

ON THE SIGNATURE OF KNOTS AND LINKS⁽¹⁾

BY
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Abstract. In 1965, K. Murasugi introduced an integral matrix M of a link and defined the signature of the link by the signature of $M + M'$. In this paper, we study some basic properties of the signature of links. We also describe the effect produced on the signature of a knot contained in a solid torus by a further knotting of the solid torus.

1. Introduction. A link l of multiplicity μ is the union of μ ordered, oriented and pairwise disjoint polygonal simple closed curves l_i in the oriented 3-sphere S^3 . In particular, if $\mu = 1$, it is called a *knot*. Two links l and l' of multiplicity μ are said to be of the *same link type* if there exists an orientation preserving homeomorphism h of S^3 onto itself such that $h|l_i$ is an orientation preserving homeomorphism of l_i onto l'_i , $i = 1, 2, \dots, \mu$.

Now let l be a link and L a projection of l , that is, the image of l under a regular projection. In 1965, K. Murasugi associated an integral matrix M to L , called the *L-principal minor of l* , and he showed that the signature of $M + M'$ is an invariant of the link type of l [8]. The signature of $M + M'$ will be called the *signature* of the link l and denoted by $\sigma(l)$.

H. F. Trotter [12] in 1962 and J. W. Milnor [6] in 1968 also defined the signature of a knot in different ways, and D. Erle [1] showed that Milnor's definition is equivalent to Trotter's. The author of the paper proved in [11] that for the case of a knot Murasugi's definition is equivalent to Trotter's.

In §2 we will state some known results which will be used in later sections.

Let l be a link of multiplicity μ and $\Delta_l(t)$ the reduced Alexander polynomial of l . In §3 we first prove that if $\Delta_l(-1) \neq 0$, then $\sigma(l)$ is even or odd according as μ is odd or even (Corollary 2). Furthermore we will show that if $\Delta_l(t) \neq 0$, then the absolute value of $\sigma(l)$ is not greater than the degree of $\Delta_l(t)$ (Theorem 3). In [8] Murasugi showed that if k is a knot, then $|\Delta_k(-1)| \equiv 1$ or $3 \pmod{4}$ according as $\sigma(k) \equiv 0$ or $2 \pmod{4}$. We will generalize this result to the case of a link by using the Hosokawa polynomial (Theorem 4).

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Let k be a knot, V a tubular neighborhood of k and V^* a trivial solid torus in S^3 . Let $f: V^* \rightarrow V$ be a faithful homeomorphism of V^* onto V . Let l^* be a knot in V^* and $l=f(l^*)$. Then l is homologous to some multiple of k , say $l \sim nk$, in V . In §4 we will prove that

$$\begin{aligned} \sigma(l) &= \sigma(l^*) && \text{if } n \text{ is even,} \\ &= \sigma(l^*) + \sigma(k) && \text{if } n \text{ is odd} \end{aligned}$$

(Theorem 9).

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2. Preliminaries. Let l be a link, L a projection of l and M the L -principal minor of l . We first state the following two results proved by Murasugi [8]:

(a) nullity $(M + M')$ is an invariant of the link type of l . nullity $(M + M') + 1$ is called the *nullity* of l and denoted by $n(l)$.

(b) Let $\Delta_i(t)$ be the reduced Alexander polynomial of l . Then

$$(2.1) \quad \Delta_i(t) = \pm t^\lambda \det(M - tM') \quad \text{for some integer } \lambda.$$

Now let l be a link of multiplicity μ and F an orientable surface in S^3 whose boundary is l . Let $T: F \rightarrow S^3 - F$ be a translation in the positive normal direction of F and $\alpha_1, \alpha_2, \dots, \alpha_{2h+\mu-1}$ a homology basis on F , where h denotes the genus of F . The matrix

$$\|\text{Link}(T\alpha_i, \alpha_j)\|_{i,j=1,2,\dots,2h+\mu-1}$$

is called a *Seifert matrix of l with respect to F* [5].

Let L be a projection of a link l . The orientable surface with boundary l which is constructed by using L as shown in §1 of [9] is called the *orientable surface associated with L* .

THEOREM A. *If l is a link with a connected and special projection L , then the L -principal minor of l is a Seifert matrix of l with respect to the orientable surface associated with L .*

For the definition of a special projection, see Definition 3.1 of [8]. The detailed proof of Theorem A may be found in §2 of [11]. We note that every link can be deformed isotopically to a link with a connected and special projection.

