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META-ANALYTIC FUNCTIONS

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1. **Introduction.** The concept of analyticity for a complex-valued function defined on an open subset of the plane is usually introduced by making one of the following three equivalent definitions, namely:

A complex-valued function $f=u+iv$ is analytic in an open subset D ($\subset$ domain f) of the plane if

(i) the mapping $(x, y) \to (u, v)$ is Fréchet differentiable in D and the partial derivatives of u and v satisfy the Cauchy-Riemann equations there (for a definition of Fréchet differentiability see §2), *or*

(ii) f is differentiable in D, *or*

(iii) f is representable by a power series in a neighborhood of each point of D.

Therefore to extend the concept of analyticity to a broader class of functions we can start with any one of the above three definitions and generalize it in an appropriate way to the situation under study. As is well known (see e.g. [2]) the third definition has been chosen to define analyticity for complex-valued functions in several real variables. But this definition is not well suited to define analyticity for functions mapping a subset of a finite-dimensional vector space into another finite-dimensional vector space whose real dimension is greater than 2. Hence in such cases the alternative seems to be to define analyticity via the Cauchy-Riemann equations (C-R equations). In their paper *Fonctions holomorphes dans l'espace*, Moisil and Theodoresco [5] have shown that this approach is fruitful at least in some cases.

Moisil and Theodoresco considered functions from $\boldsymbol{R}^3$ into $\boldsymbol{R}^4$ whose components have continuous first partial derivatives. Assuming that these partial derivatives satisfy a system of equations, which can be regarded as generalized C-R equations, they showed that these functions exhibit some properties analogous to those of an ordinary analytic function.

It should be noted that in [5] they did not generalize definition (i) but rather they generalized the definition:

A complex-valued function is analytic in an open subset of D of the plane if its real and imaginary parts admit continuous first partial derivatives in D and they satisfy the C-R equations in D.

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(1) These results have appeared in the author's Ph.D. thesis submitted to the University of Rochester, Rochester, New York, under the direction of Professor W. F. Eberlein.

In his paper *Cauchy-Riemann operator*, Eberlein [4] has introduced a certain matrix operator ∇ (the Cauchy-Riemann operator), and has shown that the null space of ∇ consists of pairs of smooth functions (u, v) which satisfy the C-R equations. Since the partial derivatives commute on smooth functions,

$$\nabla^2 = \nabla\nabla = \Delta I,$$

where Δ is the 2-dimensional Laplace operator and I is the 2×2 identity matrix.

In §3 we will exploit the above connection between ∇ and the C-R equations to generalize definition (i) to define a concept of analyticity for a class of vector-valued functions defined on an open subset of a 3-dimensional Euclidean space. Specifically, we will consider functions ψ defined on an open subset (of the Spin Model) of Euclidean 3-space and mapping into a 2-dimensional unitary space. We assume that ψ is Fréchet differentiable and that it is in the null space of a matrix operator which will also be denoted by ∇. Again it turns out that $\nabla^2 = \Delta I$, where, now, Δ is the 3-dimensional Laplace operator and I is as before. Under this hypothesis we will show that ψ exhibits some properties analogous to those of an ordinary analytic function.

It should be emphasized that our hypothesis, unlike in [5], does not require the continuity of the partial derivatives of the components of ψ. Our results in §3 yield the corresponding results in [5].

From a look at some of our results in §3, say e.g. Cauchy's theorem, it may not be entirely obvious that these are indeed generalizations of the results about ordinary analytic functions the way they usually appear in books on complex analysis. In a paper to be published, the author was able to rewrite the statements of Cauchy's theorem and Cauchy integral formula for ordinary analytic functions in an equivalent form as results concerning functions mapping an open subset of a Euclidean 2-space into another 2-dimensional Euclidean space. From these results one immediately sees that our results in §3 are indeed generalizations of the corresponding results about ordinary analytic functions.

2. In this section we will introduce the space $\mathfrak{E}_3$, "The Spin Model of Euclidean 3-Space", state some of its properties, set up notation and give some definitions. (For a detailed study of the space $\mathfrak{E}_3$ see [3].)

