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REPRESENTATIONS OF METABELIAN GROUPS REALIZABLE IN THE REAL FIELD

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Abstract. A necessary and sufficient condition is found such that all the nonlinear irreducible representations of a metabelian group are realizable in the real field, and all such groups with cyclic commutator subgroups are determined.

1. **Introduction.** The purpose of this paper is to characterize finite metabelian groups whose all nonlinear irreducible representations (over the complex field) are realizable in the real field. This is done in two steps. Using factor groups which are dihedral we prove in §2 a necessary and sufficient condition for some monomial representation to be realizable in the real field. In a remark we show that this result can be applied to give a partial solution to a question raised by Brauer in [2]. Some notations are introduced in §3 and the main result is proved in §4. This is used in §5 to determine all such finite groups with cyclic commutator subgroup G'. In particular we get that such a metacyclic group with G' of even order should be dihedral.

2. **Representations realizable in the real field.** Let K be a proper normal subgroup of a finite group G, P a linear representation of K with kernel D, and $N(D)$ the normalizer of D in G. Assume the induced representation P^G is irreducible. Define the representation P^* of G by replacing every entry of $P^G(g)$, $g \in G$, by its complex conjugate. The notation $G=\langle a, \ldots\rangle$ means G is generated by $a, \ldots$ and $|G|$ denotes its order. A dihedral group is given by $\langle a, b \mid a^n=b^2=(ab)^2=1\rangle$ and a quaternion group is given by $\langle a, b \mid a^{2n}=b^4=1,\ a^n=b^2,\ b^{-1}ab=a^{-1}\rangle$.

LEMMA 1. *Assume the above. Then P^G is equivalent to P^* if and only if there exists $y \in N(D)$, unique modulo K, such that $\bar{H}=\langle K, y\rangle/D$ is dihedral or quaternion or $|\bar{H}|=|K/D|=2$. Moreover P^G is realizable in the real field $\mathbf{R}$ if and only if $\bar{H}$ is dihedral or $|\bar{H}|=|K/D|=2$.*

Proof. Let χ be the character of P and let $\theta=\chi^G$ be the character of P^G. Then $\theta(g)=0$ for $g \notin K$ and $\theta_K=\sum_{y\in G/K}\chi^y$ where $\chi^y(k)=\chi(y^{-1}ky)$, $k \in K$. Since θ is

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irreducible, from [3, (45.4)], we have $K=\{g\in G \mid \chi^g=\chi\}$. Thus θ is real-valued, (i.e., $\theta=\bar{\theta}$ and P^G is equivalent to P^*) if and only if $\bar{\chi}=\chi^y$ for some $y\in G$.

Assume y exists, then by [3, (45.4)] y is unique modulo K. Since χ and $\bar{\chi}=\chi^y$ have the same kernel D, it follows that $y\in N(D)$. Also for $h\in G$, $\bar{\chi}^h=\chi^{hz}$ if and only if $z^{-1}y\in K$. Now $y\in K$ if and only if $\bar{\chi}=\chi$, i.e., χ is real-valued and $|K/D|=2$. If $y\notin K$, then $\chi^{y^2}=\chi$ or $y^2\in K$ and $\chi^y(k)=\bar{\chi}(k)=\chi(k^{-1})$ for all $k\in K$ or $y^{-1}(kD)y=k^{-1}D$. Thus $\bar{H}$ is quaternion or dihedral whenever $|K/D|\neq 2$.

Now assume $\theta=\bar{\theta}$ and let $H=\langle K, y\rangle$. Then $\xi=\chi^H$ is irreducible and $\xi=\bar{\xi}$. Hence $\boldsymbol{R}(\theta)=\boldsymbol{R}(\xi)$ and $m_{\boldsymbol{R}}(\theta)\leqq m_{\boldsymbol{R}}(\xi)\leqq 2$ where $m_{\boldsymbol{R}}(\theta)$ and $m_{\boldsymbol{R}}(\xi)$ denote the Schur indices of θ and ξ respectively. If $\bar{H}$ is dihedral or $|K/D|=2$ then $m_{\boldsymbol{R}}(\xi)=1$ and hence θ is realizable in $\boldsymbol{R}$. If $\bar{H}$ is quaternion then $m_{\boldsymbol{R}}(\xi)=2$. The proof will be complete if we show $m_{\boldsymbol{R}}(\theta)=2$. If $g^2\notin K$ then $\theta(g^2)=0$. If $g\in zK$ and $g^2\in K$, then $g=zk$, $k\in K$, and $\chi^h(g^2)=\chi^h(z^2)\chi^{hz}(k)\chi^h(k)$. Hence

$$\sum_{g\in zK}\theta(g^2) = |K|\sum_{h\in G/K}\chi^h(z^2)(\chi^{hz}, \bar{\chi}^h)_K$$

which is zero if $z^{-1}y\notin K$ and $-|G|$ if $z^{-1}y\in K$. The result follows from the Frobenius-Schur Theorem [4, (3.5)].

REMARK. Using [1] and the above lemma we can find all the nonlinear irreducible representations of the metabelian groups realizable in the real field and hence give a partial solution to Problem 14 stated by Brauer in [2]. As an example we consider the metacyclic group

$$G = \langle a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r\rangle$$

where $r^t-1\equiv kr-k\equiv 0 \pmod{n}$ and $t\mid m$. Let s be a positive divisor of n and t_s be the smallest positive integer such that $r^{t_s}\equiv 1 \pmod{s}$. Then the linear representations P of $K_s=\langle a, b^{t_s}\rangle$ with kernel D, $D\cap\langle a\rangle=\langle a^s\rangle$, induce all the irreducible representations P^G of G. Their number is $\sum_{s\mid n} t\phi(s)/t_s^2$. If $s\mid(k, n)$, t_s is even, $r^{t_s/2}\equiv -1 \pmod{s}$, and $D=\langle a^s, b^{t_s}\rangle$, then P^G is realizable in the real field. Therefore the number of all such nonlinear irreducible representations of G is $\sum' \phi(s)/t_s$, where $\sum'$ is over all s with the above conditions. A simpler formula for a special case is given in [5].

