ON THE COMPLEX BORDISM OF EILENBERG-MAC LANE SPACES AND CONNECTIVE COVERINGS OF $BU$

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Abstract. Explicit computations show that the universal coefficient spectral sequence from complex bordism to integral homology collapses for the spectra $K(Z)$ and $bu$, and also for their mod $p$ reductions. Moreover the complex bordism modules of these spectra have infinite projective dimension.

1. Introduction. The aim of this paper is to continue the study of the complex bordism modules $\Omega_u^*(K(Z, n))$ and $\Omega_u^*(BU(2n, \ldots, \infty))$ which was begun in the earlier note [5]. Since our interest is in the stable ranges, it is convenient to introduce the Eilenberg-Mac Lane spectrum $K(Z) = \{K(Z, n)\}$ and the connective $BU$-spectrum (see [1] or [3, §10])

$$bu = \{\ldots, BU(2n, \ldots, \infty), (2n+1, \ldots, \infty), BU(2n+2, \ldots, \infty), \ldots\}.$$ 

Thus the objects of study are the bordism modules $\Omega_u^*(K(Z))$ and $\Omega_u^*(bu)$; in general if $M = \{M_n\}$ is a spectrum, we define

$$\Omega_u^*(M) = \lim_{n \to \infty} \Omega_u^*(M_n).$$

In [5] we determined the images of the Thom homomorphisms

$$\Omega_u^*(K(Z)) \overset{\mu}{\to} H_*(K(Z)), \quad \Omega_u^*(bu) \overset{\mu}{\to} H_*(bu).$$

We are now able to obtain a more complete understanding of the relation between the complex bordism and homology of $K(Z)$ and $bu$. In [3, §4] P. Conner and L. Smith introduce a natural spectral sequence for finite (CW) complexes

$$E^2_{p,q}(X) \Rightarrow H_*(X)$$

with

$$E^2_{p,q}(X) = \text{Tor}^\Omega_u^p(\Omega_u^q(X), Z).$$
where \( Z \) is made a module over \( \Omega_*^U \) by the augmentation. By taking limits there are also spectral sequences (1.1) for the spectra \( K(Z) \) and \( bu \), as well as for their mod \( p \) reductions \( K(Z) \wedge Z_p = K(Z_p) \) and \( bu \wedge Z_p \). We shall compute the \( E^2 \)-terms and prove the following:

**Theorem.** For each of the spectra \( K(Z) \), \( K(Z_p) \), \( bu \) and \( bu \wedge Z_p \) (\( p \) a prime) the spectral sequence (1.1) of [3, §4] collapses.

Thus in each case \( H_\ast(X) \) has a filtration by graded subgroups \( 0 \subset F_0 \subset F_1 \subset \cdots \), \( \bigcup F_n = H_\ast(X) \), such that \( F_p/F_{p-1} \) is isomorphic to \( \text{Tor}_{p, \ast}^\Omega_*^U(\Omega_*^U(X), Z) \). The edge homomorphism of (1.1) is the reduced Thom homomorphism

\[
\overline{\rho}: \Omega_*^U(X) \otimes \Omega_*^U Z \to H_\ast(X),
\]

hence \( \overline{\rho} \) is an isomorphism onto \( F_0 \subset H_\ast(X) \) in each case of the theorem.

The steps involved in the proof of the theorem are outlined in §2. The analysis for \( K(Z) \) and \( K(Z_p) \) is made in §3, and in §4 we carry out the study of \( bu \) and \( bu \wedge Z_p \).

From the computation of the \( E^2 \)-terms (see (3.3) and (4.6)) we conclude the following:

**Corollary.** The complex bordism of each spectrum \( K(Z) \), \( K(Z_p) \), \( bu \) and \( bu \wedge Z_p \) (\( p \) a prime) is an \( \Omega_*^U \)-module of infinite projective dimension.