Finally let k be a knot and V a Seifert matrix of k . It was shown in [12] that the signature of $V + V'$ is an invariant of the knot type of k . Further, as a consequence of Theorem A, it was proved in [11] that

THEOREM B. *The signature $\sigma(k)$ of k is equal to the signature of $V + V'$.*

Throughout the paper Z and Q will denote the ring of integers and the field of rational numbers respectively.

3. Some properties of the signature.

THEOREM 1. *If l is a link of multiplicity μ , then $\sigma(l) \equiv \mu - n(l) \pmod{2}$.*

Proof. First we deform l isotopically to a link l_0 which has a connected and special projection L_0 . Let M be the L_0 -principal minor of l_0 . It follows from Theorem A that $M + M'$ is a $(2h + \mu - 1) \times (2h + \mu - 1)$ matrix, where h is the genus of the orientable surface associated with L_0 . Since l and l_0 belong to the same link type, we have $\sigma(l) = \text{signature}(M + M')$ and $n(l) = \text{nullity}(M + M') + 1$.

Now $M + M'$ is congruent over \mathcal{Q} to a diagonal matrix with r positive, s negative and $n(l) - 1$ zero entries on the diagonal. Since $2h + \mu - 1 = r + s + n(l) - 1$ and $\sigma(l) = r - s$, it follows that

$$\begin{aligned} \sigma(l) &= \mu - n(l) + 2h - 2s \\ &\equiv \mu - n(l) \pmod{2}. \end{aligned}$$

COROLLARY 2. *If l is a link of multiplicity μ such that $\Delta_l(-1) \neq 0$, then $\sigma(l)$ is even or odd according as μ is odd or even.*

Proof. (2.1) implies that $\Delta_l(-1) \neq 0$ if and only if $n(l) = 1$. Hence Corollary 2 is an immediate consequence of Theorem 1.

THEOREM 3. *If l is a link with $\Delta_l(t) \neq 0$, then $|\sigma(l)| \leq \text{the degree of } \Delta_l(t)$.*

Proof. Let M be the L -principal minor of l and m the number of rows of M . Since $\Delta_l(t) \neq 0$, (2.1) implies $\det(M - tM') \neq 0$.

Now if M is a singular matrix, it is congruent over \mathcal{Q} to the matrix

$$\begin{bmatrix} 0 & 0 \\ a & M_1 \end{bmatrix},$$

where M_1 is $(m - 1) \times (m - 1)$ and a is $(m - 1) \times 1$. Moreover, $\det(M - tM') \neq 0$ yields that a has at least one nonzero entry. Therefore M is congruent over \mathcal{Q} to the matrix

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & M_2 \end{bmatrix},$$

where M_2 is $(m - 2) \times (m - 2)$ and q is $(m - 2) \times 1$. It is easy to show that $\tilde{M} + \tilde{M}'$ is congruent over \mathcal{Q} to the direct sum of $\begin{bmatrix} 0 & \\ 1 & 0 \end{bmatrix}$ and $M_2 + M_2'$ and that $\tilde{M} - t\tilde{M}'$ is equivalent over the rational group ring of $H = (t:)$ (in the sense of Fox [2]) to the direct sum of $\begin{bmatrix} 0 & \\ 1 & -t \end{bmatrix}$ and $M_2 - tM_2'$.

From this it follows that

$$\text{signature}(M + M') = \text{signature}(M_2 + M_2')$$

and

$$\det(M - tM') = c_1 t \det(M_2 - tM_2')$$

for some nonzero rational number c_1 .

Since $\det(M_2 - tM'_2) \neq 0$, if M_2 is singular, by applying the preceding argument to M_2 we can obtain an $(m-4) \times (m-4)$ matrix M_3 such that

$$\text{signature}(M_2 + M'_2) = \text{signature}(M_3 + M'_3)$$

and

$$\det(M_2 - tM'_2) = c_2 t \det(M_3 - tM'_3)$$

for some nonzero rational number c_2 .

By repeating this process several times, if necessary, we can show that there exists a nonsingular $(m-2n) \times (m-2n)$ matrix N for some n such that

$$\sigma(l) = \text{signature}(N + N')$$

and

$$\Delta_l(t) = ct^\lambda \det(N - tN')$$

for some integer λ and some nonzero rational number c .