3. **Notations.** The notations of this section will be used in the remainder of the paper. Let G be a finite metabelian group and G' its commutator group. Let D be a subgroup of G' such that G'/D is cyclic and let $N(D)$ be the normalizer of D in G. Let $K(D)/D$ be a maximal abelian subgroup of $N(D)/D$ containing G'/D. Let $z_1G',\ldots, z_wG'$ be a basis of $K(D)/G'$ where z_iG' is of order t_i. Let P be a linear representation of G' with kernel D and $\bar{P}$ be a linear representation of $K(D)$ which is an extension of P to all $K(D)$, where $\bar{P}(z_i)^{t_i}=P(z_i^{t_i})$. From [1] *$\bar{P}^G$ is irreducible and every irreducible representation of G is equivalent to some $\bar{P}^G$ for some P with some kernel D and any $K(D)$*. Let $\bar{D}$ be the kernel of $\bar{P}$ and note that $\bar{D}\cap G'=D$.

For any finite abelian group A and an abelian subgroup Λ of Aut A denote $\Lambda(A)$ the extension of A by Λ with the coset representatives x_σ, $\sigma \in \Lambda$, where $x_\sigma^{-1} a x_\sigma = a^\sigma$ for all $a \in A$. For σ and τ in Λ we have $x_\sigma x_\tau = x_{\sigma\tau}(\sigma, \tau)$ where (σ, τ) denote the elements of the factor set. We set $(1, 1) = (1, \sigma) = (\sigma, 1) = 1$ for all $\sigma \in \Lambda$. If $A = \langle a, \ldots \rangle$ and $\Lambda = \langle \sigma, \ldots \rangle$ then a^σ will be given only when $a^\sigma \neq a$ or whenever confusion is likely.

Lemma 2. *Let $\Lambda = \langle \sigma_1, \ldots, \sigma_t \rangle$ be an elementary abelian 2-group (of order 2^t) and let B be a subgroup of A invariant under Λ. Then, with a possible change to an equivalent factor set, the following statements are equivalent.*

(i) $(\sigma_i, \sigma_i) = 1$ *and* $(\sigma, \tau) \in B$ *for all* i *and* $\sigma, \tau \in \Lambda$.

(ii) $(\sigma_i, \sigma_i) = 1$ *and* $(\sigma, \tau)^{-1}(\tau, \sigma) \in B$ *for all* i *and all* $\sigma, \tau \in \Lambda$.

(iii) $x_{\sigma_i}^2 = 1$ *and* $x_\sigma^{-1} x_\tau^{-1} x_\sigma x_\tau \in B$ *for all* i *and all* $\sigma, \tau \in \Lambda$.

Proof. From the definition, (ii) and (iii) are equivalent since Λ is abelian. To prove (i) from (iii) consider $\bar{\Lambda} = \langle x_{\sigma_i} \mid i = 1, \ldots, t \rangle$, then $\bar{\Lambda} \cap A \subseteq B$. Picking a suitable set of coset representatives of A in $\Lambda(A)$ we have (i).

Let $A = A(s, m, \delta, g) = \langle a, d, b, b_1, \ldots, b_g, c_1, \ldots, c_t \rangle$ be abelian where $a^{2^s} = d^m = b_i^4 = c_j^2 = 1$, $i = 1, \ldots, g$, $j = 1, \ldots, t$, m odd, and $b = 1$ if $\delta = 0$ and $b^4 = a^2$ if $\delta = 1$. Note that if $s = 1$ then $\delta = 0$.

Let $\Gamma = \langle \sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_g \rangle$ be the subgroup of Aut A, $A = A(s, m, \delta, g)$ where $a^{\sigma_i} = a^{-1}$, $i \geqq -1$, $b^{\sigma_i} = ba^{-1}$, $i \geqq 0$, $b^{\sigma_{-1}} = ba^{2^{s-1}-1}$, $b_i^{\sigma_i} = b_i a^{2^{s-1}}$, $i \geqq 1$, and $d^{\sigma_0} = d^{-1}$. Note that Γ depends on A, for instance if $\delta = 0$ then σ_{-1} does not exist and b^{σ_i}, $i \geqq 0$, are not defined.

Let $\Gamma(A)$ be an extension of A by Γ such that $(\sigma, \sigma) = 1$ if $a^\sigma = a^{-1}$ and for all $\sigma, \tau \in \Gamma$, $(\sigma, \tau) \in \langle a \rangle$. Hence $[\Gamma(A)]' \subseteq \langle a, d \rangle$ and (i)–(iii) of Lemma 2 are equivalent for $B = \langle a \rangle$. Note that $\Gamma(A)$ is the semidirect product $A \circ \Gamma$ when $A = A(s, m, 0, 0)$ or $A = A(1, 1, 0, g)$.

Lemma 3. *Let P be a linear representation of A which is faithful on $\langle a, d \rangle$ where $A = A(s, m, 0, 0)$ or $A = A(s, 1, \delta, g)$. Then the induced representation $P^{\Gamma(A)}$ is (absolutely) irreducible and is realizable in the real field.*

Proof. Irreducibility follows since P is an extension of its restriction on $\langle a, d \rangle$ and $K(1) = A$. Let D be the kernel of P. If $A = A(s, m, 0, 0)$ then $\Gamma = \langle \sigma_0 \rangle$ and $\langle A, x_{\sigma_0} \rangle / D$ is dihedral. Let $A = A(s, 1, \delta, g)$ and let $b_1, \ldots, b_v$ be all the b_i's such that $b_i^2 \notin D$. If $\delta = 0$ then let $\sigma = \sigma_1 \cdots \sigma_v$ when v is odd and let $\sigma = \sigma_0 \cdots \sigma_v$ when v is even. Hence $P(k^\sigma) = P(k^{-1})$ for all $k \in A$ and $\langle A, x_\sigma \rangle / D$ is dihedral. For $\delta = 1$ let ζ be a primitive 2^{s+1}th root of unity and assume $P(a) = \zeta^2$. Then $P(b) = \pm\zeta$ or $\pm(-1)^{1/2}\zeta$. If $P(b) = \pm\zeta$ let $\sigma = \sigma_1 \cdots \sigma_v$ when v is odd and $\sigma = \sigma_0 \sigma_1 \cdots \sigma_v$ when v is even. If $P(b) = \pm(-1)^{1/2}\zeta$ let $\sigma = \sigma_{-1}\sigma_0\sigma_1 \cdots \sigma_v$ when v is odd and $\sigma = \sigma_{-1}\sigma_1 \cdots \sigma_v$ when v is even. Then $\langle A, x_\sigma \rangle / D$ is dihedral and the result follows from Lemma 1.

Note that every irreducible nonlinear representation of $\Gamma(A(s, m, 0, 0))$ and $\Gamma(A(1, 1, 0, g))$ is realizable in the real field.