In fact L. Smith has pointed out that from [3, §5] one can obtain the more precise result that \( \Omega_*^U(K(Z_p, n)) \) has projective dimension \( \geq n \).

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2. **Sketch of the argument.** The first step is to note that, by switching the roles of the spectra involved, one obtains as in [5] the following isomorphisms:

\[
\Omega_*^U(K(Z)) \cong H_\ast(MU),
\]
\[
\Omega_*^U(K(Z_p)) \cong H_\ast(MU; Z_p) = H_\ast(MU) \otimes Z_p,
\]
\[
\Omega_*^U(bu) \cong k_\ast(MU)
\]

and

\[
\Omega_*^U(bu \wedge Z_p) \cong k_\ast(MU; Z_p) = k_\ast(MU) \otimes Z_p.
\]

These are isomorphisms of \( \Omega_*^U \)-modules if the graded rings on the right are made \( \Omega_*^U \)-modules via the following diagram of Hurewicz homomorphisms and reductions mod \( p \).

\[
\begin{array}{ccc}
\Omega_*^U &=& \pi_*^U(MU) \\
H_\ast(MU) &\xrightarrow{p_*} & H_\ast(MU; Z_p) \\
\end{array}
\]

\[
\begin{array}{ccc}
k_\ast(MU) &\xrightarrow{\rho_\ast} & k_\ast(MU; Z_p) \\
\end{array}
\]

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Next we describe convenient choices of polynomial generators for $\Omega^U_*, H_*(MU)$ and $k_*(MU)$ following the detailed study of the complex bordism ring made by R. Stong in [8, Chapter 7] (see (3.1) and (4.3)). However it is rather difficult to deal directly with the $\Omega^U_*$-modules $H_*(MU)$ and $k_*(MU)$ for the computations we have in mind. Thus we first compute the bigraded groups (see (3.3) and (4.6))

$$\text{Tor}^U_*, (*) (H_*(MU; Z_p), Z), \quad \text{Tor}^U_*, (*) (k_*(MU; Z_p), Z)$$

and by a comparison of their totalizations with $H_*(K(Z_p))$ and $H_*(bu \wedge Z_p)$ we conclude that the spectral sequences (1.1) for $K(Z_p)$ and $bu \wedge Z_p$ must collapse.

Recall from [5, p. 526] that for each prime $p$ the homology groups $H_*(K(Z))$ and $H_*(bu)$ have no elements of order $p^2$, and that this property implies that a homology class vanishes if all its reduction mod $p$ vanish. In order to show that the spectral sequences (1.1) for $K(Z)$ and $bu$ collapse, it suffices to show that also

$$\text{Tor}^U_*, (*) (H_*(MU), Z), \quad \text{Tor}^U_*, (*) (k_*(MU), Z)$$

have no elements of order $p^2$. This we do by applying the elementary result:

**Lemma 2.1.** Let \( \ldots \rightarrow C_n \xrightarrow{\partial} C_{n-1} \rightarrow \ldots \) be a complex of free abelian groups, and let $\partial_p$ denote the Bockstein homomorphism

$$H(C; Z_p) \xrightarrow{\partial} H(C) \xrightarrow{p} H(C; Z_p)$$

where $\partial$ is the boundary associated to the coefficient sequence $0 \rightarrow Z \xrightarrow{p} Z \rightarrow Z_p \rightarrow 0$ and $p$ is reduction mod $p$. If $\partial_p$ has zero homology in dimension $n+1$ then $H_n(C)$ has no elements of order $p^2$.

**Proof.** Let $x \in H_n(C)$ with $p^2x = 0$ but $px \neq 0$; we shall reach a contradiction. Since $p(px) = 0$ we have $px = \partial y$ for a class $y \in H_{n+1}(C; Z_p)$. Then $\partial_p y = p\partial y = ppx = 0$, hence $y \in \text{Ker}(\partial_p)$. By assumption $y = \rho \partial z$ for some class $z \in H_{n+2}(C; Z_p)$. But then $px = \partial y = \partial \rho \partial z = 0$ since $\partial_p = 0$, violating the assumption that $px \neq 0$. Q.E.D.

