, S_2, \ldots$ converge to $M$. Therefore $U = \bigcup_{i=0}^{\infty} Q_i$. Hence (4) and (5) imply that $U$ is an open 3-cell [5, p. 813]. Since a 2-manifold in $S^3$ is a 2-sphere if it is the boundary of an open 3-cell, we obtain the contradiction that $M$ is a 2-sphere. Thus $p_1 a_1 \cap D \neq \varnothing$. This completes the proof of Lemma 3.

We require that $M$ not be a 2-sphere in the hypothesis of Theorem 1 because the 2-sphere $M$ in Example 3.2 of [13, p. 990] is not tame from one component $U$ of $S^3 - M$, but $H(1)$ holds.

4. Some corollaries. For a 2-manifold $M$ which separates a 3-manifold $M^3$, we say that $M$ can be pierced on an arc $A \subset M$ with a disk $D$ if $\text{Int } A \subset \text{Int } D$, $\text{Bd } A \subset \text{Bd } D$, and the two components of $D - A$ lie in different components of $M^3 - M$; and we call $D$ a piercing disk. Eaton [11, p. 510] proved that a 2-sphere $S$ in $E^3$ is tame if $S$ can be pierced on each of its arcs with a tame disk.

Let us remove from $H(1)$ the condition that $M$ can be locally peripherally collared from $U$, and let us call the remaining hypothesis $H'(1)$. The following corollary is an extension of [7, Theorem 9, p. 329] because we do not require that the piercing disks be tame.

**Corollary 1.** If

(i) $M$ is not a 2-sphere,
(ii) $H'(1)$ holds for each component $U$ of $M^3 - M$, and
(iii) $M$ can be pierced on each of its arcs with a disk,

then $M$ is tame.

**Proof.** The proof is essentially the same as Burgess' proof in [7, Theorem 8, p. 329] with out Theorem 1 used in place of his Theorem 7.

In [1], Alexander proved that if $S$ is a polyhedral 2-sphere in $S^3$, then each component of $S^3 - S$ has a closure which is a 3-cell. Harrold and Moise [15, Theorems
I, II, p. 577] generalized this result by showing that if $S$ is a 2-sphere in $S^3$ which is locally polyhedral mod one point, then one component of $S^3 - S$ has a closure which is a 3-cell and the other component is simply connected. In fact, Cantrell [9, Theorem 1, p. 250] proved that this other component is an open 3-cell. Thus, the components of $S^3 - S$ are open 3-cells; however, the analogous situation is different for a torus. One of the complementary domains of a polyhedral torus in $S^3$ has a closure which is a solid torus [1], and therefore the other component is homeomorphic to the complement of a tame knot in $S^3$. But if $M$ is a torus in $S^3$ which is locally polyhedral mod one point $p$ and wild from $U$ at $p$, then it follows from the next corollary that one and only one component of $S^3 - M$ is homeomorphic to the complement of a tame knot in $S^3$ (Daverman [10] has independently proved such a result; for completeness, we give here an alternative but similar proof).

**Corollary 2.** Let $M$ be a torus in $S^3$, $U$ a component of $S^3 - M$, and $p \in M$. If $M$ is locally polyhedral mod $p$ and if $U$ is homeomorphic to the complement of a tame knot in $S^3$, then $M$ is tame from $U$.

**Proof.** Since $U$ is homeomorphic to the complement of a tame knot in $S^3$, it follows from [20, Theorem 2, p. 97] that there exists a sequence $M_1, M_2, M_3, \ldots$ of disjoint polyhedral tori converging to $M$ such that, for some $y \in U$, each $M_i$ separates $y$ from $M$ in $S^3$.

Let $s_1$ and $s_2$ be disjoint polyhedral simple closed curves each nonnullhomologous on $M$, and let $A_1$ and $A_2$ be disjoint polyhedral annuli such that, for $i = 1, 2$,

$$A_i \cap M = s_i \subset \text{Bd } A_i \quad \text{and} \quad A_i - s_i \subset U.$$  

We may assume that each $M_i$ separates $s_1 \cup s_2$ from $[(\text{Bd } A_1) - s_1] \cup [(\text{Bd } A_2) - s_2]$ in $S^3$ and that $A_1 \cup A_2$ and each $M_i$ are in general position.