Assume σ is as in the above proof and let $x=x_{\sigma_1}\cdots x_{\sigma_v}$, etc. Then since $x_{\sigma_i}^2=1$, we have $x^2 \in [\Gamma(A)]' \subseteq \langle a, d\rangle$. Since $x^{-1}kx=k^\sigma \in k^{-1}D$ we have $x^2 \in D \cap \langle a, d\rangle$ or $x^2=1$. We may take $x_\sigma=x$.

Now consider $\Gamma(A)$ where $A=A(s, 1, \delta, g)$, $s>1$. Assume $(x_\sigma x_\tau)^2=a^r$ where $\sigma=\sigma_0$ and $\tau=\sigma_i$, $i\neq 0$, and $r=r(\tau)$ is even. Let $y_t=x_\tau a^{-r/2}$, then $y_\tau^2=1$ and $x_\sigma y_\tau =y_\tau x_\sigma$. Hence $1=(x_\sigma y_\tau y_\omega)^2=(y_\tau y_\omega)^2$, $\omega=\sigma_j$, $j\neq i$, and hence $\Gamma(A)=A\circ\Gamma$. If for some $\tau=\sigma_i$, $i\neq 0$, $(x_\sigma x_\tau)^2=a$, $\sigma=\sigma_0$, let $\bar{a}=x_\sigma x_\tau$. For $\delta=0$ we have $\langle \bar{a}, b_1, \dots, b_i^2, \dots, b_g, c_1, \dots, c_t\rangle=A(s+1, 1, 0, g-1)$ and $\langle A(s+1, 1, 0, g-1), x_{\sigma_1}, \dots, x_{\sigma_g}\rangle \cong A(s+1, 0, g-1)\circ\Gamma_1$. Similarly for $\delta=1$ we have $A(s+1, 1, 0, g)\circ\Gamma_1$. Here Γ_1 is defined for $A(s+1, 1, 0, g-1)$ or $A(s+1, 1, 0, g)$ as Γ is done for $A(s, 1, \delta, g)$. The above implies that either $\Gamma(A)=A\circ\Gamma$ or a subgroup of $\Gamma(A)$ is a semidirect product.

4. **Metabelian groups.** Assume the notations of §3 and $D\nsubseteq G'$.

Theorem 1. *Assume every nonlinear irreducible representation of G is realizable in the real field. Then the following hold for every D and any corresponding $K(D)$.*

(i) *The exponent of $K(D)/G'$ divides* 2 *or divides* 4 *depending whether an odd prime divides $|G'/D|$ or $|G'/D|$ is a power of* 2.

(ii) *If $K(D)/G'$ is of exponent dividing* 2, *then there exists $x\in N(D)$ such that $\langle K(D), x\rangle/D\cong A(s, m, 0, 0)\circ\Gamma$, and if $K(D)/G'$ is of exponent* 4, *then there exist $x_{-1}, x_0, \dots, x_g$ in $N(D)$ such that $\langle K(D), x_{-1}, \dots, x_g\rangle/D\cong\Gamma(A(s, 1, \delta, g))$.*

Conversely if (i) *and* (ii) *hold for every D and some corresponding $K(D)$, then every nonlinear representation of G is realizable in the real field, and hence* (i) *and* (ii) *hold for every D and any corresponding $K(D)$.*

Proof. (i) Since $\bar{D}\cap G'=D$ from Lemma 1 we may assume $D=1$ and $G=N(D)$. Also if $D\subseteq D_1\subseteq G'$ then given $K(D)$ some $K(D_1)$ can be found such that $K(D_1)\supseteq K(D)$. Hence for this part of the proof we may also assume $G'=\langle a\rangle$ is cyclic of prime order p. Let zG' be a basis element of $K(D)/G'$ of order m, then z is of order fm, $f=1$ or p. From Lemma 1, there exists $x\in N(\bar{D})$ such that $\bar{P}(x^{-1}zx) =\bar{P}(z^{-1})$. But $x^{-1}zx=zg$ for some $g\in G'$ and hence $\bar{P}(z^{-1})=\bar{P}(zg)$ or $\bar{P}(z^2)=P(g^{-1})$. Assume $\bar{P}(z)$ is a primitive fmth root of unity. Since $\bar{P}(g^{-1})$ is a pth root of unity we have $\bar{P}(z)$ a primitive 2nd or pth or $(2p)$th root of unity. Hence $fm=2$ or $fm=p$ or $fm=2p$. If $p=2$ then $K(D)/G'$ is of exponent dividing 4.

Assume $p\neq 2$ and $fm=p$ or $2p$. Letting $b=z$ or $b=z^2$ we have $b\in K(D)-G'$ and $b^p=1$. Let ζ be a primitive pth root of unity and let $P(a)=\zeta$. Consider two extensions $\bar{P}_1$ and $\bar{P}_2$ of P to $K(D)$ with $\bar{P}_1(b)=1$ and $\bar{P}_2(b)=\zeta$. For $\bar{P}_1(b)=1$, from Lemma 1, there exists x such that $\bar{P}_1(x^{-1}kx)=\bar{P}_1(k^{-1})$ for all $k\in K(D)$, and hence $x^{-1}ax=a^{-1}$, $x^{-1}bx=b$, and $x^2\in K(D)$. Similarly if $\bar{P}_2(b)=\zeta$ we have y such $y^{-1}ay=a^{-1}$, $y^{-1}by=ba^{-2}$, and $y^2\in K(D)$. Let $w=y^{-1}x^{-1}yx$, then $w^{-1}bw=ba^4\neq b$, a contradiction, since $w\in G'$. Hence $K(D)/G'$ is of exponent 2.

(ii) We may assume $D=1$, $G'=\langle a\rangle$, not necesarily of prime order, and hence $G=N(D)$. Assume $K(D)/G'$ is of exponent dividing 2 and let $\bar{D}$ be the kernel of $\bar{P}$.