Throughout H_2 denotes a 2-dimensional unitary space and we identify H_2 with the set of 2-rowed complex column matrices. The inner product in H_2 is the one that induces the Euclidean norm, namely

$$\|h\|^2 = |h_1|^2 + |h_2|^2 \quad \text{where} \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in H_2.$$

Let

$$\mathfrak{E}_3 = \{A \in \operatorname{End}(H_2) : A \text{ self-adjoint and trace } A = 0\}.$$

The Pauli matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis over $\boldsymbol{R}$ for $\mathfrak{E}_3$ and so $\mathfrak{E}_3$ is a 3-dimensional real vector space.

It can be shown that if $A, B \in \mathfrak{E}_3$ and I is the 2×2 identity matrix, then $AB+BA=kI$, where $k \in \boldsymbol{R}$.

We define the scalar product of A and B, denoted $(A\cdot B)$, by $AB+BA=2(A\cdot B)I$. We define a norm via this scalar product and thus $\mathfrak{E}_3$ becomes a metric space. Also we see that relative to this scalar product the Pauli matrices form an orthonormal (o.n.) basis for $\mathfrak{E}_3$.

DEFINITION. Let $\zeta_1, \zeta_2, \zeta_3$ be an o.n. basis for $\mathfrak{E}_3$. Then

$$\nabla = \zeta_1\partial_1+\zeta_2\partial_2+\zeta_3\partial_3,$$

where ∂_j denotes the directional derivative in the direction ζ_j.

It is clear from the definition that ∇ can act on functions ψ mapping a subset D of $\mathfrak{E}_3$ into (i) $\boldsymbol{C}$ or (ii) H_2 or (iii) $B(H_2)$—the algebra of linear transformations of H_2 into itself. Thus

$$\nabla\psi = \zeta_1\,\partial_1\psi+\zeta_2\,\partial_2\psi+\zeta_3\,\partial_3\psi$$

maps D, respectively, into (i) the complexification $\mathfrak{E}_3$, (ii) H_2, (iii) $B(H_2)$, respectively.

It can be shown (see [3]) that the definition of ∇ given above is independent of the choice of the o.n. basis in $\mathfrak{E}_3$.

If we choose the Pauli matrices e_1, e_2, e_3 as a basis for $\mathfrak{E}_3$ and if we write

$$\partial/\partial x_j = \partial_{e_j} \qquad (j = 1, 2, 3)$$

we have a matrix representation for ∇, namely

$$\nabla = \begin{pmatrix} \partial/\partial x_3 & \partial/\partial x_1 - i\,\partial/\partial x_2 \\ \partial/\partial x_1 + i\,\partial/\partial x_2 & -\partial/\partial x_3 \end{pmatrix}.$$

On sufficiently smooth functions $\partial/\partial x_j$ and $\partial/\partial x_k$ commute and so on sufficiently smooth functions

$$\nabla^2 = \nabla\nabla = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},$$

where Δ is the 3-dimensional Laplace operator.

DEFINITION. Let E and F be two Banach spaces over the same field and let f be a map from a nonempty open subset D of E into F. f is said to be *Fréchet*

differentiable (FD) in D if for each $p \in D$ there exists a continuous linear transformation of E into F, denoted $f'(p)$, such that

$$f(p+x) = f(p)+f'(p)x+\|x\|\theta(x,p) \qquad (p+x \in D)$$

where $\lim_{x\to 0} \theta(x,p)=0$ (2).

f is said to be *FD at* $p \in D$ if f is FD in some open neighborhood of p.

DEFINITION. A nonempty connected open subset of $\mathfrak{E}_3$ will be called a *region* in $\mathfrak{E}_3$. The closure of a region will be called a closed region.

DIVERGENCE THEOREM. *Let R be a suitably well-behaved region in $\mathfrak{E}_3$. Let the boundary ∂R of R be a surface. Let R_1 be a region in $\mathfrak{E}_3$ containing $\bar{R}$. Let F: $R_1 \to \mathfrak{E}_3 + i\mathfrak{E}_3$ with $F(x)=F_1(x)+iF_2(x)$ where F_1 and F_2 are vector fields of class C^1 on R_1. If n is the outward unit normal vector to ∂R then*

$$\int_{\partial R} F\cdot n\, ds = \int_R \operatorname{div} F\, dv,$$

where

$$F\cdot n = F_1\cdot n+iF_2\cdot n, \qquad \operatorname{div} F = \operatorname{div} F_1+i \operatorname{div} F_2,$$

and ds and dv are the surface and volume elements resp.