Since $M$ is an absolute neighborhood retract, it is a retract of one of its neighborhoods $V$ in $S^3$. Neither $s_1$ nor $s_2$ can be shrunk to a point in $V$. There is a number $\delta > 0$ such that if $K$ is a set of diameter less than $\delta$, then either $K \cap A_1 = \emptyset$ or $K \cap A_2 = \emptyset$. Let $C_1, C_2, \ldots, C_n$ be 3-cells in $V$ each of diameter less than $\delta$ and let $W$ be a neighborhood of $M$ in $S^3$ such that $\text{cl } W \subseteq \bigcup_{i=1}^{n} \text{Int } C_i$. There is a positive number $\gamma < \delta$ such that if $K$ is a subset of $W$ of diameter less than $\gamma$, then $K \subseteq \text{Int } C_m$ for some $m$ ($1 \leq m \leq n$). Without loss of generality, we may assume

$$A_1 \cup A_2 \cup \left( \bigcup_{i=1}^{\infty} M_i \right) \subseteq W.$$  

We now show that $H(1)$ holds. Let $j$ be an arbitrary positive integer and $s$ an unknotted simple closed curve in $M_j$ of diameter less than $\gamma$. We suppose $s$ is not the boundary of a disk in $M_j$. Since $s \subseteq \text{Int } C_m$ for some $m$, $s$ can be shrunk to a point in $\text{Int } C_m$. Since $\text{diam } C_m < \delta$, either $C_m \cap A_1 = \emptyset$ or $C_m \cap A_2 = \emptyset$. We may assume $C_m \cap A_1 = \emptyset$. Let $s'$ be a component of $M_j \cap A_4$ that is nonnullhomologous on $M_j$. Since $s \cap s' = \emptyset$, $s$ and $s'$ bound an annulus $A$ on $M_j$. Let $A'$ be the annulus in $A_1$.
such that \( \text{Bd } A' = s_1 \cup s' \). Then \( s_1 \) can be shrunk to a point in the subset \( A' \cup A \cup \text{Int } C_m \) of \( V \), a contradiction. Therefore \( s \) must be the boundary of a disk in \( M \). According to Theorem 1, \( M \) is tame from \( U \).

5. 2-manifolds that are almost tame. We state the following lemma without proof.

Lemma 4. Let \( H \) be a closed 2-manifold and \( J \) a simple closed curve which lies in a 3-manifold \( M^3 \) in such a way that \( J \) intersects and pierces \( H \) at exactly one point. Then \( J \) cannot be shrunk to a point in \( M^3 \).

Now, by replacing the hypothesis \( H(1) \) of [7, Theorem 7, p. 328] with the weaker condition \( H(2) \), we obtain the following stronger result. The symbol \( \sim \) is used to stand for "is homologous to."

Theorem 2. If \( H(2) \) holds, then there is a point \( p \) such that \( M \) is locally tame from \( U \mod p \).

Proof. Let \( p_1 \) and \( p_2 \) be two arbitrary points in \( M \). If we show that \( M \) can be locally spanned from \( U \) at either \( p_1 \) or \( p_2 \), then it follows from [6, Theorem 10, p. 88] that there must be a point \( p \) such that \( M \) is locally tame from \( U \mod p \).

Let \( K_0 \) be a disk in \( M \) such that \( p_1 \cup p_2 \subseteq \text{Int } K_0 \). Since \( K_0 \) is an absolute neighborhood retract, there is a neighborhood \( V_0 \) of \( K_0 \) in \( M^3 \) that retracts onto \( K_0 \). It follows from [22, Lemma 1, p. 5] that there is a neighborhood \( W_0 \) of \( K_0 \) in \( V_0 \) such that each simple closed curve in \( W_0 \) can be shrunk to a point in \( V_0 \).

For \( i = 1, 2 \), there are a disk \( K_i \) and a polyhedral 3-cell \( C_i \) in \( W_0 \) such that

\[
p_i \subseteq \text{Int } K_i \subseteq K_i \subseteq M \cap \text{Int } C_i.
\]

Let \( \varepsilon, \delta, \) and \( y' \) be positive numbers such that

\[
7\delta < \varepsilon, \quad 6\delta < y', \quad 3\delta + y' < \gamma, \quad \text{and } N(K_i, 7\delta) \subseteq C_i.
\]

Since \( M \) can be locally peripherally collared from \( U \), there exist (for \( i = 1, 2 \)) disjoint disks \( D_i \) and disjoint annuli \( A_i \) such that

\[
p_i \subseteq \text{Int } D_i \subseteq D_i \subseteq \text{Int } K_i, \quad A_i \cap M = \text{Bd } D_i \subseteq \text{Bd } A_i, \quad A_i - \text{Bd } D_i \subseteq U, \quad \text{and diam } (A_i \cup D_i) < \delta.
\]