Then from Lemma 1 there exists $x \in N(\bar{D})$ such that $x^2 \in \bar{D}$ and $\bar{P}(x^{-1}kx) = \bar{P}(k^{-1})$ for all $k \in K(D)$. Hence $x^{-1}kx = k^{-1}d$ for some $d \in \bar{D}$. But $k^{-1}x^{-1}kx = k^{-2}d \in G'$ and, $k^{-2} \in G'$ which implies that $d \in \bar{D} \cap G' = D = 1$. Hence $d = 1$ and $x^{-1}kx = k^{-1}$ for all $k \in K(D)$. Assume $x^2 \notin D$ then $x^4 \in G' \cap \bar{D} = D = 1$ or $x^4 = 1$. Define another extension $\bar{P}_1$ of P to $K(D)$ with kernel $\bar{D}_1$ such that $\bar{P}_1(x^2) = -1$. This implies that $\langle K(D), x\rangle/\bar{D}_1$ is quaternion and from Lemma 1, $\bar{P}_1^G$ is not realizable in the real field, a contradiction. Hence $x^2 \in D$ and $\langle K(D), x\rangle/D \cong A(s, m, 0, 0) \circ \Gamma$.

Now assume G' is cyclic of order a power of 2, $D = 1$, $G = N(D)$, $K = K(D)$, and K/G' of exponent 4. From Lemma 1 there exists $x \in G$ such that $\bar{P}(x^{-1}kx) = \bar{P}(k^{-1})$ for all $k \in K$. Since $x^{-1}kx = kd$, $d \in G'$, we have after some calculations $K = A(s, 1, \delta, g)$ where $\delta = 0$ if $s = 1$, $G' = \langle a\rangle$ if $s = 1$ or $\delta = 1$, and $G' = \langle a\rangle$ or $\langle a^2\rangle$ if $\delta = 0$ and $s > 1$.

Assume $\delta = 0$ and let ζ be a primitive 2^sth root of unity with $P(a) = \zeta$. For $0 \leqq i \leqq g$ define the extensions P_i of P to K by $P_i(b_i) = (-1)^{1/2}$ and $P_i(b_j) = P_i(c_r) = 1$, $i \neq j$, $r = 1, \ldots, t$. (Note that $i \geqq 1$ if $s = 1$.) Hence from Lemma 1 there exists x_i such that $P_i(x_i^{-1}kx_i) = P_i(k^{-1})$ for all $k \in K$ and $P_i(x_i^2) = 1$. Since $x_i^{-1}kx_i = kd$, $d \in G'$, we have $x_i^{-1}ax_i = a^{-1}$, $x_i^{-1}b_ix_i = b_ia^{2^{s-1}}$, $x_ib_j = b_jx_i$, $i \neq j$, and $x_ic_r = c_rx_i$, $r = 1, \ldots, t$. For fixed i, let R_i be other extensions of P to K defined by $R_i(k) = \pm P_i(k)$ where $k = b_j$, $j = 1, \ldots, g$, or $k = c_j$, $j = 1, \ldots, t$. Then $R_i(x_i^{-1}kx_i) = R_i(k^{-1})$ for all $k \in K$. From Lemma 1, $R_i(x_i^2) = 1$ for all R_i, and hence $x_i^2 \in T_i$, where T_i is the intersection of the kernels of all different R_i for fixed i. Now since $x_i^2 \in K(D)$, $x_i^2 = a^eb_1^{e_1}\cdots b_g^{c_g}c_1^{h_1}\cdots c_t^{h_t}$. The fact that $R_i(x_i^2) = 1$ for all R_i, i fixed, implies that $h_i = 0$, $i = 1, \ldots, t$, $e_j = 0$ or $e_j = 2$, $j = 1, \ldots, g$, $e = 0$ when $e_i = 0$, and $e = 2^{s-1}$ when $e_i = 2$. Let $y_i = x_ib_1^{f_1}\cdots b_g^{f_g}$ where $2f_j = e_j$. Then under the various cases above, $y_i^2 = 1$, i.e., x_i can be chosen such that $x_i^2 = 1$. Now let x be a product of an odd number of $x_0, \ldots, x_g$, say $x = x_0\cdots x_v$. Note that $x^{-1}ax = a^{-1}$. Define an extension $\bar{P}$ of P to all K by $\bar{P}(b_j) = (-1)^{1/2}$, $0 < j \leqq v$, and $\bar{P}(b_j) = \bar{P}(c_r) = 1$, $j > v$, $1 \leqq r \leqq t$. Then $\bar{P}(x^{-1}kx) = \bar{P}(k^{-1})$ for all $k \in K$. Since $x^2 \in G'$, if $x^2 \neq 1$, then $\bar{P}(x^2) = -1$ and $\langle K, x\rangle/\bar{D}$ quaternion, where $\bar{D}$ is the kernel of $\bar{P}$. By Lemma 1, $\bar{P}^G$ is not realizable in the real field, a contradiction. Hence $x^2 = 1$ and $\langle K, x_0, x_1, \ldots, x_g\rangle/D \cong \Gamma(A(s, 1, 0, g))$. Note that if $s = 1$ then v may be taken even or odd and hence $\Gamma(A) = A \circ \Gamma$.

Assume $\delta = 1$ and $s > 1$. Let ζ be a primitive 2^{s+1}th root of unity and $P(a) = \zeta^2$. Consider the extensions P_i, $i \geqq -1$, of P to K, defined $P_i(b) = \zeta$, $i \geqq 0$, $P_{-1}(b) = (-1)^{1/2}\zeta$, $P_i(b_i) = (-1)^{1/2}$, $i \geqq 1$, $P_i(b_j) = P_i(c_r) = 1$, $j \neq i$, $r = 1, \ldots, t$. As above this implies the existence of $x_{-1}, x_0, x_1, \ldots, x_g$ such that $x_i^{-1}ax_i = a^{-1}$, $i \geqq -1$, $x_i^{-1}bx_i = ba^{-1}$, $i \geqq 0$, $x_{-1}^{-1}bx_{-1} = ba^{2^{s-1}-1}$, $x_i^{-1}b_ix_i = b_ia^{2^{s-1}}$, $i \geqq 1$, $x_ib_j = b_jx_i$, $j \neq i$, and $x_ic_r = c_rx_i$, $r = 1, \ldots, t$. For fixed i, define other extensions R_i of P to K by $R_i(k) = \pm P_i(k)$ where $k = b$ or $k = b_j$, $j = 1, \ldots, g$, or $k = c_r$, $r = 1, \ldots, t$. As above we have $x_i^2 = a^eb^{e_0}b_1^{e_1}\cdots b_g^{e_g}$ where $e_j = 0$ or $e_j = 2$, $j = 1, \ldots, g$. Also for $i > 0$, if $e_i = 0$ then $R_i(a^eb^{e_0}) = 1$ and hence $e = 0$ and $e_0 = 0$ or $e = -1$ and $e_0 = 2$ and if $e_i = 2$ then $R_i(a^eb^{e_0}) = -1$ and hence $e = 2^{s-1}$ and $e_0 = 0$ or $e = 2^{s-1} - 1$ and $e_0 = 2$. For $i = 0$, $R_0(a^eb^{e_0}) = 1$ and hence $e = 0$ and $e_0 = 0$ or $e = -1$ and $e_0 = 2$. For $i = -1$, $R_{-1}(a^eb^{e_0})$