The proof of the theorem follows by the application of the usual divergence theorem for real vector fields to F_1 and F_2.

In what follows we will be concerned only with functions ψ mapping a nonempty open subset of $\mathfrak{E}_3$ into H_2. Hence we adopt the following

Convention: ψ always denotes a function mapping a nonempty open subset D of $\mathfrak{E}_3$ into H_2. Specifically, ψ: $D\,(\subset \mathfrak{E}_3) \to H_2$ with

$$\psi(x) = \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} \qquad (x \in D)$$

where U, V: $D \to \boldsymbol{C}$.

DEFINITION. Let ψ be continuous and let $S \subset D$. If m is a measure on S we define

$$\int_S \psi(x)\, dm(x) = \begin{pmatrix} \int_S U(x)\, dm(x) \\ \int_S V(x)\, dm(x) \end{pmatrix}.$$

We claim that

$$\left\| \int_S \psi(x)\, dm(x) \right\| \leqq \int_S \|\psi(x)\|\, dm(x). \tag{2.1}$$

(2) Here 0 in $\lim_{x\to 0}$ denotes the null vector in E and 0 on the right-hand side of the equation denotes the null vector in F. We will use the same symbol 0 for the null vectors in the various vector spaces we come across. The exact interpretation will be clear from the context.

Let $w=\int_S \psi(x)\,dm(x)$. Since $w \in H_2$ and since H_2 is a Hilbert space there exists a unique linear functional, say T, on H_2 such that $T(h)=\langle h, w\rangle$ $(h \in H_2)$ where $\langle\ ,\ \rangle$ denotes the inner product in H_2. It is easily seen that the linear functional defined by $T(h)=\int_S \langle h, \psi(x)\rangle\,dm(x)$ has the above property. Hence

$$\|w\|^2 = \langle w, w\rangle = T(w) = \int_S \langle w, \psi(x)\rangle\,dm(x) \leqq \|w\| \int_S \|\psi(x)\|\,dm(x)$$

and this proves our claim.

DEFINITION. By a *box* in $\mathfrak{E}_3$ we always mean a rectangular parallelepiped. By definition a box is always a closed and bounded set.

If B is a box in $\mathfrak{E}_3$ then ∂B denotes the boundary of B—i.e., ∂B is the union of the six faces of B.

DEFINITION. Let $n\colon \mathfrak{E}_3 \to \mathfrak{E}_3$ with $n(x)=\sum_{j=1}^{3} n_j(x)e_j$. Then $n\psi\colon D \to H_2$ with $(n\psi)(x)=n(x)\psi(x)$ $(x \in D)$ where the multiplication on the right side is the ordinary matrix multiplication—i.e.,

$$\begin{aligned}(n\psi)(x) &= \begin{pmatrix} n_3(x) & n_1(x)-in_2(x) \\ n_1(x)+in_2(x) & -n_3(x)\end{pmatrix}\begin{pmatrix} U(x) \\ V(x)\end{pmatrix} \\ &= \begin{pmatrix} n_3(x)U(x)+(n_1(x)-in_2(x))V(x) \\ (n_1(x)+in_2(x))U(x)-n_3(x)V(x)\end{pmatrix}.\end{aligned}$$

Separating $U(x)$ and $V(x)$ into their real and imaginary parts and using the definition of the norm on H_2 we can show that

$$\|(n\psi)(x)\| = \|n(x)\|\,\|\psi(x)\| \qquad (x \in D). \tag{2.2}$$

In what follows we will be using (2.1) and (2.2) freely and will never explicitly refer to them.

3. We will introduce the concept of a meta-analytic function in this section and will show that a meta-analytic function has some properties analogous to those of an ordinary analytic function in the plane. We will begin with a

DEFINITION. Let $G \subset D$ be a nonempty open subset of $\mathfrak{E}_3$. ψ is *meta-analytic* in G if

(i) ψ is FD in G,

(ii) $(\nabla\psi)(x)=0$ for each $x \in G$.

