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ALMOST LOCALLY TAME 2-MANIFOLDS IN A 3-MANIFOLD(1)

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Abstract. Several conditions are given which together imply that a 2-manifold M in a 3-manifold is locally tame from one of its complementary domains, U, at all except possibly one point. One of these conditions is that certain arbitrarily small simple closed curves on M can be collared from U. Another condition is that there exists a certain sequence $M_1, M_2, \dots$ of 2-manifolds in U converging to M with the property that each unknotted, sufficiently small simple closed curve on each M_i is nullhomologous on M_i. Moreover, if each of these simple closed curves bounds a disk on a member of the sequence, then it is shown that M is tame from U ($M \neq S^2$). As a result, if U is the complementary domain of a torus in S^3 that is wild from U at just one point, then U is not homeomorphic to the complement of a tame knot in S^3.

1. **Introduction.** We let M always denote a compact connected 2-manifold which lies in the interior of a connected triangulated (cf. [2], [20]) 3-manifold M^3 with or without boundary and which separates M^3, and we let U denote a component of $M^3 - M$. In Theorem 7 of [7], Burgess gave a sufficient condition for M to be locally tame from U mod one point. In §3, we prove that if we add to his condition the requirement that M not be a 2-sphere, then M is tame from U. This suggests that there is a weaker sufficient condition for M to be locally tame from U mod one point. In §5, we give such a condition.

2. **Notations and definitions.** We use the abbreviations Bd, cl, diam, dist, Ext, and Int for "boundary," "closure," "diameter," "distance," "exterior," and "interior," respectively. $N(K, \varepsilon)$ denotes the ε-neighborhood of a set K, and d denotes the metric of M^3. We let ab denote an arc with a and b as endpoints and let $I = [0, 1]$.

Let K be a subset of M^3. If there exists a homeomorphism g of M^3 onto itself such that $g(K)$ is a subpolyhedron of M^3, then we say that K is *tame*. M is said to

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be *locally tame from U at* a point p of M if there are a 3-cell C and a disk D such that

$$p \in \operatorname{Int} D \subset D \subset \operatorname{Bd} C, \qquad C \cap M = D, \quad \text{and} \quad C - D \subset U.$$

We say that M is *tame from U* if M is locally tame from U at all its points. If M fails to be locally tame from U at a point p, then M is said to be *wild from U at p*. The next definition is equivalent to the one given in [7]. We say that M can be *locally peripherally collared from U at* a point p of M if for each $\varepsilon > 0$ there exist a disk D and an annulus A such that

$$p \in \operatorname{Int} D \subset D \subset M, \qquad A \cap M = \operatorname{Bd} D \subset \operatorname{Bd} A,$$
$$A - \operatorname{Bd} D \subset U, \quad \text{and} \quad \operatorname{diam}(A \cup D) < \varepsilon.$$

If we substitute the word *disk* for *annulus* in the preceding definition, then we obtain what is meant by M can be *locally spanned from U at p*. If M can be locally peripherally collared from U at all its points, then we say that M can be *locally peripherally collared from U.*

For a subset K of M^3 and a point p of K, we say that K is *locally polyhedral at p* if there is a neighborhood W of p in M^3 such that $K \cap \operatorname{cl} W$ is a polyhedron. In particular, whenever we say that K is *locally polyhedral* mod X, we mean that $K - X$ is locally polyhedral at all its points.

3. **2-manifolds that are tame.** The following hypothesis is called $H(k)$, $k = 1$ or 2, if condition (k) below always holds:

Let M be a compact connected 2-manifold which separates a connected 3-manifold M^3, and let U be a component of $M^3 - M$. Suppose M can be locally peripherally collared from U. Furthermore, suppose there exists a sequence $M_1, M_2, M_3, \ldots$ of polyhedral 2-manifolds converging to M such that for some $y \in U$ each M_j separates y from M in M^3 and, for some $\gamma > 0$ and for each positive integer j, every unknotted simple closed curve in M_j of diameter less than γ is either

(1) the boundary of a disk in M_j or

(2) homologous to 0 on M_j.

Clearly, $H(1)$ implies $H(2)$. In [7, Theorem 7, p. 328], Burgess proved that if $H(1)$ holds, then M is locally tame from U modulo one point. Our proofs of Theorem 1 and Theorem 2 rely heavily on Burgess' proofs of [7, Theorem 1, p. 322; Theorem 7, p. 328].

Now let M^3, M, M_1, and U be as in $H(k)$, $k = 1$ or 2. Choose two distinct points $p_1, p_2 \in M$ and a number $\delta > 0$. There is a disk K in M which can be chosen in either of two ways: either (i) Int K contains both p_1 and p_2 or (ii) Int K contains at least p_1, and some polyhedral 3-cell C contains $N(K, \delta)$. For convenience in stating the following two lemmas, we assume (i) and (ii) simultaneously hold. Since M can be locally peripherally collared from U, there exist (for $i = 1, 2$) disjoint disks D_i and disjoint annuli A_i such that

$$p_i \in \operatorname{Int} D_i \subset D_i \subset \operatorname{Int} K, \qquad A_i \cap M = \operatorname{Bd} D_i \subset \operatorname{Bd} A_i,$$
$$A_i - \operatorname{Bd} D_i \subset U, \quad \text{and} \quad \operatorname{diam}(A_i \cup D_i) < \delta.$$

Then the following result is due to Burgess [7, p. 329].