We use this result as follows: Let

$$\ldots \rightarrow F_n \xrightarrow{d} F_{n-1} \rightarrow \ldots \rightarrow F_0 \xrightarrow{e} Z \rightarrow 0$$

be a free $\Omega^U_*$-resolution of $Z$, and let $M_*$ denote either of the $\Omega^U_*$-modules $H_*(MU)$, $k_*(MU)$ (both are free abelian). Then $\text{Tor}^U_*(M_*, Z)$ is the homology of the complex of free abelian groups

$$(2.2) \ldots \rightarrow M_* \otimes \Omega^U_* F_n \rightarrow M_* \otimes \Omega^U_* F_{n-1} \rightarrow \ldots \rightarrow M_* \otimes \Omega^U_* F_0 \rightarrow 0$$

and hence $\text{Tor}^U_*(M_* \otimes Z_p, Z)$ is the mod $p$ homology of the complex (2.2) (for clarity we indicate only the homological degree). We shall compute the Bockstein
homomorphism $\partial_p$ in $\text{Tor}^{\mathbb{Z}_p}_* (\mathcal{M}_*, \mathbb{Z})$ and show that its homology is zero in positive dimensions (see (3.6) and (4.8)), hence by the lemma $\text{Tor}^{\mathbb{Z}_p}_* (\mathcal{M}_*, \mathbb{Z})$ has no elements of order $p^2$.

Finally we exploit the commutative diagrams of differentials

$$
\begin{array}{c}
E_r(X) \xrightarrow{d^r} E_r(X) \\
\downarrow \rho_p \quad \downarrow \rho_p \\
E_r(X \wedge \mathbb{Z}_p) \xrightarrow{d^r} E_r(X \wedge \mathbb{Z}_p)
\end{array}
$$

to conclude, by an induction on $r \geq 2$, that the spectral sequences (1.1) for $K(Z)$ and $bu$ must collapse. That is, if we suppose that $d^2 = \cdots = d^{r-1} = 0$ for $X = K(Z)$ or $bu$ then $E_r(X) = E_2(X)$ and we find a commutative diagram

$$
\begin{array}{c}
E_r(X) \xrightarrow{d^r} E_r(X) \\
\downarrow \prod \rho_p \quad \downarrow \prod \rho_p \\
\prod_p E_r(X \wedge \mathbb{Z}_p) \xrightarrow{\prod d^r} \prod_p E_r(X \wedge \mathbb{Z}_p)
\end{array}
$$

in which $\prod \rho_p$ is a monomorphism and $\prod d^r = 0$, hence also $d^r = 0$ for the spectrum $X$.

3. $H^*(MU)$ and $H_*(MU; \mathbb{Z}_p)$ as $\Omega_*^U$-modules. We begin by recalling the relation of the complex bordism ring $\Omega_*^U$ to the ring $H_*(MU) \cong H_*(BU)$ (multiplication is provided by the Whitney sum). The Hurewicz homomorphism

$$
\Omega_*^U = \pi_*(MU) \xrightarrow{\mathcal{H}} H_*(MU)
$$

is a monomorphism ($\Omega_*^U$ has no torsion) which records the Chern numbers of stably complex manifolds. The following result of J. Cohen, taken from [8, p. 130], provides all the information we need.

**Proposition 3.1.** There exist polynomial generators $x_i (i \geq 1)$ of $\Omega_*^U$ and $z_i (i \geq 1)$ of $H_*(MU)$, $\dim x_i = \dim z_i = 2i$ such that $\mathcal{H} x_i = m_i z_i$ where $m_i = p$ if $i + 1 = p^k$ for some prime $p$ and $m_i = 1$ otherwise.