It is clear that

$$(3.1) \quad |\sigma(l)| \leq \text{the number of rows of } N.$$

Since the constant term and the leading term of $\det(N - tN')$ are $\det N$ and $\pm \det N$ respectively, these are nonzero rational numbers. Hence we have

$$(3.2) \quad \begin{aligned} \text{the degree of } \Delta_l(t) &= \text{the degree of } \det(N - tN') \\ &= \text{the number of rows of } N. \end{aligned}$$

Theorem 3 is a consequence of (3.1) and (3.2), which completes the proof.

Let l be a link of multiplicity μ and l_1, l_2, \dots, l_μ the components of l . If $\mu \geq 2$, we define the matrix $A = \|l_{ij}\|_{i,j=1,2,\dots,\mu}$ by the formula

$$l_{ij} = \text{Link}(l_i, l_j) \quad \text{for } i \neq j, \quad l_{ii} = - \sum_{j=1; j \neq i}^{\mu} l_{ij}.$$

Let A_i be the matrix obtained from A by deleting the i th row and the i th column. Clearly $\det A_i$ does not depend on the choice of i or the order of l_1, l_2, \dots, l_μ . Then we define

$$(3.3) \quad \begin{aligned} D(l) &= 1 && \text{if } \mu = 1, \\ &= \det A_i && \text{if } \mu \geq 2. \end{aligned}$$

It is easy to see that $D(l)$ is an invariant of the link type of l .

The polynomial

$$(3.4) \quad \nabla_l(t) = \Delta_l(t)/(1-t)^{\mu-1}$$

is called the *Hosokawa polynomial* of l [4].

THEOREM 4. *Let l be a link of multiplicity μ . If $\nabla_l(-1) \neq 0$ and $\sigma(l) = \epsilon m$, where $\epsilon = \pm 1$ and $m \geq 0$, then*

$$|\nabla_l(-1)| \equiv \epsilon^m (-1)^{(m-u+1)/2} D(l) \pmod{4}.$$

Proof. We may assume that l has a connected and special projection L . Let M be the L -principal minor of l , F the orientable surface associated with L and h the genus of F . Then by Theorem A M is a Seifert matrix of L with respect to some homology basis on F .

Now it follows from the argument given in §3 of [4] that we can choose a homology basis on F with respect to which the Seifert matrix V of l is of the form

$$V = \begin{bmatrix} C & B' \\ B & A_1 \end{bmatrix},$$

where C is a $2h \times 2h$ matrix such that $C - C'$ is the direct sum of h copies of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and A_1 is a $(\mu - 1) \times (\mu - 1)$ matrix appearing in (3.3). Therefore there exists a unimodular integral matrix P such that $PMP' = V$.

Let

$$f(t) = \det(V - tV') / (1 - t)^{\mu - 1} = \begin{vmatrix} C - tC' & B' \\ (1 - t)B & A_1 \end{vmatrix}.$$

Note that since A_1 is symmetric, each of the last $\mu - 1$ columns of $V - tV'$ has a factor $1 - t$.

By (2.1) and (3.4) we have $f(t) = \pm t^\lambda \nabla_l(t)$ for some integer λ . Since $f(t)$ is a polynomial of degree at most $2h$ and satisfies the condition $f(t) = t^{2h} f(t^{-1})$, we may put

$$f(t) = c_0 + c_1 t + \dots + c_{2h} t^{2h},$$

where $c_i = c_{2h-i}$ for $0 \leq i \leq h - 1$. Using the facts $\det(C - C') = 1$ and $\det A_1 = D(l)$, we obtain

$$f(1) = D(l) = 2 \sum_{i=0}^{h-1} c_i + c_h.$$

From this and the fact $(-1)^i - (-1)^h \equiv 0 \pmod{2}$ it follows that

$$\begin{aligned} f(-1) &= 2 \sum_{i=0}^{h-1} (-1)^i c_i + (-1)^h c_h \\ (3.5) \qquad &= (-1)^h D(l) + 2 \sum_{i=0}^{h-1} \{(-1)^i - (-1)^h\} c_i \\ &\equiv (-1)^h D(l) \pmod{4}. \end{aligned}$$

Since $\nabla_l(-1) \neq 0$ and $\sigma(l) = \epsilon m$, for some unimodular rational matrix, Q , $QP(M + M')P'Q' = Q(V + V')Q'$ is a diagonal matrix with diagonal entries

$$(a_1, \dots, a_n, -a'_1, \dots, -a'_n, \epsilon b_1, \dots, \epsilon b_m),$$

where a_i, a'_j and b_k are positive rational numbers. Therefore we have

$$f(-1) = \det(M + M')/2^{\mu-1}$$

$$= (-1)^n \varepsilon^m a_1 \cdots a_n a'_1 \cdots a'_n b_1 \cdots b_m / 2^{\mu-1}$$

and

$$(3.6) \quad \text{sign } f(-1) = (-1)^n \varepsilon^m.$$

Since $2n + m = 2h + \mu - 1$, it follows from (3.5) and (3.6) that

$$|\nabla_i(-1)| = |f(-1)| = f(-1) \cdot \text{sign } f(-1)$$

$$\equiv \varepsilon^m (-1)^{(m-\mu+1)/2} D(I) \pmod{4}.$$

Thus the proof is completed.