ψ is meta-analytic at $p \in D$ if ψ is meta-analytic in some open neighborhood of p.

LEMMA 1. *Let $B \subset D$ be a box in $\mathfrak{E}_3$. Let ψ be a constant function and let n denote the outward unit normal vector to the boundary ∂B of B. Then*

$$\int_{\partial B} (n\psi)(x)\,ds(x) = 0,$$

where ds denotes the surface element.

Proof. If Int B denotes the interior of B, it follows from the Divergence Theorem, since ψ is a constant function, that

$$\int_{\partial B} (n\psi)(x)\, ds(x) = \int_{\operatorname{Int} B} 0\, dv = 0. \quad \text{(Q.E.D.)}$$

LEMMA 2. *Let ψ be FD at $p \in D$ and let $(\nabla\psi)(p)=0$. Let B and n be the same as in Lemma* 1. *Then*

$$\int_{\partial B} (n\psi'(p))(x)\, ds(x) = 0.$$

Proof. Choose the Pauli matrices e_1, e_2, e_3 as a basis for $\mathfrak{E}_3$. Let $x=(x_1, x_2, x_3)$ denote an arbitrary point in $\mathfrak{E}_3$. For $j=1, 2, 3$ let

$$\alpha_j = \left.\frac{\partial U}{\partial x_j}\right|_p \equiv (\partial_{e_j} U)(p) \quad \text{and} \quad \beta_j = \left.\frac{\partial V}{\partial x_j}\right|_p \equiv (\partial_{e_j} V)(p).$$

Then

$$(\psi'(p))(x) = \begin{pmatrix} \sum_{j=1}^{3} \alpha_j x_j \\ \sum_{k=1}^{3} \beta_k x_k \end{pmatrix}.$$

Hence

$$\int_{\partial B} (n\psi'(p))(x)\, ds = \int_{\partial B} \begin{pmatrix} n_3 \sum_{j=1}^{3} \alpha_j x_j + (n_1 - in_2) \sum_{k=1}^{3} \beta_k x_k \\ (n_1 + in_2) \sum_{j=1}^{3} x_j - n_3 \sum_{k=1}^{3} \beta_k x_k \end{pmatrix} ds.$$

Now, by the Divergence Theorem, the integral on the right side is equal to

$$\int_{\operatorname{Int} B} \begin{pmatrix} \alpha_3 + \beta_1 - i\beta_2 \\ \alpha_1 + i\alpha_2 - \beta_3 \end{pmatrix} dv = \int_{\operatorname{Int} B} (\nabla\psi)(p)\, dv.$$

But $(\nabla\psi)(p)=0$ by hypothesis and hence the conclusion of the lemma. (Q.E.D.)

THEOREM 1 (CAUCHY'S THEOREM FOR A BOX). *Let B be a box in $\mathfrak{E}_3$ and let ψ be meta-analytic in a region containing B. Then*

$$\int_{\partial B} n\psi\, ds = 0,$$

where, as usual, n denotes the outward unit normal vector to ∂B.

Proof. Let $I(B)=\int_{\partial B} n\psi\, ds$. Divide B into eight congruent boxes, say $B^{(j)}$, $j=1, 2, \ldots, 8$. For $1 \leqq j \leqq 8$ let

$$I(B^{(j)}) = \int_{\partial B^{(j)}} n\psi\, ds.$$

Since the integrals on the common faces cancel out we get that

$$I(B) = \sum_{j=1}^{8} I(B^{(j)}).$$

Hence there exists at least one k $(1 \leqq k \leqq 8)$ such that $8\|I(B^{(k)})\| \geqq \|I(B)\|$. Denote this $B^{(k)}$ by B_1. Then $B_1 \subset B_0 \equiv B$. Repeat the above process with B_1 in place of B_0 and obtain a $B_2 \subset B_1$ such that $8\|I(B_2)\| \geqq \|I(B_1)\|$. Thus by induction we obtain a sequence of nested boxes $\{B_n : n=0, 1, 2, \ldots\}$ such that for each positive integer n we have that

$$8^n\|I(B_n)\| \geqq \|I(B)\|.$$

Let $\bigcap_{n=0}^{\infty} B_n = \{p\}$. Let $\varepsilon > 0$ be any preassigned number. Since ψ is FD at p we have

$$\psi(x) = \psi(p) + \psi'(p)(x-p) + \|x-p\|\theta(x, p), \tag{3.1}$$

where $\theta(x, p) \to 0$ as $x \to p$.