Recall that we regard $\mathbb{Z}$ as a module over $\Omega_*^U = \mathbb{Z}[x_1, x_2, \ldots]$ by means of the augmentation. The **Koszul resolution** (see [6, p. 204]) of the $\Omega_*^U$-module $\mathbb{Z}$ consists of the bigraded exterior algebra

$$
E_{*,*} = E_{0,0}[y_1, y_2, \ldots]
$$

where $y_i$ has bidegree $(1, 2i)$ and elements of $\Omega_*^U$ have bidegree $(0, 2i)$, and homomorphisms of $\Omega_*^U$-modules

$$
\cdots \xrightarrow{d} E_{n,*} \xrightarrow{e} E_{n-1,*} \cdots \xrightarrow{e} E_{0,*} \rightarrow \mathbb{Z} \rightarrow 0
$$
such that $d$ is a derivation satisfying $d(y_i) = x_i \cdot 1$ on the generators and $e$ is the augmentation $E_{0,*} = \Omega^U_* \to \mathbb{Z}$. This is a free resolution.

We now fix a prime $p$ and continue to denote by $z_i$ the images of the polynomial generators in $H_*(MU; \mathbb{Z}_p)$, hence $H_*(MU; \mathbb{Z}_p) = \mathbb{Z}_p[z_1, z_2, \ldots]$. Then the bigraded algebra

(3.2) $\text{Tor}^\mathbb{Z}_*(H_*(MU; \mathbb{Z}_p), \mathbb{Z})$

is the homology of

$H_*(MU; \mathbb{Z}_p) \otimes \mathbb{E}_p[y_1, y_2, \ldots]$

under a derivation $d'$ of $H_*(MU; \mathbb{Z}_p)$-modules which satisfies $d'(1 \otimes y_i) = \rho \rho^* x_i \otimes 1$ if $i+1 \neq p^s$ (note that $\rho \rho^* x_i$ is then a generator of $H_*(MU; \mathbb{Z}_p)$ in dimension $2i$) and $d'(1 \otimes y_i) = 0$ if $i+1 = p^s$ for some $s > 0$. Thus (3.2) is the homology of the tensor product of the complexes

$\mathbb{Z}_p[\rho \rho^* x_i; i+1 \neq p^s] \otimes \mathbb{E}_p[z_1, i > 0],$

$\mathbb{Z}_p[z_{p^s-1}; s > 0] \otimes \mathbb{E}_p[y_{p^s-1}; s > 0]$

under a derivation $d'$ which satisfies $d'(y_i) = \rho \rho^* x_i$ if $i+1 \neq p^s$ and which annihilates $z_{p^s-1}$ and $y_{p^s-1}$ for $s > 0$. This is simply the tensor product of a Koszul resolution for $\mathbb{Z}_p$ and a complex with zero differential, hence the Künneth formula yields

**Proposition 3.3.** There is an algebra isomorphism

$\text{Tor}^\mathbb{Z}_*(H_*(MU; \mathbb{Z}_p), \mathbb{Z}) \cong \mathbb{Z}_p[z_{i > 0}] \otimes \mathbb{E}_p[y_{i > 0}]$

where $y_i$ has bidegree $(1, 2i)$ and $z_i$ has bidegree $(0, 2i)$ for $i = p^s - 1$.

Thus we have computed the $E^2$-term of the spectral sequence

(3.4) $E^2(K(Z_p)) \Rightarrow H_*(K(Z_p))$.