$=1$ and hence $e=0$ and $e_0=0$ or $e=2^{s-1}-1$ and $e_0=2$. Letting $y_i=x_i b^{-f_0} b_1^{f_1}\cdots b_g^{f_g}$, $2f_j=e_j$, $j=0,\ldots,g$, $i=-1,\ldots,g$, we have, for all the above cases, $y_i^2=1$, or x_i can be chosen such that $x_i^2=1$. Using the same argument as above we have $x^2=1$ if x is the product of an odd number of $x_{-1}, x_0,\ldots,x_g$. Hence

$$\langle K(D), x_{-1}, x_0,\ldots,x_g\rangle/D \simeq \Gamma(A(s,1,1,g)).$$

The converse follows at once from Lemma 3 and the proof is complete.

In the proof of (i) above we did not assume $x^2=y^2=1$, and hence by Lemma 1,

COROLLARY. *Assume every nonlinear irreducible character of G is real valued. Then* (i) *of Theorem* 1 *holds. Moreover we have*

(ii) *Any nonlinear irreducible character of G is induced by some linear character of a subgroup $K\supseteq G'$ such that K/G' is of exponent dividing* 4.

(iii) *If H is a subgroup of G, $H\supseteq G'$, $H'\neq G'$, then H/G' is of exponent dividing* 4. *Moreover if an odd prime divides $|G'/H'|$, then H/G' is of exponent dividing* 2.

From the first part of §3, for each D we can find a $K(D)$ and fix it. Using all possible D's and the corresponding $K(D)$ we can find all the irreducible representations of G. But $K(D)$ for each D may not be unique. Now using the last paragraph of §3 we have

COROLLARY. *Assume G as in Theorem* 1. *Then for each D some $K(D)$ can be chosen such that $\langle K(D), x_{-1},\ldots,x_g\rangle/D\cong A\circ\Gamma$.*

5. **Cyclic commutator group.** We introduce further notations. The order of $r \pmod m$, $(r,m)=1$, is the smallest positive integer t such that $r^t\equiv 1 \pmod m$.

If $D=\langle a,\ldots\rangle$ then set $K(D)=K(a,\ldots)$.

Let $A=A(s,m,0,0)$, $m>1$ and m odd. Let $\Gamma(2h)=\langle\sigma'\rangle$ be the subgroup of Aut A such that $d^{\sigma'}=d^r$, $(\sigma')^{2h}=1$, and r is of order $2h \pmod p$ for every prime p dividing m. Hence $2h|p-1$ and $(2h,m)=1$.

Let $\Omega=\langle\tau_1,\ldots,\tau_f\rangle$ be a subgroup of Aut A, $A=A(s,m,\delta,g)$, such that $a^{\tau_i}=a$ or (for $s\geqq 3$) $a^{\tau_i}=a^{2^{s-1}+1}$, $d^{\tau_i}=d^r$, $r^2\equiv 1 \pmod m$, and $c_j^{\tau_i}=c$ or $c_j^{\tau_i}=c_j a^{2^{s-1}}$ $j=1,\ldots,t$. Here Ω is an elementary abelian 2-group. Let $\Omega_1=\{\tau\in\Omega \mid a^\tau=a$ and $c_j^\tau=c_j\}$ and set $\Omega=\Omega_1\times\Omega_2$ for some subgroup Ω_2 of Ω. Let ω_1 and ω_2 be in Ω_1 and $d^{\omega_i}=d^{r_i}$, $i=1$ or 2. Define $\omega_1<\omega_2$ if $\omega_1\neq\omega_2$ and $r_1\equiv -1 \pmod p$ implies $r_2\equiv -1 \pmod p$ where p runs over all primes dividing m. Hence if $\omega_1<\omega_2$ then $\omega_1\omega_2<\omega_2$. Two elements ω_1 and ω_2 are disjoint if $\omega_1<\omega_1\omega_2$. A nonidentity element ω of Ω_1 is minimal if there exists no $\tau\in\Omega_1$, $\tau\neq 1$, such that $\tau<\omega$. Clearly every $\tau\in\Omega_1$, $\tau\neq 1$, is a product of (not necessarily unique) disjoint minimal elements of Ω_1. Unless otherwise stated, $\tau_i\in\Omega_1$ will mean τ_i is a minimal element of Ω_1 and $\tau_i\in\Omega_2$ will mean τ_i is an element of some basis of Ω_2. Using these notations, consider the following statements.

(i) For all $\sigma,\tau\in\Gamma$, $(\sigma,\tau)=1$.

(ii) For all $\sigma,\tau\in\Lambda$, $(\sigma,\tau)\in\langle a^{2^{s-1}}\rangle$.

(iii) For some $\{\tau_i\}$, a basis of Ω_2, $(\tau_i, \tau_i)=1$.

(iv) For $\{\tau_i\}$, set of minimal elements of Ω_1, $(\tau_i, \tau_i)=1$.

(v) For $\tau_i \in \Omega_1$ if $(\tau_i, \tau_i)=1$ then $(\sigma_0, \tau_i)=(\tau_i, \sigma_0)$ and if $(\tau_i, \tau_i)=a^{2^{s-1}}$ then $(\sigma_0, \tau_i)^{-1}(\tau_i, \sigma_0)=a^{2^{s-1}}$.

(vi) For every element σ of order 4 we have either $c_i^\sigma=c_i a^{2^{s-1}}$, for some $i=1,\ldots,t$, or $(\sigma, \tau_i)^{-1}(\tau_i, \sigma)=a^{2^{s-1}}$ for every minimal element $\tau_i \in \Omega_1$, and hence $t\neq 0$ or $\Omega_1\neq 1$.