Hence there exists a $\delta > 0$ such that $\|x-p\| < \delta \Rightarrow \|\theta(x, p)\| < \varepsilon$. Since the B_n's are nested there exists an integer $N > 0$ such that for $n \geqq N$ we have

$$B_n \subset \{x : \|x-p\| < \delta\}.$$

Choose $m \geqq N$. Then by (3.1) we have

$$\begin{aligned} I(B_m) &= \int_{\partial B_m} n\psi \, ds \\ &= \int_{\partial B_m} \{n(x)\psi(p) + n(x)\psi'(p)(x-p) + n(x)\|x-p\|\theta(x, p)\} \, ds. \end{aligned}$$

On writing the right side as the sum of four integrals we see that the first and the third integrals are each equal to 0 by Lemma 1 and the second integral, namely $\int_{\partial B_m} n(x)(\psi'(p))(x) \, ds(x)$ is equal to 0 by Lemma 2. Hence

$$\|I(B_m)\| \leqq \int_{\partial B_m} \|x-p\|\theta(x, p) \, ds.$$

Let d_m (resp. d) and S_m (resp. S) denote the diameter and surface area respectively of B_m (resp. B). Then

$$d = 2^m d_m \qquad \text{and} \qquad S = 4^m S_m.$$

Hence

$$\|I(B)\| \leqq 8^m\|I(B_m)\| \leqq 8^m d_m S_m \varepsilon = dS\varepsilon.$$

Since ε was arbitrary we conclude that $\|I(B)\| = 0$ and so $I(B) = 0$. (Q.E.D.)

DEFINITION. Let $p \in \mathfrak{E}_3$. Then we define the map $g(\cdot, p)$ as follows:

$$g(\cdot, p) \colon \mathfrak{E}_3 - \{p\} \to \mathfrak{E}_3$$

defined by $g(x, p) = \text{grad}\,(-1/r)$ where $r = \|x-p\|$ and grad denotes the gradient.

If, relative to a basis, $p = (p_1, p_2, p_3)$ and $x = (x_1, x_2, x_3)$ then

$$g(x, p) = \left(\frac{x_1-p_1}{r^3}, \frac{x_2-p_2}{r^3}, \frac{x_3-p_3}{r^3}\right).$$

LEMMA 3. *Let the components U, V of a function ψ have continuous first partial derivatives. Let R be a suitably well-behaved region whose boundary ∂R is a surface. Also let $\bar{R} \subset D$. Then, for $p \notin \bar{R}$,*

$$\int_{\partial R} g(x, p)(n\psi)(x)\, ds = \int_R g(x, p)(\nabla\psi)(x)\, dv.$$

Proof. Choose the Pauli matrices e_1, e_2, e_3 as a basis for $\mathfrak{E}_3$ and let x $(=\sum_{j=1}^3 x_j e_j)$ denote an arbitrary point of D. Let $p = \sum_{j=1}^3 p_j e_j$. On performing the matrix multiplications we will see that

$$g(x, p)n(x)\psi(x) = r^{-3}\begin{pmatrix} n_1\{(x_1-p_1)U - i(x_2-p_2)U + (x_3-p_3)V\} \\ \quad + n_2\{i(x_1-p_1)U + (x_2-p_2)U - i(x_3-p_3)V\} \\ \quad + n_3\{(x_3-p_3)U - (x_1-p_1)V + i(x_2-p_2)V\} \\ n_1\{(x_1-p_1)V + i(x_2-p_2)V - (x_3-p_3)U\} \\ \quad + n_2\{-i(x_1-p_1)V + (x_2-p_2)V - i(x_3-p_3)U\} \\ \quad + n_3\{(x_1-p_1)U + i(x_2-p_2)U + (x_3-p_3)V\} \end{pmatrix}.$$