It follows that \( A_i \cup D_i \subseteq C_i \). It is clear from the proof of Lemma 2 that there exist positive numbers \( \sigma < \sigma' < \delta \) such that, for \( i = 1, 2 \), each simple closed curve in \( N(K_i, \sigma') \) can be shrunk to a point in \( C_i - M_1 \) and such that, for each arc \( p_i x \) in \( N(M, \sigma) \) with \( \text{diam } p_i x \geq 3\delta \), there are a subarc \( p_i b \) of \( p_i x \), a point \( c \in K_i - D_i \), and an arc \( bc \) for which

\[
p_i b \subseteq N(K_i, \sigma), \quad \text{diam } bc < \sigma', \quad \text{and } bc \cap (A_i \cup D_i) = \emptyset.
\]

For \( i = 1, 2 \), we let \( J_i = (\text{Bd } A_i) - \text{Bd } D_i \) and \( a_i \in J_i \). There is an arc \( a_1 a_2 \) such that

\[
a_1 a_2 - (a_1 \cup a_2) \subseteq (U \cap W_0) - (A_1 \cup A_2).
\]
For each $i$, let $A'_i$ be an annulus in $A_i$ such that

$$A'_i \cap M = \text{Bd} \; D_i, \quad A'_i \cap M_2 = \emptyset, \quad \text{and} \quad A'_i \subseteq N(K_i, \sigma).$$

We may assume, without loss of generality, that

$A_i$ is locally polyhedral mod Bd $D_i$ [3, Theorem 7, p. 478],

$M_1 \subseteq N(M, \delta)$,

$M_1$ separates $J_1 \cup J_2 \cup a_1a_2$ from $M$ in $M^3$,\n
$M_2 \subseteq N(M, \sigma)$,

$M_2$ separates $(M_1 \cup A_1 \cup A_2) - (A'_1 \cup A'_2)$ from $M$ in $M^3$, and

$M_2$ and $A_1 \cup A_2$ are in relative general position.

Now, $	ext{diam} \; A_i < \delta < \gamma'$. Therefore since each component of $(A_1 \cup A_2) \cap M_2$ is an unknotted simple closed curve in $A'_1 \cup A'_2$ of diameter less than $\gamma$, then according to $H(2)$, each such simple closed curve is homologous to 0 on $M_2$. For $i = 1, 2$, some component of $A_i \cap M_2$ separates $J_i$ from Bd $D_i$ in $A_i$ because $M_2$ separates $J_i$ from Bd $D_i$ in $M^3$; it follows that one such component $s_0$ is a chain which bounds a 2-manifold $H_0$ in $M_2$ such that each component of $(A_1 \cup A_2) \cap (H_0 - \text{Bd} \; H_0)$ is the boundary of a disk in $A'_1 \cup A'_2$. We may assume $s_0 \subseteq A_i$.

Suppose $s_i$ is a simple closed curve in $M_2 \cap A_i$ ($i = 1$ or 2) which is the boundary of a disk $D$ in $A_i$. We may assume $M_2 \cap \text{Int} \; D = \emptyset$. Slightly thicken $D$ to obtain a polyhedral 3-cell $B$ in $U \cap N(K_i, \sigma)$ such that

$\text{Int} \; D = \text{Int} \; B$,\n
$\text{Bd} \; D = \text{Bd} \; B$,\n
$B \cap A_i = D_i$, \n
$B \cap M_2 = (\text{Bd} \; B) \cap M_2 - A$, an annulus with $s_1$ as center line,\n
$\text{Bd} \; B - \text{Int} \; A = E_1 \cup F_1$, where $E_1$ and $F_1$ are disjoint disks, and

$B \subseteq N(A_i, \delta)$. \n
Let $R = (M_2 - A) \cup E_1 \cup F_1$. $R \subseteq N(M, \sigma)$ and $R \cap A_i = (M_2 \cap A_i) - s_i$. Since $s_i \sim 0$ on $M_2$, $R$ consists of two components $R_1$ and $T_1$ with $E_1 \subseteq R_1$ and $F_1 \subseteq T_1$. We may assume $s_0 \subseteq R_1$.