COROLLARY 5. *Let k be any knot. If $|\sigma(k)| = 2m$, then $|\Delta_k(-1)| \equiv (-1)^m \pmod{4}$.*

This corollary was first proved by Murasugi (Theorem 5.6 in [8]).

Now it follows from Corollary 5 and Theorem 1 of [6] that

$$(3.7) \quad |\Delta_k(-1)| \equiv (-1)^m \pmod{4} \text{ if and only if } |\sigma(k)| \equiv 2m \pmod{4},$$

and if $|\Delta_k(-1)| = 1$ then $\sigma(k) \equiv 0 \pmod{8}$.

Moreover the condition (3.7) is the best possible in the following sense:

THEOREM 6. *Let m and n be nonnegative integers. Then there exists a knot k such that*

- (1) $|\Delta_k(-1)| = 4m + 1$ and $|\sigma(k)| = 8n$,
- (2) $|\Delta_k(-1)| = 8m + 5$ and $|\sigma(k)| = 8n + 4$,
- (3) $|\Delta_k(-1)| = 4m + 3$ and $|\sigma(k)| = 4n + 2$.

REMARK. The existence of a knot k such that

- (4) $|\Delta_k(-1)| = 8m + 1$ ($m > 0$) and $|\sigma(k)| = 8n + 4$

can be proved for the case $m \equiv \pm 1 \pmod{3}$. The case $m \equiv 0 \pmod{3}$ still remains open, though the affirmative answer is expected.

Proof of Theorem 6. We will use the following facts (a)–(e):

(a) If $k = k_1 \# k_2$ is the composition of two knots k_1 and k_2 , then $\sigma(k) = \sigma(k_1) + \sigma(k_2)$ and $|\Delta_k(-1)| = |\Delta_{k_1}(-1)| \cdot |\Delta_{k_2}(-1)|$.

(b) If k^{-1} is the mirror image of a knot k , then $\sigma(k^{-1}) = -\sigma(k)$ and $|\Delta_{k^{-1}}(-1)| = |\Delta_k(-1)|$.

Let $K(p, q)$ denote the torus knot of type (p, q) .

(c) $\sigma(K(5, 3)) = 8$ and $|\Delta_{K(5,3)}(-1)| = 1$.

(d) $\sigma(K(2p+1, 2)) = 2p$ and $|\Delta_{K(2p+1,2)}(-1)| = 2p + 1$.

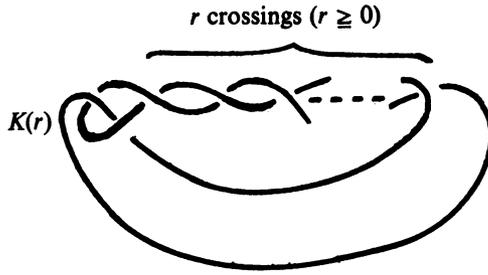


FIGURE 1

Let $K(r)$ be the knot shown in Figure 1.

(e) $\sigma(K(r))=0$ or 2 according as r is even or odd and $|\Delta_{K(r)}(-1)|=2r+1$.

Now let k_1, k_2 and k_3 be knots defined by

$$k_1 = K(2m) \# K(5, 3) \# \cdots \# K(5, 3) \quad (n \text{ times});$$

$$k_2 = K(8m+5, 2) \# K(5, 3)^\varepsilon \# \cdots \# K(5, 3)^\varepsilon$$

$$(|n-m| \text{ times}), \quad \text{where } \varepsilon = \text{sign}(n-m);$$

$$k_3 = K(2m+1) \# K(5, 3) \# \cdots \# K(5, 3) \quad (s \text{ times}) \quad \text{if } n = 2s,$$

$$= K(2m+1)^{-1} \# K(5, 3) \# \cdots \# K(5, 3) \quad (s+1 \text{ times}) \quad \text{if } n = 2s+1.$$

Then, by using (a)–(e), we can show easily that k_i satisfies the condition (i) of Theorem 6 for $i=1, 2, 3$.