LEMMA 1. *There exists a positive number* $\sigma < \delta$ *such that each simple closed curve in* $N(K, \sigma)$ *can be shrunk to a point in the component of* $M^3 - M_1$ *that contains* K *and such that for each arc* $p_i x$ $(i=1, 2)$ *in* $N(M, \sigma)$ *with* $\operatorname{diam} p_i x \geqq 3\delta$ *there are a subarc* $p_i b$ *of* $p_i x$, *a point* $c \in K-(D_1 \cup D_2)$ *and an arc* bc *for which*

$$p_i b \subset N(K, \sigma), \quad \operatorname{diam} bc < \sigma, \quad \text{and} \quad bc \cap (A_1 \cup D_1 \cup A_2 \cup D_2) = \varnothing.$$

We now prove a similar lemma.

LEMMA 2. *There exist positive numbers* $\sigma < \sigma' < \delta$ *such that each simple closed curve in* $N(K, \sigma')$ *can be shrunk to a point in* $C - M_1$ *and such that, for each arc* $p_1 x$ *in* $N(M, \sigma)$ *with* $\operatorname{diam} p_1 x \geqq 3\delta$, *there are a subarc* $p_1 b$ *of* $p_1 x$, *a point* $c \in K - D_1$, *and an arc* bc *for which*

$$p_1 b \subset N(K, \sigma), \qquad \operatorname{diam} bc < \sigma', \quad \text{and} \quad bc \cap (A_1 \cup D_1) = \varnothing.$$

Proof. We may assume, by choosing a smaller number δ if necessary, that there is a disk D_1' such that

$$D_1 \subset \operatorname{Int} D_1' \subset \operatorname{Int} K \quad \text{and} \quad \operatorname{dist}(\operatorname{cl}(M - D_1'), A_1 \cup D_1) \geqq 3\delta.$$

Choose a positive number $\alpha < \delta/2$ such that $N(K, \alpha) \subset C$ and $M_1 \cap N(K, \alpha) = \varnothing$. Since K is an absolute neighborhood retract, there exists a neighborhood V of K in $N(K, \alpha)$ and a retraction $r\colon V \twoheadrightarrow K$. By [22, Lemma 1, p. 5], there is a neighborhood W of K in V and a homotopy $h\colon W \times I \to V$ such that $h_0 =$ the identity map and $h_1 = r$. Choose a positive number $\sigma' < \alpha$ such that $N(K, \sigma') \subset W$. It follows that each simple closed curve in $N(K, \sigma')$ can be shrunk to a point in V. Since $V \subset C - M_1$, each simple closed curve in $N(K, \sigma')$ can be shrunk to a point in $C - M_1$. Since C is uniformly locally arcwise connected, there exists a positive number $\sigma < \sigma'$ such that if y and z are two points in C for which $d(y, z) < \sigma$, then there is an arc yz for which $\operatorname{diam} yz < \sigma'$.

Now let $p_1 x$ be an arc in $N(M, \sigma)$ with $\operatorname{diam} p_1 x \geqq 3\delta$. Let b be the first point of $p_1 x$, as one goes from p_1 to x, such that $b \in \operatorname{Fr}[N(p_1, 3\delta/2)]$ where Fr denotes frontier. Then $p_1 b - b \subset N(p_1, 3\delta/2)$. Since $\operatorname{dist}(\operatorname{cl}(M - D_1'), A_1 \cup D_1) \geqq 3\delta$ and $\sigma < \delta/2$, $N(\operatorname{cl}(M - D_1'), \sigma) \cap \operatorname{cl} N(p_1, 3\delta/2) = \varnothing$. Therefore

$$p_1 b \subset N(M, \sigma) \cap \operatorname{cl} N(p_1, 3\delta/2) \subset N(K, \sigma).$$

Since $b \in N(K, \sigma)$, there is a point $c \in K$ such that $d(b, c) < \sigma$. Assume $c \in D_1$. Then

$$d(p_1, b) \leqq d(p_1, c) + d(c, b) < \delta + \sigma < \delta + \delta/2 = 3\delta/2.$$

Therefore $b \in N(p_1, 3\delta/2)$, a contradiction to the way b was chosen. Hence $c \notin D_1$. This shows that $c \in K - D_1$.

Since $d(b, c) < \sigma$, there is an arc bc for which $\operatorname{diam} bc < \sigma'$. Let $y \in bc$. By the triangle inequality,

$$d(p_1, y) \geqq d(p_1, b) - d(y, b) > 3\delta/2 - \sigma' > 3\delta/2 - \delta/2 = \delta.$$

Therefore since $p_1 \in A_1 \cup D_1$ and $\operatorname{diam}(A_1 \cup D_1) < \delta$, $y \notin A_1 \cup D_1$. This shows that $bc \cap (A_1 \cup D_1) = \varnothing$.