By interchanging the roles of the Eilenberg-Mac Lane spectra $K(Z)$ and $K(Z_p)$ as in [5] we find that $H_*(K(Z_p)) \cong H_*(K(Z); \mathbb{Z}_p)$. It is well known that

$H_*(K(Z); \mathbb{Z}_p) \to H_*(K(Z_p); \mathbb{Z}_p)$

is a monomorphism whose image we now describe. Recall from [7] that $H_*(K(Z_p); \mathbb{Z}_p)$ is a Hopf algebra dual to the mod $p$ Steenrod algebra and that there is an algebra isomorphism

(3.5) $H_*(K(Z_p); \mathbb{Z}_p) = E_{\mathbb{Z}_p}[\eta_i; i \geq 0] \otimes \mathbb{Z}_p[\zeta_i; i > 0]$

where $\deg \eta_i = 2p^i - 1$ and $\deg \zeta_i = 2(p^i - 1)$. Then the image of $H_*(K(Z); \mathbb{Z}_p)$ is the subalgebra generated by the $\eta_i$ and $\zeta_i$ for $i > 0$, and so we conclude by a comparison of (3.3) and (3.5) that $H_*(K(Z_p))$ is (algebra) isomorphic to the totalization of $E^2_*(K(Z_p))$. Since both are $\mathbb{Z}_p$-vector spaces of finite type, the spectral sequence (3.4), i.e. the spectral sequence (1.1) for $K(Z_p)$, must collapse.
It remains to compute the Bockstein $\partial_p$ in $\text{Tor}_{\pi_*}^{\Omega_*^U}(H_*(MU; \mathbb{Z}_p), \mathbb{Z})$, as in §2, in order to show that also the spectral sequence (1.1) for $K(Z)$ collapses. We shall prove

**Proposition 3.6.** The Bockstein $\partial_p$ is a derivation of the algebra

$$\text{Tor}_{\pi_*}^{\Omega_*^U}(H_*(MU; \mathbb{Z}_p), \mathbb{Z})$$

which satisfies $\partial_p(y^*_pz_{s-1}) = z^*_p z_{s-1}$ and $\partial_p(z^*_p z_{s-1}) = 0$ on the generators.

It then follows that the homology of $\partial_p$ is isomorphic to $\mathbb{Z}_p$ (in bidegree $(0,0)$) since we have simply the Koszul resolution for $\mathbb{Z}_p$ over the polynomial algebra $\mathbb{Z}_p[z^*_p z_{s-1}; s > 0]$. We then argue as in §2 that the spectral sequence (1.1) for $K(Z)$ must also collapse.

**Proof of (3.6).** We omit the standard argument that $\partial_p$ is a derivation and concentrate on identifying $\partial_p(y_i)$, where the $y_i$ are the exterior algebra generators of (3.3). Thus we consider the projection

$$H_*(MU) \otimes E_\mathbb{Z}[y_1, y_2, \ldots] \to H_*(MU; \mathbb{Z}_p) \otimes E_\mathbb{Z}[y_1, y_2, \ldots]$$

of complexes, lift $1 \otimes y_i (i = p^s - 1)$ back to $1 \otimes y_i$ in $H_*(MU) \otimes E_\mathbb{Z}[y_1, y_2, \ldots]$, apply the differential to obtain $x_1 \otimes 1 = p z_1 \otimes 1$ (see (3.1)), divide by $p$ to obtain $z_1 \otimes 1$ and finally apply reduction mod $p$ which yields $z_1$ as desired. Q.E.D.

4. $k_*(MU)$ and $k_*(MU; \mathbb{Z}_p)$ as $\Omega_*^U$-modules. First recall that $k_*(\ )$ is the multiplicative homology theory represented by the connective $BU$-spectrum $bu$, and that the coefficient ring $k_* = \mathbb{Z}[\beta], \dim \beta = 2$ (for example see [3, §10]). For a spectrum $M = \{M_n\}$ we define

$$k_*(M) = \lim_{n \to \infty} k_* + n(M).$$

Maps which result from the Whitney sum make $k_*(MU)$ an algebra over $k_*$. Since $H_*(MU)$ is a polynomial algebra over $\mathbb{Z}$, hence has no torsion, it follows easily that