Now we prove

THEOREM 2. *Let G be a finite group with cyclic commutator subgroup G' and assume all its nonlinear irreducible representations are realizable in the real field. Then $G=\Lambda(A)$, where A and Λ are given below with the factor set satisfying the indicated conditions above.*

I. $A=A(s, m, 0, 0)$, $\Lambda=\Gamma\times\Omega$, $\Gamma=\langle\sigma_0\rangle$, (i)–(v).

II. $A=A(0, m, 0, 0)$, $\Lambda=\Gamma\times\Omega$, $\Gamma=\Gamma(h)$, *h even, and* $\Lambda(A)=A\circ\Lambda$.

III. $A=A(s, 1, 0, g)$, $s=1$ *or* 2, $\Lambda=\Gamma\times\Omega$, $\Gamma=\langle\sigma_0,\ldots,\sigma_g\rangle$, *satisfies* (i), $(\tau_i, \tau_i)\in\langle a^{2^{s-1}}, c_1,\ldots,c_\tau\rangle$, $\tau_i\in\Omega$, *and* $(\sigma, \tau)^{-1}(\tau, \sigma)\in\langle a^{2^{s-1}}\rangle$ *for all* $\sigma, \tau\in\Lambda$.

IV. $A=A(2, 1, 1, 0)$, $\Lambda=\Gamma\times\Omega$, $\Gamma=\langle\sigma_{-1}, \sigma_0\rangle$, (i)–(iii).

V. $A=A(1, m, 0, 0)$, $\Lambda=\Gamma\times\Omega$, $\Gamma=\langle\sigma_0\rangle$, (i), (ii), (v).

VI. $A=A(1, m, 0, 0)$, $\Lambda=\Gamma\times\Omega$, $\Gamma=\Gamma(4)=\langle\sigma'\rangle$, (i), (ii), (iv), (v) *and* (vi).

VII. $A=A(2, m, 0, 0)$, $\Lambda=\Gamma\times\Gamma(4)\times\Omega$, $\Gamma=\langle\sigma_0\rangle$, $\Gamma(4)=\langle\sigma'\rangle$, $(\sigma, \tau)=1$ *for all* $\sigma, \tau\in\Gamma(4)$ *and* (i)–(vi).

VIII. $A=A(2, m, 0, 0)$, $\Lambda=\Gamma\times\Gamma(4)\times\Omega$, $\Gamma=\langle\sigma_0\rangle$, $\Gamma(4)=\langle\sigma'\rangle$, (i), (iii)–(vi), $((\sigma')^2, (\sigma')^2)=a^2$, $(\sigma, \tau)\in\langle a^2\rangle$ *for all* $\sigma, \tau\in\Gamma\times\Omega$ *or* $\sigma, \tau\in\Gamma(4)\times\Omega$ *and*

$$(\sigma_0, \sigma')^{-1}(\sigma', \sigma_0) = a.$$

Proof. First we consider the case where G' is cyclic of order a power of 2, i.e., if $a^{2^s}=1$ we let $G'=\langle a\rangle$ or $G'=\langle a^2\rangle$. Second we assume $G'=\langle d\rangle$, $d^m=1$, m odd. Third we assume $G'\subseteq\langle a, d\rangle$. If $G/A\cong\Lambda\subseteq\operatorname{Aut} A$ and $\sigma\in\Lambda$, then x_σ is a coset representative of A such that $x_\sigma^{-1}kx_\sigma=k^\sigma$ for all $k\in A$. We omit some of the calculations, in particular we omit some of those that prove $K(1)$ and Λ are of the types indicated.

Assume $s=1$ and $G'=\langle a\rangle$, then $K(1)=A(1, 1, 0, g)$ and $G/A\cong\Lambda=\Gamma\times\Omega$. If $\tau\in\Omega$ and x_τ^2 is of order 4, then $K(1)$ can be chosen containing x_τ and hence either $K(1)/G'$ is of exponent 8 or $G'\not\supseteq\langle a\rangle$, a contradiction. Therefore x_τ^2 is of order dividing 2 and $x_\tau^2=a^e b_1^{2e_1}\cdots b_g^{2e_g}c_1^{f_1}\cdots c_t^{f_t}$. Setting $y_\tau=x_\tau b_1^{e_1}\cdots b_g^{e_g}$, we have $y_\tau^2=a^e c_1^{f_1}\cdots c_t^{f_t}$ and hence x_{τ_i} can be chosen such that III is satisfied for $s=1$. The same method gives III for $s=2$ and $G'=\langle a^2\rangle$.

Assume $s=2$ and $G'=\langle a\rangle$ and assume there exists $k\in K(1)$, $k^2\notin\langle a\rangle$ and $k^4=1$. From Lemma 1, there exists x such that $x^{-1}ax=a^{-1}$, $x^{-1}kx=k$ and $x^2=1$ (here $\bar P(k)=1$, for some $\bar P$). Considering $K(a^2)\supseteq K(1)$ we have y such that $y^{-1}ay=a^e$, $e=1$ or -1, $y^{-1}ky=ka^f$, $f=1$ or -1 and $y^2\in\langle a^2\rangle$, (here $\bar P(k)=(-1)^{1/2}$ for some $\bar P$). This implies that $e=-1$ and letting $z=xyx^{-1}y^{-1}$ we have $z^{-1}kz=ka^2\neq k$, a contradiction, since $z \in G'$. Hence $K(1)=A(2, 1, \delta, 0)$. Assume $\delta=0$ and G' is not a proper subgroup of a cyclic subgroup of G. Then $G/A \cong \Lambda = \Gamma \times \Omega$, $\Gamma = \langle \sigma_0 \rangle$. If $\tau \in \Omega$ and $x_\tau^2 \notin \langle a \rangle$ then $K(1)$ can be chosen such that $x_\tau \in K(1)$ or $x_\tau^4 = a^2$ which is the case when $\delta=1$. Otherwise for a basis $\{\tau_i\}$ of Ω, $x_{\tau_i}^2=1$ and hence $G'=\langle a^2 \rangle$, a contradiction. Hence $\delta=1$, $K(1)=A(2, 1, 1, 0)$ and $G/A \cong \Lambda = \Gamma \times \Omega$, $\Gamma = \langle \sigma_{-1}, \sigma_0 \rangle$. If $\tau \in \Omega$ and $x_\tau^4 \notin \langle a \rangle$ then either a different coset representative can be chosen such that $x_\tau^2=1$ or $K(1)$ can be chosen containing x_τ and hence $x_\tau^4=a^2$ or $(x_\tau b)^4=1$, the latter is a contradiction. Hence IV follows.