THEOREM 1. *If $H(1)$ holds and if M is not a 2-sphere, then M is tame from U.*

Proof. By [19, Theorem 2, p. 166], there exists in an arbitrary neighborhood of M in M^3 a polyhedral subset L that is homeomorphic to $M \times I$ and there exists a finite disjoint collection $H_1, H_2, \ldots, H_q$ of arbitrarily small polyhedral cubes with handles in M^3 such that

each H_i meets L precisely in a disk in $(\operatorname{Bd} H_i) \cap \operatorname{Bd} L$,
$M \subset \operatorname{Int}(L \cup H_1 \cup H_2 \cup \cdots \cup H_q)$, and $y \notin L \cup H_1 \cup H_2 \cup \cdots \cup H_q$,
where y is some point of U which each M_j separates from M in M^3.

By [7, Theorem 7, p. 328], there exists a point $p_1 \in M$ such that M is locally tame from U at each point of $M - p_1$. Let ε and δ be positive numbers such that

$$\delta < \gamma, \qquad 7\delta < \varepsilon,$$

and

$$N(M, \delta) \subset \operatorname{Int}(L \cup H_1 \cup H_2 \cup \cdots \cup H_q)$$

and such that there exist a disk K and a polyhedral 3-cell C for which

$$p_1 \in \operatorname{Int} K \subset K \subset M,$$
$$N(K, \delta) \subset C \subset \operatorname{Int}(L \cup H_1 \cup H_2 \cup \cdots \cup H_q),$$

and

$$C \cap \bigcup_{i=1}^{q} [(\operatorname{Bd} H_i) \cap \operatorname{Bd} L] = \varnothing.$$

For this last condition to hold, it might be necessary to make a slight adjustment of $\operatorname{Bd} L$ near p_1.

Since M can be locally peripherally collared from U, there exist a disk D_1 and an annulus A_1 such that

$$p_1 \in \operatorname{Int} D_1 \subset D_1 \subset \operatorname{Int} K, \qquad A_1 \cap M = \operatorname{Bd} D_1 \subset \operatorname{Bd} A_1,$$
$$A_1 - \operatorname{Bd} D_1 \subset U, \quad \text{and} \quad \operatorname{diam}(A_1 \cup D_1) < \delta.$$

Therefore $A_1 \cup D_1 \subset C$. Let σ and σ' be as in Lemma 2. Let $J = (\operatorname{Bd} A_1) - \operatorname{Bd} D_1$, $a_1 \in J$, and A_1' be an annulus in A_1 such that

$$A_1' \cap M = \operatorname{Bd} D_1, \qquad A_1' \cap M_1 = \varnothing, \quad \text{and} \quad A_1' \subset N(K, \sigma).$$

By [3, Theorem 7, p. 478], we may assume A_1 is locally polyhedral mod $\operatorname{Bd} D_1$. Without loss of generality, we assume

$M_1 \subset N(M, \delta)$,
M_1 separates J from M in M^3,

there is a 2-manifold M_0 which separates M_1 from M_2 in M^3 such that for each point w in U such that dist $(w, M) \geqq \sigma$, there is an arc wy such that $wy \cap M_0 = \varnothing$,

$M_0 \cap A_1' = \varnothing$,

$M_2 \subset N(M, \sigma)$,

M_2 separates $M_0 \cup M_1 \cup (A_1 - A_1')$ from M in M^3, and

M_2 and A_1 are in relative general position.

It follows that each component of $A_1 \cap M_2$ is a simple closed curve in A_1'. Since each simple closed curve in A_1 is unknotted and has diameter less than γ, then according to $H(1)$, each simple closed curve in $A_1 \cap M_2$ is the boundary of a disk in M_2. Furthermore, since M_2 separates J from Bd D_1 in M^3, some component of $A_1 \cap M_2$ separates J from Bd D_1 in A_1. Therefore there exists a disk $D' \subset M_2$ such that

Bd $D' \subset A_1'$,

Bd D' is not the boundary of a disk in A_1, and

each component of $A_1 \cap \text{Int } D'$ is the boundary of a disk in A_1'.

Thus, by replacing certain subdisks of D' by disks near A_1', we can adjust D' to a polyhedral disk D such that

$$\text{Bd } D = \text{Bd } D', \qquad \text{Int } D \subset U-(A_1 \cup M_0 \cup M_1), \quad \text{and} \quad D \subset D' \cup N(K, \sigma).$$

The last inclusion follows from the fact that $A_1' \subset N(K, \sigma)$.

Let A' be the annulus in A_1' such that $\text{Bd } A' = (\text{Bd } D_1) \cup \text{Bd } D$, and let $S = A' \cup D \cup D_1$. S is a 2-sphere which is locally polyhedral mod D_1, and $S - D_1 \subset U$.

Now suppose M is orientable. Therefore $L \cup H_1 \cup H_2 \cup \cdots \cup H_q$ can be considered to be already imbedded in S^3. It then follows from the next lemma (Lemma 3) that M is tame from U.