$$k_*(MU) = k_*(t_1, t_2, \ldots), \quad \dim t_i = 2i.$$

It is possible to choose the generators so that the Hurewicz homomorphism

$$(4.2) \quad \Omega_*^U = \pi_*(MU) \xrightarrow{\mathcal{H}} k_*(MU),$$

induced by the map of the sphere spectrum $S = \{S^n\}$ into $bu$ which consists of maps $S^{2n} \to BU(2n, \ldots, \infty)$ and $S^{2n+1} \to U(2n+1, \ldots, \infty)$ that lift generators of $\pi_{2n}(BU)$ and $\pi_{2n+1}(U)$, takes a convenient form. Namely, the generators can be chosen so that we have

$$\mathcal{H}[M^{2n}] = \sum [M^{2n}]_{t_1, \ldots, t_i, \beta^n t_i^{-1}, \ldots, t_i^{-1}}.$$
where the coefficient is the tangential $K$-theory characteristic number (see [2] or [8])

\[ [M^{2n}]_{\lambda_1, \ldots, \lambda_r} = s_{\lambda_1, \ldots, \lambda_r}(\gamma(\tau))[M^{2n}] \]

of the $U$-manifold $M^{2n}$ determined by the partition $(\lambda_1, \ldots, \lambda_r)$. (We choose tangential rather than normal $K$-theory numbers in order to agree with [8].)

It follows directly from the celebrated theorem of A. Hattori [4] and R. Stong [8, p. 129] that the Hurewicz homomorphism (4.2) is (additively) a split mono-
morphism. We shall deduce the following further information from [8, Chapter 7].

**Proposition 4.3.** There exist polynomial generators $x_i$ $(i \geq 1)$ of $\Omega^U_*$ and $z_i$ $(i \geq 1)$ of $k_*(MU)$ over $k_*$ that $\lambda$ is such that $H_{x_{p-1}} = \beta_{p-1}$, and on the coefficient rings $\lambda$ is the augmentation $k_* = Z[\beta] \to Z$ (so $\lambda(0) = 0$).

When $i+1 = p$ and $s > 1$ we have an equality $H_x = \beta_{p-1} w(\lambda)$ where $w(\lambda) \in k_{2p^s-p}(MU)$. Although this knowledge is sufficient for our purposes, it would be interesting to have a better grasp on the elements $w(\lambda)$. We conjecture that it is possible to choose the generators so that

\[ H_{x_{p^s-1}} = \beta_{p^s-1} + (H_{x_{p-1}}(z_{p^s-1}-1))^p \]

for $s > 1$. The best supporting evidence is the lemma on p. 121 of [8].

**Proof of (4.3).** We assume familiarity with the relevant portion of [8, Chapter 7].

If $i+1$ is not a prime power let $x_i = [M^{2i}]$ be any generator of $\Omega^U_*$ in dimension $2i$; then by [8, p. 128] we have $s_{0}(c(\tau))[M^{2i}] = \pm 1$ and therefore $H_x$ is a generator of $k_*(MU)$ over $k_*$ in dimension $2i$. We put $z_i = H_{x_i}$.

If $i+1 = p$ then put $x_i = [CP(p-1)]$. Since $s_{0}(c(\tau))[CP(p-1)] = p$ it follows from [8, p. 128] that $[CP(p-1)]$ is a generator of $\Omega^U_*$ in dimension $2(p-1)$. From [2, (14.1)] we see that $s_{0}(c(\tau))[CP(p-1)]$ is zero mod $p$ unless $\omega$ is the empty partition, and then as is well known we obtain the Todd genus of $CP(p-1)$ which is 1. The equation $H_{x_{p^s-1}} = \beta_{p^s-1}$ now defines the generator $z_{p^s}$.