Assume $s=3$ and $G'=\langle a^2 \rangle$. From the above there exists no $k \in K(1)$, $k^2 \notin \langle a \rangle$, or $K(1)=A(3, 1, 0, 0)$. The result, I, for $s=3$ and $m=1$, is immediate when $a^\tau = a$ for all $\tau \in \Omega$. Let $\Omega = \langle \tau_1, \ldots, \tau_f \rangle$, $a^{\tau_1}=a^5$ and $a^{\tau_i}=a$, $i>1$. Then we may choose x_{τ_i}, $i>1$, such that $x_{\tau_i}^2=1$ and hence $K(a^4)=\langle A, x_{\tau_1}, \ldots, x_{\tau_f} \rangle$ and hence $G/K(a^4)$ is of order 2. But if $x_{\tau_1}\langle a^4 \rangle$ is of order 4 in $K(a^4)/\langle a^4 \rangle$, then from Theorem 1, $G/K(a^4)$ should be at least of order 4, a contradiction. Hence $x_{\tau_1}^2 \in \langle a^4 \rangle$ and x_{τ_1} can be chosen such that $x_{\tau_1}^2=1$ and I follows for $s=3$ and $m=1$.

Assume $s=3$, $G'=\langle a \rangle$, and G' is not a proper subgroup of a cyclic subgroup. If $K(1)=A(3, 1, 1, 0)$, then x exists such that $x^{-1}ax=a^{-1}$, $x^{-1}bx=ba^{-1}$, and $x^2=1$. (This follows from Lemma 1 after taking $P(a)=\zeta^2$, $\bar{P}(b)=\zeta$ and we should have $\bar{P}(x^{-1}bx)=\bar{P}(b^{-1})$, ζ is a primitive 2^4th root of unity.) Taking $K(a^4) \supseteq K(1)$ we have y such that $y^{-1}ay=a^e$, $e=-1$ or 3, $y^{-1}by=ba^f$, $f=1$ or 5, and $y^2 \in \langle a^4 \rangle$. (This follows after taking $P(a)=-1$, $\bar{P}(b)=(-1)^{1/2}$.) It follows that $e=-1$ and letting $z=xyx^{-1}y^{-1}$ we have $z^{-1}bz \neq b$, a contradiction. Hence $K(1)=A(3, 1, 0, 0)$ which implies $G'=\langle a^2 \rangle$, another contradiction. Hence for $s \geqq 3$ we may assume $G'=\langle a^2 \rangle$.

If $s=4$ and $G'=\langle a^2 \rangle$ then $K(1)=A(4, 1, 0, 0)$ and $G/A \cong \Lambda = \Gamma \times \Omega$. By a method similar to the one preceding the above paragraph, the existence of $\tau \in \Omega$ such that $a^\tau = a^5$ leads to a contradiction. Hence $a^\tau = a^9$ or $a^\tau = a$ for all $\tau \in \Omega$. The case when $s>4$ can be proved by a similar method. This completes the proof of I when $m=1$.

Assume $G'=\langle d \rangle$ of order m, m odd, then $K(1)=A(0, m, 0, 0)$ and $G/A \cong \Lambda$, then $c_i^\sigma = c_i$ for $i=1, \ldots, t$. Let $d^\sigma = d^r$, σ of order h. If h is odd and r is of order $h_1 \pmod p$, p some prime dividing m, and $h_1<h$, then $K(d^p)$ can be chosen such that h/h_1 divides the exponent of $K(d^p)/G'$, a contradiction. Furthermore if $4|h$ and $r^{h/2} \equiv 1 \pmod p$, $p|m$, then a $K(d^p)$ can be chosen such that $x_\omega^2 \in K(d^p)$, $\omega = \sigma^{h/4}$. Choosing $\bar{P}$ such that $\bar{P}(x_\omega^2)=-1$ we have $\langle K(d^p), x_\omega \rangle / \bar{D}$ quaternion, a contradiction. We have a similar contradiction when $\sigma^2=1$ and $x_\sigma^2 \in \langle c_1, \ldots, c_t \rangle - \langle 1 \rangle$. Hence II follows.

Now assume $G' \subseteq \langle a, d \rangle$ is of even order, $a^{2^s}=d^m=1$, m odd, then $K(1)=A(s, m, 0, 0)$. Considering $K(d)$ we have, from the above cases, G/G' of exponent 2 if $s>2$ and of exponent dividing 4 if $s \leqq 2$.

Assume G/G' is of exponent 2, $G/A \cong \Lambda = \Gamma \times \Omega$, where $\Gamma = \langle \sigma_0 \rangle$. We have $x_\tau^2 \in \langle a \rangle$ for all $\tau \in \Omega$. Assume $x_\tau^2 \in \langle a^2 \rangle$ for all $\tau \in \Omega$. Let $\{\tau_i\}$ be a basis of Ω_2 then x_{τ_i} can be chosen such that $x_{\tau_i}^2=1$. Similarly if $\{\tau_i\}$ is the set of minimal elements of Ω_1 we may let $x_{\tau_i}^2=1$. Now assume $x_{\sigma_0}^{-1}x_{\tau_i}x_{\sigma_0}=x_{\tau_i}a^{2^{s-1}}$, $\tau_i \in \Omega_1$. Let $d^{\tau_i}=d^r$ and

q be the product of all p^e, $p^e \| m$, and $r \equiv 1 \pmod p$, p a prime. Then $K(d^q)=\langle A, x_{\tau_i}\rangle$. Now $\{x_\omega \mid \omega \in \Lambda\}$ contains a set of coset representatives of $K(d^q)$ and there exists no ω such that $x_\omega^{-1}kx_\omega \in k^{-1}\langle d^q\rangle$ for all $k \in K(d^q)$, a contradiction. Hence $x_{\sigma_0}^{-1}x_{\tau_i}x_{\sigma_0}=x_{\tau_i}$ and I follows.

Now if $s=1$ and $x_{\tau_i}^2=a$, $\tau_i \in \Omega_1$, then it can be shown that $x_{\sigma_0}^{-1}x_{\tau_i}x_{\sigma_0}=x_{\tau_i}a$. Hence V follows.