Next, suppose M is nonorientable and has genus p. For convenience, we identify L with a polyhedron $B^p \times I \subset M^3$, where B^p is homeomorphic to M. Then

$$L \cup H_1 \cup H_2 \cup \cdots \cup H_q = (B^p \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q.$$

The following argument, which shows that S separates M^3, is analogous to the proof of [12, Lemma 15, p. 414].

Let J_p be a median of a Moebius band N in B^p. Since each $H_i \cap (B^p \times I)$ is a disk which does not intersect C and since $D_1 \subset C$, it may be assumed that

$$N \cap S = \varnothing \quad \text{and} \quad (J_p \times I) \cap \left(D_1 \cup \bigcup_{i=1}^{q} [(\text{Bd } H_i) \cap \text{Bd } L]\right) = \varnothing.$$

By relative general position, it may be assumed that each component of $S \cap (J_p \times I)$ is a polyhedral simple closed curve. Let F be such a component which is the boundary of a disk E in S such that

$$(\text{Int } E) \cap (J_p \times I) = \varnothing \quad \text{and} \quad E \cap D_1 = \varnothing.$$

F must be nullhomotopic in $J_p \times I$; otherwise, there is an annulus A in $J_p \times I$ such that Bd $A = F \cup J_p$. Then $A \cup E$ is a disk such that $(A \cup E) \cap N = J_p$, a situation which contradicts [12, Lemma 7, p. 409]. Thus F must be the boundary of a disk E' in $J_p \times I$. It follows from the proof of [12, Lemma 15, p. 414] that the 2-sphere $E \cup E'$ is the boundary of a 3-cell G in $(B^p \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q$. By deforming E across G onto E' and then off $J_p \times I$, we can eliminate the component F of $S \cap (J_p \times I)$. Therefore we may assume a priori that $S \cap (J_p \times I) = \varnothing$. By cutting B^p along J_p, we obtain a 2-manifold B^{p-1} of genus $p-1$ with one contour such that

$$S \subset \operatorname{Int}\,[(B^{p-1} \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q].$$

By induction, we obtain a 2-manifold B^0 of genus zero with p contours such that

$$S \subset \operatorname{Int}\,[(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q].$$

Since B^0 is orientable, $(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q$ can be considered to be already imbedded in S^3. The closure of each component of $S^3 - S$ is a crumpled cube (by definition). Since Bd $B^0 \neq \varnothing$, Bd $[(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q]$ is connected and therefore contained in a single component of $S^3 - S$. Then the closure of the other component is a crumpled cube Q in

$$\operatorname{Int}\,[(B^0 \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_q]$$

and hence in Int $(L \cup H_1 \cup H_2 \cup \cdots \cup H_q)$. Consequently S separates M^3.

Assume S separates y from $M - D_1$ in M^3. Then since $y \notin \operatorname{Int} Q$, $M - D_1 \subset \operatorname{Int} Q$. Since $D_1 \subset S \subset Q$, $M \subset Q$. Therefore since Q is a crumpled cube, it along with M can be imbedded in S^3. This is a contradiction to the fact that a closed nonorientable 2-manifold like M cannot be imbedded in S^3 [8, Theorem 22, p. 182]. Therefore S does not separate y from $M - D_1$ in M^3.

We now show that diam $D < 6\delta$. Assume otherwise. Using the methods of [23, p. 66], we construct an arc p_1a_1 such that $p_1a_1 - (p_1 \cup a_1) \subset U - A_1$. There exists an arc a_1y such that $a_1y - a_1 \subset U - (A_1 \cup M_1 \cup p_1a_1)$. Since y and $M - D_1$ are in the same component of $M^3 - S$, there is an arc yp_2 such that

$$p_2 \in M - D_1 \quad \text{and} \quad yp_2 - (y \cup p_2) \subset U - (A_1 \cup D \cup p_1a_1 \cup a_1y).$$

Let p_2p_1 be an arc such that

$$p_2p_1 - (p_2 \cup p_1) \subset M^3 - (M \cup U),$$

and let J^* denote the simple closed curve $p_1a_1 \cup a_1y \cup yp_2 \cup p_2p_1$. By construction $J^* \cap D = p_1a_1 \cap D$. $J^* \cap D \neq \varnothing$ because

$$J^* \cap (S - D) = p_1, \quad J^* \text{ pierces } S \text{ at } p_1, \quad \text{and} \quad S \text{ separates } M^3.$$

Therefore $p_1a_1 \cap D \neq \varnothing$. Let a be the first point of p_1a_1 (as one goes from p_1 to a_1) such that $a \in D$, and let p_1a denote the subarc of p_1a_1. Since it is assumed that diam $D \geqq 6\delta$, there is a point $x \in D$ such that $d(a, x) \geqq 3\delta$. Let ax be an arc in D, and let $p_1x = p_1a \cup ax$. Then $\operatorname{diam} p_1x \geqq 3\delta$. Assume $p_1a \not\subset N(M, \sigma)$. Then there exists a point $w \in \operatorname{Int} p_1a$ such that $\operatorname{dist}(w, M) \geqq \sigma$. Therefore there is an arc wy such that

$$wy-(w \cup y) \subset U-(A' \cup D \cup p_1a \cup yp_2).$$

Let p_1w be the subarc of p_1a. Then $J^{**} = p_1w \cup wy \cup yp_2 \cup p_2p_1$ is a simple closed curve which intersects and pierces S precisely at p_1, a contradiction. Hence $p_1a \subset N(M, \sigma)$. Since $ax \subset D \subset N(M, \sigma)$, $p_1x \subset N(M, \sigma)$. Therefore by Lemma 2, there exist a subarc p_1b of p_1x, a point $c \in K-D_1$, and an arc bc such that

$$p_1b \subset N(K, \sigma), \qquad \operatorname{diam} bc < \sigma', \quad \text{and} \quad bc \cap (A_1 \cup D_1 \cup \operatorname{Int} p_1b) = \varnothing.$$