Finally suppose $i+1 = p^{s+1}$ and $s > 0$. Let $x_i = [M^{2i}]$ be a generator of $\Omega^U_*$ in dimension $2i = 2(p^{s+1} - 1)$ which is congruent to a multiple of $[H_{p^s, \ldots, p^s}]$ mod $p$ (see [8, p. 121]). Then the key lemma on p. 121 of [8] implies that $H_x$ is divisible by $\beta_{p^s-1}$ mod $p$; for the mod $p$ $K$-theory characteristic numbers of $M^{2i}$ are a multiple of those of $H_{p^s, \ldots, p^s}$, and therefore $s_{0}(c(\tau))[M^{2i}] = 0$ mod $p$ if $i_1 + \cdots + i_r > p^{s+1} - p = i - (p-1)$. From [8, p. 128] we have $s_{0}(c(\tau))[M^{2i}] = \pm 1$, hence any solution of the congruence $H_x = \beta_{p^s-1}$ is a generator of $k_*(MU)$ in dimension $2i$. Q.E.D.

We are now ready to compute the bigraded algebra

\[ \text{Tor}^U_\ast_\ast (k_*(MU; Z_p), Z) \]
for a fixed prime $p$. We continue to denote by $z_i$ the reductions of the generators for $k_\ast(MU)$, so that $k_\ast(MU; \mathbb{Z}_p) = \mathbb{Z}_p[\beta; z_1, z_2, \ldots]$. Then, as in §3, (4.4) is the homology of

\[(4.5) \quad k_\ast(MU; \mathbb{Z}_p) \otimes_{\mathbb{Z}} E_{\beta}[y_1, y_2, \ldots]\]

under a derivation $d'$ of $k_\ast(MU; \mathbb{Z}_p)$-modules which satisfies $d'(1 \otimes y_1) = \rho H x_1 \otimes 1$ if $i+1 \neq p^i$ (note that $\rho H x_1$ is then a generator of $k_\ast(MU; \mathbb{Z}_p)$ in dimension $2i$) and

\[d'(1 \otimes y_{p^i-1}) = \rho H x_{p^i-1} \otimes 1 = \beta^{p-1} w^{(a)} \otimes 1 \quad \text{for some } w^{(a)} \in k_{2(a)p^i-p}(MU; \mathbb{Z}_p).
\]

In particular we have $d'(1 \otimes y_{p^i-1}) = \beta^{p-1} \otimes 1$.

For $s > 1$ we shall replace the exterior algebra generator $1 \otimes y_{p^s-1}$ by the cycle $y'_{p^s-1} = 1 \otimes y_{p^s-1} - w^{(a)} \otimes y_{p^s-1}$. One checks easily that these cycles generate an exterior algebra (each $y_i$ has odd total degree). Hence the complex (4.5) is the tensor product of the complexes

\[Z_p[\rho H x_i; i+1 \neq p^i] \otimes E_{\beta}[y_i; i+1 \neq p^i], \quad Z_p[\beta] \otimes E_{\beta}[y_{p^s-1}]
\]

and

\[Z_p[z_{p^s-1}; s > 0] \otimes E_{\beta}[y'_{p^s-1}; s > 1]
\]

under a derivation $d'$ which satisfies $d'(y_i) = \rho H x_i$ if $i+1 \neq p^i$, $d'(\beta) = 0$ and $d'(y_{p^s-1}) = \beta^{p-1}$, and which annihilates $z_{p^s-1}$ and $y'_{p^s-1}$. In view of the Koszul resolution and the Künneth formula we now find

**Proposition 4.6. There is an isomorphism**

\[\text{Tor}^q_{*,*}(k_\ast(MU; \mathbb{Z}_p), \mathbb{Z}) = B_\ast \otimes P_\ast \otimes E_\ast
\]

with the tensor product of the truncated polynomial ring $B_\ast = Z_p[\beta]/(\beta^{p-1})$, the polynomial ring $P_\ast = Z_p[z_{p^s-1}; s > 0]$ and the exterior algebra $E_\ast = E_{\beta}[y'_{p^s-1}; s > 1]$.