In the above fashion, we may inductively construct closed 2-manifolds $R_k$ and disjoint disks $E_k$ ($k \geq 1$) such that

(1) $E_k \subseteq R_k \subseteq N(M, \sigma)$,

(2) $R_k - E_k = R_{k-1}$ (we define $R_0 = M_2$),

(3) the chain $s_0$ bounds a 2-manifold $H_k$ in $R_k$ such that each component of $(A_1 \cup A_2) \cap (H_k - \text{Bd} \; H_k)$ is the boundary of a disk in $A'_1 \cup A'_2$,

(4) $(A_1 \cup A_2) \cap R_k$ has fewer components than $(A_1 \cup A_2) \cap R_{k-1}$, and

(5) either $E_k \subseteq N(A_1, \delta)$ or $E_k \subseteq N(A_2, \delta)$.

For each $k$, let $E_k^* = \bigcup_{j=1}^{k} E_j$.

Let $s$ be an arbitrary unknotted simple closed curve in $R_k$ such that $\text{diam} \; s < \gamma'$. It is possible that $s \cap E_k^* \neq \emptyset$. Nevertheless, (5) implies that there is an unknotted simple closed curve $s'$ in $R_k - E_k^*$ such that $s' \sim s$ on $R_k$ and $\text{diam} \; s' < 3\delta + \gamma' < \gamma$.
Therefore since (2) implies \( s' \subseteq R_k - E^*_k \subseteq M_2 \), we must have \( s' \sim 0 \) on \( M_2 \) and thus on \( R_k \). Then \( s \sim 0 \) on \( R_k \) because \( s \sim s' \) on \( R_k \).

Now, (3) and (4) imply that the inductive construction stops at some positive integer \( n \) for which

\[
(A_1 \cup A_2) \cap (H_n - \text{Bd } H_n) = \emptyset.
\]

Therefore \( (A_1 \cup A_2) \cap H_n = s_0 \). By (1), \( H_n \subseteq N(M, s) \). Let \( A' \) be the annulus in \( A'_1 \) such that \( \text{Bd } A' = s_0 \cup \text{Bd } D_1 \); and let \( H = H_n \cup A' \cup D_1 \), which is a closed 2-manifold.

Using the methods of [23, p. 66], we construct an arc \( p_1 a_1 \) such that

\[
p_1 a_1 - (p_1 \cup a_1) \subseteq (U \cap W_0) - (A_1 \cup A_2 \cup a_2 a_2).
\]

Let \( a_2 p_2 \) be an arc in \( A_2 \cup D_2 \), and let \( p_2 p_1 \) be an arc such that

\[
p_2 p_1 - (p_2 \cup p_1) \subseteq W_0 - (M \cup U).
\]

Let \( J \) denote the simple closed curve \( p_1 a_1 \cup a_2 a_2 \cup a_2 p_2 \cup p_2 p_1 \). By construction, \( J \) intersects and pierces \( H - H_n \) at precisely the point \( p_1 \). Since \( J \subseteq W_0 \), \( J \) can be shrunk to a point in \( V_0 \). Therefore it follows from Lemma 4 that \( J \cap H_n \neq \emptyset \). Then \( p_1 a_1 \cap H_n \neq \emptyset \). We can now use the same techniques of the proof of Theorem 1 (when we proved \( \text{diam } D < 6 \delta \) there) in order to show that \( \text{diam } H_n < 6 \delta < \gamma' \). Since \( N(K, 7 \delta) \subseteq C_1 \), \( H_n \subseteq C_1 \).

By [21, p. 1] or [3, Theorem 7, p. 478], there is a polyhedral 2-manifold \( H' \) in \( C_1 \) such that

\[
H' \text{ is homeomorphic to } H, \quad H_n \subseteq H', \quad \text{and } \text{cl } (H' - H_n) \text{ is a disk}.
\]

Suppose \( H_n \) is not a disk. Then \( H' \) has genus greater than zero. Therefore by [14, Theorem 1, p. 462] or [16, Theorem 1, p. 129], there exists an unknotted simple closed curve \( t \) in \( H' \) not homologous to 0 on \( H' \). Since \( \text{cl } (H' - H_n) \) is a disk, there is an unknotted simple closed curve \( t' \) in \( H_n \) such that \( t' \sim t \) on \( H' \). Therefore \( t' \) is not homologous to 0 on \( H' \) and thus on \( R_n \). But since \( t' \subseteq H_n \),

\[
\text{diam } t' < \text{diam } H_n < \gamma'.
\]

Then according to what was shown earlier about the simple closed curve \( s \) in \( R_k \), we must have \( t' \sim 0 \) on \( R_n \). Since we have reached a contradiction, \( H_n \) must be a disk. Therefore since \( \text{diam } H < 7 \delta < \varepsilon \), \( M \) can be locally spanned from \( U \) at \( p_1 \). Thus the conclusion of the theorem follows.

**References**