Let cp_1 be an arc in $N(K, \sigma)$ such that

$$cp_1-(c \cup p_1) \subset M^3-(M \cup U \cup bc),$$

and let J' denote the simple closed curve $p_1b \cup bc \cup cp_1$. By construction,

$$J' \subset N(K, \sigma'), \qquad J' \cap (A_1 \cup D_1) = p_1, \quad \text{and} \quad J' \text{ pierces } A_1 \cup D_1 \text{ at } p_1.$$

Therefore, since $J' \cup A_1 \cup D_1 \subset C$, J' links J in C. But since $J' \subset N(K, \sigma')$, we can apply Lemma 2 to shrink J' to a point in the component of $C-M_1$ that contains K. Since M_1 separates J from K in M^3, $C \cap M_1$ separates J from K in C. Therefore J' does not link J in C, a contradiction. Hence we must have that $\operatorname{diam} D < 6\delta$. Therefore

$$\operatorname{diam}(D_1 \cup A' \cup D) \leqq \operatorname{diam}(D_1 \cup A') + \operatorname{diam} D < \delta + 6\delta = 7\delta < \varepsilon.$$

Since the disks D_1 and $A' \cup D$ are those in the definition of local spanning, we have shown that M can be locally spanned from U at p_1. Since M is locally spanned from U at all other points, it follows from [6, Theorem 10, p. 88] and from the proof of [6, Theorem 16, pp. 95–96] that M is locally tame from U.

The proof of the following lemma finishes the proof of Theorem 1.

LEMMA 3. *If $H(1)$ holds for $M^3 = S^3$ and if M is not a 2-sphere, then M is tame from U.*

Proof. We let D_1, A_1, D, A', S, and p_1a_1 be those sets constructed in the proof of the nonorientable case of Theorem 1. In the proof of that case of Theorem 1, the nonorientability of M was used just to show that $p_1a_1 \cap D \neq \varnothing$. Therefore we only need to show again that $p_1a_1 \cap D \neq \varnothing$ for the case when M is an orientable manifold in S^3.

On the contrary, assume $p_1a_1 \cap D = \varnothing$. If we assume that S does not separate y from $M-D_1$ in S^3, then we can construct the simple closed curve J^* exactly as done in the proof of Theorem 1. But now J^* intersects and pierces S precisely at p_1, a contradiction. Therefore S must separate y from $M-D_1$ in S^3. Let Q_0 be that component of S^3-S which contains y. Then $Q_0 \subset U$.

It follows from [4, Theorem 5, p. 302] and from [17, p. 666] or [18, Theorem 2, p. 541] that we may assume M is tame from $S^3 - \mathrm{cl}\, U$. By [2, Theorem 9, p. 157], we may further assume that M is locally polyhedral mod p_1; and consequently, we may assume that S was constructed to be locally polyhedral mod p_1. It now follows from [9, Theorem 1, p. 250] that Q_0 is an open 3-cell.

Since S could have been constructed in an arbitrary neighborhood of M in S^3, it is clear that S is just one member of a sequence S_0 $(=S)$, $S_1, S_2, \ldots$ of 2-spheres such that, for each nonnegative integer i,

(1) $S_i - [S_i \cap (A_1 \cup D_1)]$ is a polyhedral disk in U,

(2) $S_i \subset N(M, 1/i)$,

(3) S_i separates y from $M - D_1$ in S^3,

(4) if Q_i is that component of $S^3 - S_i$ containing y, then Q_i is an open 3-cell in U, and

(5) $Q_i \subset Q_{i+1}$.

Now (1), (2), and (3) imply that $S_0, S_1, S_2, \ldots$ converge to M. Therefore $U = \bigcup_{i=0}^{\infty} Q_i$. Hence (4) and (5) imply that U is an open 3-cell [5, p. 813]. Since a 2-manifold in S^3 is a 2-sphere if it is the boundary of an open 3-cell, we obtain the contradiction that M is a 2-sphere. Thus $p_1 a_1 \cap D \neq \varnothing$. This completes the proof of Lemma 3.

We require that M not be a 2-sphere in the hypothesis of Theorem 1 because the 2-sphere M in Example 3.2 of [13, p. 990] is not tame from one component U of $S^3 - M$, but $H(1)$ holds.