4. Knots in solid tori. Let U and V be solid tori in S^3 . A homeomorphism of U onto V is called *faithful* if it preserves the orientations induced by the orientation of S^3 in U and V and if it carries a longitude of U onto that of V [3].

LEMMA 7. *If $f: U \rightarrow V$ is a faithful homeomorphism, then*

$$\text{Link}(\alpha, \beta) = \text{Link}(f(\alpha), f(\beta))$$

for any pair of disjoint 1-cycles α and β in $\text{Int } U$.

Proof. Let q be a longitude of U . Then α is homologous to some multiple of q , say $\alpha \sim mq$, in U and there exists a 2-chain C in U such that $\alpha - mq = \partial C$. Using the fact that f is faithful, we obtain

$$(1) f(\alpha) - mf(q) = \partial f(C),$$

$$(2) f(q) \text{ is a longitude of } V,$$

$$(3) S(C, \beta) = S(f(C), f(\beta)),$$

where $S(A^2, B^1)$ denotes the intersection number of a 2-chain A^2 and a 1-chain B^1 in S^3 .

Since β lies in $\text{Int } U$ and q bounds a 2-chain Q in $S^3 - \text{Int } U$, it follows that

$\text{Link}(q, \beta) = S(Q, \beta) = 0$. Applying the same argument to $f(q)$ and $f(\beta)$, we can show that $\text{Link}(f(q), f(\beta)) = 0$. Therefore we obtain

$$\begin{aligned} \text{Link}(\alpha, \beta) &= \text{Link}(\alpha - mq, \beta) = S(C, \beta) \\ &= S(f(C), f(\beta)) \\ &= \text{Link}(f(\alpha) - mf(q), f(\beta)) \\ &= \text{Link}(f(\alpha), f(\beta)). \end{aligned}$$

LEMMA 8. *Let q be a longitude of a solid torus U and Q a 2-chain in $S^3 - \text{Int } U$ such that $\partial Q = q$. If α is a 1-cycle in $\text{Int } U$ and β is a 1-cycle in $S^3 - U$ with $S(\beta, Q) = 0$, then $\text{Link}(\alpha, \beta) = 0$.*

Proof. Since $\alpha \sim mq$ in U for some integer m , there exists a 2-chain C in U such that $\alpha = mq + \partial C = \partial(mQ + C)$. Therefore we obtain

$$\text{Link}(\alpha, \beta) = S(mQ + C, \beta) = mS(Q, \beta) + S(C, \beta) = 0.$$

Let k be a knot in S^3 , V a tubular neighborhood of k and V^* a tubular neighborhood of a trivial knot k^* . Let $f: V^* \rightarrow V$ be a faithful homeomorphism of V^* onto V , l^* a knot contained in $\text{Int } V^*$ and $l = f(l^*)$. Then $l^* \sim nk^*$ in $\text{Int } V^*$ for some integer n .

THEOREM 9.

$$\begin{aligned} \sigma(l) &= \sigma(l^*) && \text{if } n \text{ is even,} \\ &= \sigma(l^*) + \sigma(k) && \text{if } n \text{ is odd.} \end{aligned}$$

Proof. There is no loss of generality in supposing that n is nonnegative. Note that the signature does not depend on the choice of orientation for a knot.

Case I. $n = 0$.

Since l^* is nullhomologous in V^* , l^* bounds an orientable surface F^* in $\text{Int } V^*$. Let h be the genus of F^* and $\alpha_1^*, \alpha_2^*, \dots, \alpha_{2h}^*$ a homology basis on F^* . Clearly $F = f(F^*)$ is an orientable surface contained in $\text{Int } V$ whose boundary is l , and $\alpha_1 = f(\alpha_1^*), \alpha_2 = f(\alpha_2^*), \dots, \alpha_{2h} = f(\alpha_{2h}^*)$ is a homology basis on F . Let $T^*: F^* \rightarrow S^3 - F^*$ and $T: F \rightarrow S^3 - F$ be translations in the positive normal direction of F^* and of F respectively. We may assume that $T^*\alpha_i^*$ and $T\alpha_i$ are contained in $\text{Int } V^*$ and in $\text{Int } V$ respectively. Since $T\alpha_i \sim f(T^*\alpha_i^*)$ in $V - F$, it follows from Lemma 7 that

$$\begin{aligned} \text{Link}(T^*\alpha_i^*, \alpha_j^*) &= \text{Link}(f(T^*\alpha_i^*), \alpha_j) \\ &= \text{Link}(T\alpha_i, \alpha_j). \end{aligned}$$

Therefore we have shown that a Seifert matrix $\|\text{Link}(T^*\alpha_i^*, \alpha_j^*)\|_{i,j=1,2,\dots,2h}$ of l^* coincides with a Seifert matrix $\|\text{Link}(T\alpha_i, \alpha_j)\|_{i,j=1,2,\dots,2h}$ of l . This implies immediately that $\sigma(l) = \sigma(l^*)$.