Notice that this result immediately implies that the spectral sequence

\[(4.7) \quad E^\infty <bu \wedge \mathbb{Z}_p> \Rightarrow H_\ast(bu \wedge \mathbb{Z}_p)
\]

collapses. For J. F. Adams showed in [1] that $H^\ast(bu; \mathbb{Z}_p)$ is isomorphic to a direct sum of cyclic modules $A_0/A_0 Q_0 + A_1 Q_1$ over the mod $p$ Steenrod algebra on generators in $H^{2i}(bu; \mathbb{Z}_p)$ for $i = 0, 1, \ldots, p-2$ (recall that $Q_0 \in A^0_1$ and $Q_1 \in A^0_{p-1}$). Hence $H_\ast(bu; \mathbb{Z}_p)$ and the totalization of $E^\infty <bu \wedge \mathbb{Z}_p> \cong \text{Tor}^q_{*,*}(k_\ast(MU; \mathbb{Z}_p), \mathbb{Z})$ are graded $\mathbb{Z}_p$-modules of finite type which have the same dimension in each degree, so they are isomorphic. Therefore the spectral sequence (4.7), i.e. the spectral sequence (1.1) for $bu \wedge \mathbb{Z}_p$, must collapse.

It only remains to compute the Bockstein $\partial_p$ in $\text{Tor}^q_{*,*}(k_\ast(MU; \mathbb{Z}_p), \mathbb{Z})$, as in §2, in order to show that also the spectral sequence (1.1) for $bu$ collapses. We shall prove, in the notation of (4.6),
Proposition 4.8. The Bockstein $\partial_p$ is a derivation of the algebra

$$\text{Tor}^{\mathbb{Q}}_{*,*}(k_*(MU; \mathbb{Z}_p), \mathbb{Z})$$

which satisfies $\partial_p(y_p^{*-1}) = z_p^{*-1}$ for $s > 1$, $\partial_p(\beta) = 0$ and $\partial_p(z_p^{*-1}) = 0$ for $s > 0$.

Proof. By a standard argument $\partial_p$ is a derivation sending $\text{Tor}_r$ to $\text{Tor}_{r-1}$, and so we concentrate on identifying $\partial_p(y_p^{*-1})$. Thus we consider the projection

$$k_*(MU) \otimes E_\mathbb{Z}[y_1, y_2, \ldots] \rightarrow k_*(MU; \mathbb{Z}_p) \otimes E_\mathbb{Z}[y_1, y_2, \ldots]$$

of complexes, lift the cycle $y_p^{*-1}$ back to $1 \otimes y_p^{*-1} - w^{(s)} \otimes y_p^{-1}$ (where $w^{(s)} \in k_{2(p-1)}(MU)$ satisfies $\mathbb{H}x_p^{*-1} = p z_p^{*-1} + \beta^{p-1} w^{(s)}$), apply the differential to obtain $(\mathbb{H}x_p^{*-1} - \beta^{p-1} w^{(s)}) \otimes 1 = p z_p^{*-1} \otimes 1$, divide by $p$ to obtain $z_p^{*-1} \otimes 1$, and finally apply reduction mod $p$ which yields $z_p^{*-1}$ as desired. Q.E.D.

We now obtain immediately

Corollary 4.9. The homology of the Bockstein $\partial_p$ in $\text{Tor}^{\mathbb{Q}}_{*,*}(k_*(MU; \mathbb{Z}_p), \mathbb{Z})$ is algebra isomorphic to $\mathbb{Z}_p[\beta, z_p^{-1}]/(\beta^{p-1})$, where $\beta$ has bidegree $(0, 2)$ and $z_p^{-1}$ has bidegree $(0, 2p - 2)$.

Thus the homology of $\partial_p$ is concentrated in bidegrees $(0, *)$, and then (2.1) implies that $\text{Tor}^{\mathbb{Q}}_{*,*}(k_*(MU), \mathbb{Z})$ has no elements of order $p^2$. We now may conclude as in §2 that the spectral sequence (1.1) for $bu$ must also collapse.

References