4. **Some corollaries.** For a 2-manifold M which separates a 3-manifold M^3, we say that M can be *pierced on an arc* $A \subset M$ *with a disk* D if $\mathrm{Int}\, A \subset \mathrm{Int}\, D$, $\mathrm{Bd}\, A \subset \mathrm{Bd}\, D$, and the two components of $D - A$ lie in different components of $M^3 - M$; and we call D a *piercing disk*. Eaton [11, p. 510] proved that a 2-sphere S in E^3 is tame if S can be pierced on each of its arcs with a tame disk.

Let us remove from $H(1)$ the condition that M can be locally peripherally collared from U, and let us call the remaining hypothesis $H'(1)$. The following corollary is an extension of [7, Theorem 9, p. 329] because we do not require that the piercing disks be tame.

COROLLARY 1. *If*

(i) *M is not a 2-sphere,*

(ii) *$H'(1)$ holds for each component U of $M^3 - M$, and*

(iii) *M can be pierced on each of its arcs with a disk,*

then M is tame.

Proof. The proof is essentially the same as Burgess' proof in [7, Theorem 8, p. 329] with out Theorem 1 used in place of his Theorem 7.

In [1], Alexander proved that if S is a polyhedral 2-sphere in S^3, then each component of $S^3 - S$ has a closure which is a 3-cell. Harrold and Moise [15, Theorems

I, II, p. 577] generalized this result by showing that if S is a 2-sphere in S^3 which is locally polyhedral mod one point, then one component of S^3-S has a closure which is a 3-cell and the other component is simply connected. In fact, Cantrell [9, Theorem 1, p. 250] proved that this other component is an open 3-cell. Thus, the components of S^3-S are open 3-cells; however, the analogous situation is different for a torus. One of the complementary domains of a polyhedral torus in S^3 has a closure which is a solid torus [1], and therefore the other component is homeomorphic to the complement of a tame knot in S^3. But if M is a torus in S^3 which is locally polyhedral mod one point p and wild from U at p, then it follows from the next corollary that one and only one component of S^3-M is homeomorphic to the complement of a tame knot in S^3 (Daverman [10] has independently proved such a result; for completeness, we give here an alternative but similar proof).

COROLLARY 2. *Let M be a torus in S^3, U a component of S^3-M, and $p \in M$. If M is locally polyhedral* mod *p and if U is homeomorphic to the complement of a tame knot in S^3, then M is tame from U.*

Proof. Since U is homeomorphic to the complement of a tame knot in S^3, it follows from [20, Theorem 2, p. 97] that there exists a sequence $M_1, M_2, M_3, \ldots$ of disjoint polyhedral tori converging to M such that, for some $y \in U$, each M_j separates y from M in S^3.

Let s_1 and s_2 be disjoint polyhedral simple closed curves each nonnullhomologous on M, and let A_1 and A_2 be disjoint polyhedral annuli such that, for $i=1, 2$,

$$A_i \cap M = s_i \subset \operatorname{Bd} A_i \quad \text{and} \quad A_i - s_i \subset U.$$

We may assume that each M_j separates $s_1 \cup s_2$ from $[(\operatorname{Bd} A_1)-s_1] \cup [(\operatorname{Bd} A_2)-s_2]$ in S^3 and that $A_1 \cup A_2$ and each M_j are in general position.

Since M is an absolute neighborhood retract, it is a retract of one of its neighborhoods V in S^3. Neither s_1 nor s_2 can be shrunk to a point in V. There is a number $\delta > 0$ such that if K is a set of diameter less than δ, then either $K \cap A_1 = \varnothing$ or $K \cap A_2 = \varnothing$. Let $C_1, C_2, \ldots, C_n$ be 3-cells in V each of diameter less than δ and let W be a neighborhood of M in S^3 such that $\operatorname{cl} W \subset \bigcup_{i=1}^n \operatorname{Int} C_i$. There is a positive number $\gamma < \delta$ such that if K is a subset of W of diameter less than γ, then $K \subset \operatorname{Int} C_m$ for some m $(1 \leqq m \leqq n)$. Without loss of generality, we may assume

$$A_1 \cup A_2 \cup \left(\bigcup_{j=1}^{\infty} M_j\right) \subset W.$$

We now show that $H(1)$ holds. Let j be an arbitrary positive integer and s an unknotted simple closed curve in M_j of diameter less than γ. We suppose s is not the boundary of a disk in M_j. Since $s \subset \operatorname{Int} C_m$ for some m, s can be shrunk to a point in $\operatorname{Int} C_m$. Since $\operatorname{diam} C_m < \delta$, either $C_m \cap A_1 = \varnothing$ or $C_m \cap A_2 = \varnothing$. We may assume $C_m \cap A_1 = \varnothing$. Let s' be a component of $M_j \cap A_1$ that is nonnullhomologous on M_j. Since $s \cap s' = \varnothing$, s and s' bound an annulus A on M_j. Let A' be the annulus in A_1

such that $\mathrm{Bd}\, A' = s_1 \cup s'$. Then s_1 can be shrunk to a point in the subset $A' \cup A \cup \mathrm{Int}\, C_m$ of V, a contradiction. Therefore s must be the boundary of a disk in M_j. According to Theorem 1, M is tame from U.