Case II. n is a positive integer.

First we will construct orientable surfaces F^* and F bounded by l^* and l respectively. Our construction is the same as the one given in §4 of [10], but for the sake of completeness we include it here.

We choose n pairwise disjoint longitudes $q_1^*, q_2^*, \dots, q_n^*$ on the boundary of V^* such that q_i^* is homologous to k^* in V^* , $i=1, 2, \dots, n$. Since $l^* \sim nk^* \sim \sum_{i=1}^n q_i^*$ in V^* , there exists an orientable surface F_0^* in V^* with $F_0^* = l^* - \sum_{i=1}^n q_i^*$. Let $F_1^*, F_2^*, \dots, F_n^*$ be pairwise disjoint 2-cells in $S^3 - \text{Int } V^*$ with $\partial F_i^* = q_i^*$, $i=1, 2, \dots, n$. Then $F^* = F_0^* \cup \bigcup_{i=1}^n F_i^*$ is an orientable surface in S^3 whose boundary is l^* .

Let $F_0 = f(F_0^*)$ and $q_i = f(q_i^*)$ for $i=1, 2, \dots, n$. Then F_0 is an orientable surface contained in V whose boundary is $l - \sum_{i=1}^n q_i$. Since q_1 is a longitude of V , it bounds an orientable surface F_1 in $S^3 - \text{Int } V$. Without loss of generality we may assume that q_1, q_2, \dots, q_n are ordered in the positive normal direction of F_1 . By pushing F_1 isotopically in its positive normal direction we obtain an orientable surface F_2 in $S^3 - \text{Int } V$ which is parallel to F_1 and whose boundary is q_2 . Similarly, by pushing F_2 isotopically in its positive normal direction, we obtain an orientable surface F_3 in $S^3 - \text{Int } V$ with $\partial F_3 = q_3$ which is parallel to F_2 and intersects neither F_1 nor F_2 . By continuing this process we finally obtain n pairwise disjoint orientable surfaces F_1, \dots, F_n in $S^3 - \text{Int } V$ with $\partial F_i = q_i$, $i=1, 2, \dots, n$. Clearly $F = F_0 \cup \bigcup_{i=1}^n F_i$ is an orientable surface bounded by l .

Let g be the genus of F^* and let

$$(4.1) \quad \alpha_1^*, \alpha_2^*, \dots, \alpha_{2g}^*$$

be a homology basis on F^* . Since F_1^*, \dots, F_n^* are 2-cells, we may assume that these basis elements are lying on F_0^* . Therefore $\alpha_1 = f(\alpha_1^*), \alpha_2 = f(\alpha_2^*), \dots, \alpha_{2g} = f(\alpha_{2g}^*)$ are lying on F_0 . Let h be the genus of F_1 and $\alpha_1^1, \alpha_2^1, \dots, \alpha_{2h}^1$ a homology basis on F_1 . For $\nu = 2, 3, \dots, n$, we choose $\alpha_1^\nu, \alpha_2^\nu, \dots, \alpha_{2h}^\nu$ as a homology basis on F_ν , where α_i^ν is the image of α_i^1 under the above described isotopy which carries F_1 to F_ν . Then it is easy to show that

$$(4.2) \quad \alpha_1^1, \alpha_2^1, \dots, \alpha_{2h}^1; \dots; \alpha_1^n, \alpha_2^n, \dots, \alpha_{2h}^n; \alpha_1, \alpha_2, \dots, \alpha_{2g}$$

is a homology basis on F .