Assume $s>1$ and for some $\tau \in \Omega$, $x_\tau^2=a$. If $d^\tau=d$, then this is the case when $A=A(s+1, m, 0, 0)$. Hence assume $d^\tau=d^r \neq d$, $r \equiv -1 \pmod p$ for some prime $p|m$. Then for $\omega \in \Omega$, $x_\omega^{-1}x_\tau x_\omega=x_\tau a^e$ and hence $x_\omega^{-1}x_\tau^2 x_\omega=x_\tau^2 a^{2e}=a$ or $a^{2^{s-1}+1}$, and therefore e is even. Assume $x_\omega^2=x_\tau^{2h}$, then setting $y_\omega=x_\omega x_\tau^f$, f a solution of $f(1+e)\equiv -h \pmod{2^s}$, we have $y_\omega^2=1$. Hence a basis $\{\tau, \tau_2, \ldots, \tau_f\}$ can be found such that $x_{\tau_i}^2=1$ for $i>1$. This means the derived group of $\langle A, x_\tau, x_{\tau_2}, \ldots, x_{\tau_f}\rangle$ is included in $\langle a^2, d\rangle$. Hence $x_\tau \notin K(a^2, d^p) \supseteq K(1)$ and $x_\tau^{-1}kx_\tau \in k^{-1}\langle a^2, d^p\rangle$ for all $k \in K(a^2, d^p)$. But $x_\tau^2=a$ and hence $\langle K(a^2, d^p), x_\tau\rangle/\bar{D}$ is quaternion, a contradiction. Hence for $s>1$, $x_\tau^2 \in \langle a^2\rangle$, for all $\tau \in \Omega$.

Now assume G/G' is of exponent 4 and let $s=1$. Then $K(1)=A(1, m, 0, 0)$ and $G/A \cong \Lambda=\Gamma \times \Omega$, $\Gamma=\Gamma(4)=\langle\sigma'\rangle$. Here $x_\tau^2 \in \langle a\rangle$, $\tau \in \Omega$, and $x_{\sigma'}^4=1$. Since $x_{\sigma'}^2$ commutes with every x_τ we have $x_{\tau_i}^2=1$, $\tau_i \in \Omega_1$. Now we show that $c_i^{\sigma'}=c_i a$ for some i, or $x_{\sigma'}^{-1}x_{\tau_i}x_\sigma=x_{\tau_i}a$, for every $\tau_i \in \Omega_1$. Assume $\Omega_1=1$ and $c_i^{\sigma'}=c_i$ for all i, then $K(d)=\langle A, x_{\sigma'}\rangle$ and there exists no $\tau \in \Omega=\Omega_2$, $\tau \neq 1$, such that $c_i^\tau=c_i$ for all i. This means $x_\tau^{-1}\bar{D}x_\tau \neq \bar{D}$ for the kernel $\bar{D}$ of any $\bar{P}$, a contradiction. Let $\Omega_1 \not\supseteq 1$ and τ_i be a minimal element of Ω_1 and $c_j^{\sigma'}=c_j$ for $j=1, \ldots, t$. Let q be the product of all p^e, $p^e \| m$ and $r \equiv 1 \pmod p$. (Here $d^{\tau_i}=d^r$.) Let $H=G/\langle d^q\rangle$ then $A(s, q, 0, 0) =\langle A(s, m, 0, 0), x_{\tau_i}\rangle/\langle d^q\rangle$. Since $c_j^{\sigma'}=c_j$ for $j=1, \ldots, t$, from the above we have $x_{\sigma'}^{-1}x_{\tau_i}x_{\sigma'}=x_{\tau_i}a$. (Here x_{τ_i} is like some c_j in $A(s, q, 0, 0)$.) Hence VI follows.

Assume $s=2$ then $G/A \cong \Lambda=\Gamma \times \Gamma(4) \times \Omega$, $\Gamma=\langle\sigma_0\rangle$ and $\Gamma(4)=\langle\sigma'\rangle$. Since $s>1$ we have $x_{\tau_i}^2=1$, $\tau_i \in \Omega_j$, $j=1$ or 2. Also $x_{\sigma'}^4=1$ or $x_{\sigma'}^4=a^2$ and the rest follows as above giving VII and VIII.

To complete the proof we should show that every nonlinear irreducible representation of the above groups is realizable in the real field. It is immediate when $A=A(s, 1, 0, 0)$, $A=A(2, 1, 1, 0)$, $A=A(s, 1, 0, \mathrm{g})$, $s=1$ or 2 and $A=A(0, m, 0, 0)$. We prove for I when $s>0$ and $m>1$, and the remaining cases are similar.

We have $x_{\tau_i}^2=1$, $\tau_i \in \Omega_j$, $j=1$ or 2. Let P be a linear representation of $\langle a, d\rangle$ with kernel $D=\langle a^{2^y}, d^z\rangle$. If $y \leqq s-1$, then $K(D)=\langle A(s, m, 0, 0), x_\tau \mid \tau \in \Omega, d^\tau \in d\langle d^z\rangle\rangle$. Now $x_{\sigma_0}^{-1}kx_{\sigma_0} \in k^{-1}D$ and hence $\langle K(D), x_{\sigma_0}\rangle/D$ satisfies the conditions of Theorem 1 and the result is immediate. Now assume $D=\langle d^z\rangle$. If $\Omega_1=1$ then $K(D) =A(s, m, 0, 0)$ and considering $\langle K(D), x_{\sigma_0}\rangle/D$ the result follows from Theorem 1. Assume $\Omega_1 \not\supseteq 1$. Again $K(D)=A(s, m, 0, 0)$ if there exists no minimal element τ_i, $d^{\tau_i}=d^r$, such that $r \equiv 1 \pmod z$ and the result follows as above. Assume such a minimal element τ_i exists, i.e., $r \equiv 1 \pmod z$. Let q be as defined above, then $z|q$ and $K(d^q)=\langle A, x_{\tau_i}\rangle$. Hence considering $K(d^q)/\langle d^q\rangle \cong A(s, q, 0, 0)$ and $H=G/\langle d^q\rangle$ it follows that Δ_1, Δ_1 defined for H as Ω_1 is defined for G, has less elements than Ω_1.

Assuming the inductive hypothesis on the number of elements of Ω_1 the result follows. This completes the proof of the theorem.

We conclude this paper with

COROLLARY. *Let G be a metacyclic group as defined in the remark of* §2. *Then every nonlinear irreducible representation of G is realizable in the real field if and only if* (a) *or* (b) *below holds.*

(a) $(n, 4)|2$, $t=m=2h$, *r of order* $2h$ (mod p) *for every odd prime p dividing m.*

(b) $(n, 4)=4$, $t=m=2$, *and* $r=-1$, *i.e., G is dihedral.*

The proof is immediate from cases I and II of Theorem 2 since $A=A(s, m, 0, 0) =\langle a\rangle$, a of order n, and Λ is cyclic, i.e., $\Omega=1$. Another proof can be given by using Lemma 1.

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