5. **2-manifolds that are almost tame.** We state the following lemma without proof.

LEMMA 4. *Let H be a closed 2-manifold and J a simple closed curve which lies in a 3-manifold M^3 in such a way that J intersects and pierces H at exactly one point. Then J cannot be shrunk to a point in M^3.*

Now, by replacing the hypothesis $H(1)$ of [7, Theorem 7, p. 328] with the weaker condition $H(2)$, we obtain the following stronger result. The symbol $\sim$ is used to stand for "is homologous to."

THEOREM 2. *If $H(2)$ holds, then there is a point p such that M is locally tame from U mod p.*

Proof. Let p_1 and p_2 be two arbitrary points in M. If we show that M can be locally spanned from U at either p_1 or p_2, then it follows from [6, Theorem 10, p. 88] that there must be a point p such that M is locally tame from U mod p.

Let K_0 be a disk in M such that $p_1 \cup p_2 \subset \mathrm{Int}\, K_0$. Since K_0 is an absolute neighborhood retract, there is a neighborhood V_0 of K_0 in M^3 that retracts onto K_0. It follows from [22, Lemma 1, p. 5] that there is a neighborhood W_0 of K_0 in V_0 such that each simple closed curve in W_0 can be shrunk to a point in V_0.

For $i=1, 2$, there are a disk K_i and a polyhedral 3-cell C_i in W_0 such that

$$p_i \subset \mathrm{Int}\, K_i \subset K_i \subset M \cap \mathrm{Int}\, C_i.$$

Let ε, δ, and γ' be positive numbers such that

$$7\delta < \varepsilon, \qquad 6\delta < \gamma', \qquad 3\delta + \gamma' < \gamma, \quad \text{and} \quad N(K_i, 7\delta) \subset C_i.$$

Since M can be locally peripherally collared from U, there exist (for $i=1, 2$) disjoint disks D_i and disjoint annuli A_i such that

$$p_i \in \mathrm{Int}\, D_i \subset D_i \subset \mathrm{Int}\, K_i, \qquad A_i \cap M = \mathrm{Bd}\, D_i \subset \mathrm{Bd}\, A_i,$$
$$A_i - \mathrm{Bd}\, D_i \subset U, \quad \text{and} \quad \mathrm{diam}\,(A_i \cup D_i) < \delta.$$

It follows that $A_i \cup D_i \subset C_i$. It is clear from the proof of Lemma 2 that there exist positive numbers $\sigma < \sigma' < \delta$ such that, for $i=1, 2$, each simple closed curve in $N(K_i, \sigma')$ can be shrunk to a point in $C_i - M_1$ and such that, for each arc $p_i x$ in $N(M, \sigma)$ with $\mathrm{diam}\, p_i x \geqq 3\delta$, there are a subarc $p_i b$ of $p_i x$, a point $c \in K_i - D_i$, and an arc bc for which

$$p_i b \subset N(K_i, \sigma), \qquad \mathrm{diam}\, bc < \sigma', \quad \text{and} \quad bc \cap (A_i \cup D_i) = \varnothing.$$

For $i=1, 2$, we let $J_i = (\mathrm{Bd}\, A_i) - \mathrm{Bd}\, D_i$ and $a_i \in J_i$. There is an arc $a_1 a_2$ such that

$$a_1 a_2 - (a_1 \cup a_2) \subset (U \cap W_0) - (A_1 \cup A_2).$$

For each i, let A_i' be an annulus in A_i such that

$$A_i' \cap M = \text{Bd } D_i, \qquad A_i' \cap M_1 = \varnothing, \quad \text{and} \quad A_i' \subset N(K_i, \sigma).$$

We may assume, without loss of generality, that

A_i is locally polyhedral mod Bd D_i [3, Theorem 7, p. 478],

$M_1 \subset N(M, \delta)$,

M_1 separates $J_1 \cup J_2 \cup a_1a_2$ from M in M^3,

$M_2 \subset N(M, \sigma)$,

M_2 separates $(M_1 \cup A_1 \cup A_2) - (A_1' \cup A_2')$ from M in M^3, and

M_2 and $A_1 \cup A_2$ are in relative general position.

Now, diam $A_i < \delta < \gamma'$. Therefore since each component of $(A_1 \cup A_2) \cap M_2$ is an unknotted simple closed curve in $A_1' \cup A_2'$ of diameter less than γ, then according to $H(2)$, each such simple closed curve is homologous to 0 on M_2. For $i = 1, 2$, some component of $A_i \cap M_2$ separates J_i from Bd D_i in A_i because M_2 separates J_i from Bd D_i in M^3; it follows that one such component s_0 is a chain which bounds a 2-manifold H_0 in M_2 such that each component of $(A_1 \cup A_2) \cap (H_0 - \text{Bd } H_0)$ is the boundary of a disk in $A_1' \cup A_2'$. We may assume $s_0 \subset A_1$.