Now we want to consider the Seifert matrix of l with respect to the homology basis (4.2) on F . Let $T^*: F^* \rightarrow S^3 - F^*$ and $T: F \rightarrow S^3 - F$ be translations in the positive normal direction of F^* and of F respectively. Let

$$A^* = \|\text{Link}(T^* \alpha_i^*, \alpha_j^*)\|_{i,j=1,2,\dots,2g}$$

be the Seifert matrix of l^* with respect to homology basis (4.1) on F^* . Then the argument in Case I implies easily that

$$(4.3) \quad \|\text{Link}(T \alpha_i, \alpha_j)\|_{i,j=1,2,\dots,2g} = A^*.$$

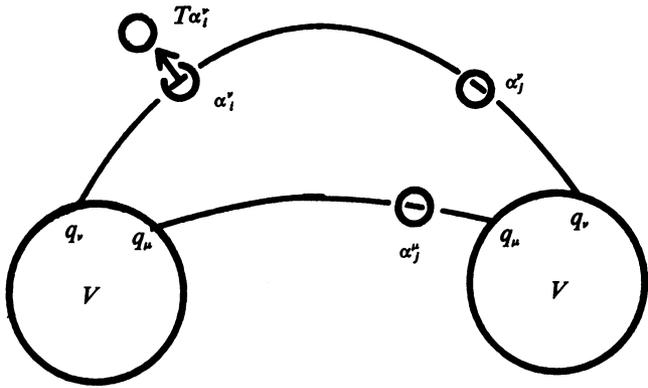
Since α_s is a 1-cycle in V and $T\alpha_i^y \cap F_1 = \emptyset$, it follows from Lemma 8 that

$$(4.4) \quad \text{Link}(T\alpha_i^y, \alpha_s) = 0 \quad \text{for } 1 \leq \nu \leq n; 1 \leq i \leq 2h; 1 \leq s \leq 2g.$$

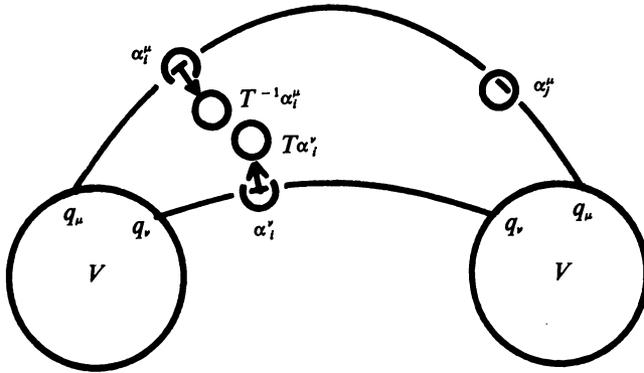
Since $\alpha_j^y \sim T\alpha_j^y$ in $S^3 - V$, we have $\text{Link}(T\alpha_r, \alpha_j^y) = \text{Link}(T\alpha_r, T\alpha_j^y)$. Applying Lemma 8 to $T\alpha_r$ and $T\alpha_j^y$, we obtain

$$(4.5) \quad \text{Link}(T\alpha_r, \alpha_j^y) = 0 \quad \text{for } 1 \leq \nu \leq n; 1 \leq j \leq 2h; 1 \leq r \leq 2g.$$

If $\nu \geq \mu$, we have $\alpha_j^y \sim \alpha_j^\mu$ in $S^3 - T\alpha_i^y$ (see Figure 2(a)), from which we obtain $\text{Link}(T\alpha_i^y, \alpha_j^\mu) = \text{Link}(T\alpha_i^y, \alpha_j^y)$. If $\nu < \mu$, $T\alpha_i^y \sim T^{-1}\alpha_i^\mu$ in $S^3 - \alpha_j^\mu$, where $T^{-1}: F \rightarrow S^3 - F$ is a translation in the negative normal direction of F (see Figure 2(b)). From this we obtain



(a) $\nu \geq \mu$



(b) $\nu < \mu$

FIGURE 2

Link $(T\alpha_i^\nu, \alpha_j^\mu) = \text{Link}(T^{-1}\alpha_i^\mu, \alpha_j^\nu) = \text{Link}(\alpha_i^\mu, T\alpha_j^\nu)$. Therefore we have shown that

$$(4.6) \quad \begin{aligned} \text{Link}(T\alpha_i^\nu, \alpha_j^\mu) &= \text{Link}(T\alpha_i^\nu, \alpha_j^\mu) \quad \text{if } \nu \geq \mu, \\ &= \text{Link}(\alpha_i^\mu, T\alpha_j^\nu) \quad \text{if } \nu < \mu. \end{aligned}$$

Since q_1 and k belong to the same knot type, the matrix

$$B = \|\text{Link}(T\alpha_i^1, \alpha_j^1)\|_{i,j=1,2,\dots,2h}$$

can be considered as a Seifert matrix of k . By the construction of $\alpha_1^\nu, \alpha_2^\nu, \dots, \alpha_{2h}^\nu$ it is clear that

$$\text{Link}(T\alpha_i^\nu, \alpha_j^\nu) = \text{Link}(T\alpha_i^1, \alpha_j^1)$$

for $\nu=1, 2, \dots, n$. From this and (4.6) we have

$$(4.7) \quad \begin{aligned} \|\text{Link}(T\alpha_i^\nu, \alpha_j^\nu)\|_{i,j=1,2,\dots,2h} &= B \quad \text{if } \nu \geq \mu, \\ &= B' \quad \text{if } \nu < \mu. \end{aligned}$$

Let A be the Seifert matrix of l with respect to the homology basis (4.2) on F . Then (4.3)–(4.7) show that A is a matrix of the following form:

$$A = \begin{bmatrix} \tilde{B} & 0 \\ 0 & A^* \end{bmatrix},$$

where $\tilde{B} = \|B_{\nu\mu}\|_{\nu,\mu=1,2,\dots,n}$,

$$\begin{aligned} B_{\nu\mu} &= B \quad \text{if } \nu \geq \mu, \\ &= B' \quad \text{if } \nu < \mu. \end{aligned}$$

To calculate signature $(A+A')$, we will make use of the facts that $A-A'$ is the matrix $S = \|S(\alpha_i^1, \alpha_j^1)\|_{i,j=1,2,\dots,2h}$ of intersection numbers of α_i^1 and α_j^1 on F_1 [5] and that S is a unimodular skew symmetric matrix [9]. In $A+A'$, we subtract the 2nd row block from the 1st and the 2nd column block from the 1st; the 3rd row block from the 2nd and the 3rd column block from the 2nd; \dots ; the n th row block from the $(n-1)$ th and the n th column block from the $(n-1)$ th, which shows that $A+A'$ is congruent over Z to the following matrix:

$$(4.8) \quad \begin{array}{c} n \text{ blocks} \\ \left[\begin{array}{ccc|c} 0 & -S & 0 & \\ S & 0 & -S & \\ & \ddots & \ddots & \ddots \\ 0 & & S & 0 & -S \\ & & & S & B+B' \\ \hline & & & 0 & A^*+A^{*'} \end{array} \right] \end{array}.$$

Furthermore, by adding the 1st row block to the 3rd and the 1st column block to the 3rd; the new 3rd row block to the 5th and the new 3rd column block to the 5th; . . . , it can be shown that (4.8) is congruent over Z to the matrix

$$(4.9) \quad \begin{aligned} & \tilde{S} \oplus \cdots \oplus \tilde{S} \oplus \bar{S} \oplus (A^* + A'^*) \quad (n/2 - 1 \text{ copies}) \quad \text{if } n \text{ is even,} \\ & \tilde{S} \oplus \cdots \oplus \tilde{S} \oplus (B + B') \oplus (A^* + A'^*) \quad ((n-1)/2 \text{ copies}) \quad \text{if } n \text{ is odd,} \end{aligned}$$

where

$$\tilde{S} = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix} \quad \text{and} \quad \bar{S} = \begin{bmatrix} 0 & -S \\ S & B + B' \end{bmatrix}.$$

Since \bar{S} is congruent to \tilde{S} over Z and signature $\tilde{S} = 0$, it follows from (4.9) that

$$\begin{aligned} \sigma(l) &= \sigma(l^*) && \text{if } n \text{ is even,} \\ &= \sigma(l^*) + \sigma(k) && \text{if } n \text{ is odd.} \end{aligned}$$

This completes the proof of Theorem 9.

REMARK 1. In [10] H. Seifert showed that $\Delta_l(t) = \Delta_{l^*}(t)\Delta_k(t^n)$, where $\Delta_l(t)$, $\Delta_{l^*}(t)$ and $\Delta_k(t)$ are the Alexander polynomials of l , l^* and k .

REMARK 2. Let $M_2(l)$, $M_2(l^*)$ and $M_2(k)$ be the 2-fold branched covering spaces of l , l^* and k . Then it follows from (4.9) that

$$\begin{aligned} H_1(M_2(l)) &= H_1(M_2(l^*)) && \text{if } n \text{ is even,} \\ &= H_1(M_2(l^*)) \oplus H_1(M_2(k)) && \text{if } n \text{ is odd,} \end{aligned}$$

where the coefficients of these homology groups are the integers.

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