Suppose s_1 is a simple closed curve in $M_2 \cap A_i$ ($i = 1$ or 2) which is the boundary of a disk D in A_i'. We may assume $M_2 \cap \text{Int } D = \varnothing$. Slightly thicken D to obtain a polyhedral 3-cell B in $U \cap N(K_i, \sigma)$ such that

Int $D \subset$ Int B,

Bd $D \subset$ Bd B,

$B \cap A_i = D$,

$B \cap M_2 = (\text{Bd } B) \cap M_2 = A$, an annulus with s_1 as center line,

$(\text{Bd } B) - \text{Int } A = E_1 \cup F_1$, where E_1 and F_1 are disjoint disks, and

$B \subset N(A_i, \delta)$.

Let $R = (M_2 - A) \cup E_1 \cup F_1$. $R \subset N(M, \sigma)$ and $R \cap A_i = (M_2 \cap A_i) - s_1$. Since $s_1 \sim 0$ on M_2, R consists of two components R_1 and T_1 with $E_1 \subset R_1$ and $F_1 \subset T_1$. We may assume $s_0 \subset R_1$.

In the above fashion, we may inductively construct closed 2-manifolds R_k and disjoint disks E_k ($k \geqq 1$) such that

(1) $E_k \subset R_k \subset N(M, \sigma)$,

(2) $R_k - E_k \subset R_{k-1}$ (we define $R_0 = M_2$),

(3) the chain s_0 bounds a 2-manifold H_k in R_k such that each component of $(A_1 \cup A_2) \cap (H_k - \text{Bd } H_k)$ is the boundary of a disk in $A_1' \cup A_2'$,

(4) $(A_1 \cup A_2) \cap R_k$ has fewer components than $(A_1 \cup A_2) \cap R_{k-1}$, and

(5) either $E_k \subset N(A_1, \delta)$ or $E_k \subset N(A_2, \delta)$.

For each k, let $E_k^* = \bigcup_{j=1}^k E_j$.

Let s be an arbitrary unknotted simple closed curve in R_k such that diam $s < \gamma'$. It is possible that $s \cap E_k^* \neq \varnothing$. Nevertheless, (5) implies that there is an unknotted simple closed curve s' in $R_k - E_k^*$ such that $s' \sim s$ on R_k and diam $s' < 3\delta + \gamma' < \gamma$.

Therefore since (2) implies $s' \subset R_k - E_k^* \subset M_2$, we must have $s' \sim 0$ on M_2 and thus on R_k. Then $s \sim 0$ on R_k because $s \sim s'$ on R_k.

Now, (3) and (4) imply that the inductive construction stops at some positive integer n for which

$$(A_1 \cup A_2) \cap (H_n - \mathrm{Bd}\, H_n) = \varnothing .$$

Therefore $(A_1 \cup A_2) \cap H_n = s_0$. By (1), $H_n \subset N(M, \sigma)$. Let A' be the annulus in A_1' such that $\mathrm{Bd}\, A' = s_0 \cup \mathrm{Bd}\, D_1$; and let $H = H_n \cup A' \cup D_1$, which is a closed 2-manifold.

Using the methods of [23, p. 66], we construct an arc p_1a_1 such that

$$p_1a_1 - (p_1 \cup a_1) \subset (U \cap W_0) - (A_1 \cup A_2 \cup a_1a_2).$$

Let a_2p_2 be an arc in $A_2 \cup D_2$, and let p_2p_1 be an arc such that

$$p_2p_1 - (p_2 \cup p_1) \subset W_0 - (M \cup U).$$

Let J denote the simple closed curve $p_1a_1 \cup a_1a_2 \cup a_2p_2 \cup p_2p_1$. By construction, J intersects and pierces $H - H_n$ at precisely the point p_1. Since $J \subset W_0$, J can be shrunk to a point in V_0. Therefore it follows from Lemma 4 that $J \cap H_n \neq \varnothing$. Then $p_1a_1 \cap H_n \neq \varnothing$. We can now use the same techniques of the proof of Theorem 1 (when we proved diam $D < 6\delta$ there) in order to show that diam $H_n < 6\delta < \gamma'$. Since $N(K_1, 7\delta) \subset C_1$, $H_n \subset C_1$.

By [21, p. 1] or [3, Theorem 7, p. 478], there is a polyhedral 2-manifold H' in C_1 such that

$$H' \text{ is homeomorphic to } H, \qquad H_n \subset H', \quad \text{and} \quad \mathrm{cl}\,(H' - H_n) \text{ is a disk.}$$

Suppose H_n is not a disk. Then H' has genus greater than zero. Therefore by [14, Theorem 1, p. 462] or [16, Theorem 1, p. 129], there exists an unknotted simple closed curve t in H' not homologous to 0 on H'. Since cl $(H' - H_n)$ is a disk, there is an unknotted simple closed curve t' in H_n such that $t' \sim t$ on H'. Therefore t' is not homologous to 0 on H' and thus on R_n. But since $t' \subset H_n$,

$$\mathrm{diam}\, t' < \mathrm{diam}\, H_n < \gamma'.$$

Then according to what was shown earlier about the simple closed curve s in R_k, we must have $t' \sim 0$ on R_n. Since we have reached a contradiction, H_n must be a disk. Therefore since diam $H < 7\delta < \varepsilon$, M can be locally spanned from U at p_1. Thus the conclusion of the theorem follows.

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