Now if we write the right side as

$$\begin{pmatrix} n\cdot F_1 \\ n\cdot F_2 \end{pmatrix}$$

then an easy but long computation will show that

$$\text{div}\, F_1 = r^{-3}[\{(x_1-p_1) - i(x_2-p_2)\}(U_1 + iU_2 - V_3) - (x_3-p_3)(V_1 - iV_2 + U_3)]$$

and

$$\text{div}\, F_2 = r^{-3}[\{(x_1-p_1) + i(x_2-p_2)\}(V_1 - iV_2 + U_3) - (x_3-p_3)(U_1 + iU_2 - V_3)]$$

where, for $j = 1, 2, 3$,

$$U_j = \partial U/\partial x_j \equiv \partial_{e_j} U \quad \text{and} \quad V_j = \partial V/\partial x_j \equiv \partial_{e_j} V.$$

Now an application of the Divergence Theorem will give that

$$\int_{\partial R} g(x, p)(n\psi)(x)\, ds = \int_R \begin{pmatrix} \text{div}\, F_1 \\ \text{div}\, F_2 \end{pmatrix} dv = \int_R g(x, p)\begin{pmatrix} V_1 - iV_2 + U_3 \\ U_1 + iU_2 - V_3 \end{pmatrix} dv$$

$$= \int_R g(x, p)(\nabla\psi)(x)\, dv. \qquad \text{(Q.E.D.)}$$

LEMMA 4. *Let B be a box in $\mathfrak{E}_3$ and let $h=\binom{h_1}{h_2}$ be an element of H_2. Then*

$$\int_{\partial B} g(x,p)n(x)h\,ds = 0 \qquad \text{if } p \in \operatorname{ext} B \text{ (exterior of } B),$$
$$= 4\pi h \quad \text{if } p \in \operatorname{Int} B \text{ (interior of } B)$$

where, as usual, n denotes the outward unit normal vector to ∂B.

Proof. Regarding h as a constant map from $\mathfrak{E}_3$ to H_2 we can write

$$\int_{\partial B} g(x,p)n(x)h\,ds = \int_{\partial B} g(x,p)(nh)(x)\,ds.$$

Now let $p \in \operatorname{ext} B$.

Then by Lemma 3 we get that

$$\int_{\partial B} g(x,p)(nh)(x)\,ds = \int_{\operatorname{Int} B} g(x,p)(\nabla h)(x)\,dv = 0.$$

Now let $p \in \operatorname{Int} B$.

Describe an open ball S with center p and radius, say $\rho>0$, such that $\bar{S}\subset \operatorname{Int} B$. Now we can apply Lemma 3 taking the region enclosed between B and S as R and get

$$\int_{\partial B} g(x,p)n(x)h\,ds - \int_{\partial S} g(x,p)n(x)h\,ds = 0.$$

But on ∂S, $n(x)=(x_1/\rho, x_2/\rho, x_3|\rho)$.

$\therefore$ On ∂S, $g(x,p)n(x)=\rho^{-2}I$, where I is the 2×2 identity matrix. Hence

$$\int_{\partial B} g(x,p)n(x)h\,ds = \rho^{-2}\int_{\partial S} h\,ds = 4\pi h. \quad \text{(Q.E.D.)}$$

THEOREM 2 (CAUCHY INTEGRAL FORMULA). *Let ψ be meta-analytic in D. If $p \in D$ then*

$$\psi(p) = \frac{1}{4\pi}\int_{\partial B} g(x,p)(n\psi)(x)\,ds,$$

where B is a box in D with $p \in \operatorname{Int} B$ and n, as usual, is the outward unit normal vector to ∂B.

Proof. Let $\varphi(p)=\int_{\partial B} g(x,p)(n\psi)(x)\,ds$. Let $\varepsilon>0$ be given.

Since ψ is uniformly continuous on B we can find an open cube, say C, with the following properties:

(i) The center of C is p, (ii) $\bar{C}\subset \operatorname{Int} B$, (iii) $p' \in \bar{C} \Rightarrow \|\psi(p)-\psi(p')\| < \varepsilon$ and (iv) the edges of C are parallel to the edges of B.

Let ρ (>0) denote the length of a side of C. If P and Q denote any two points of $\mathfrak{E}_3$ we will denote the distance between them by PQ.

Let us label the point p by P. Clearly if $O \in B \setminus C$ then $PQ \geqq \rho/2$.

Since ψ is uniformly continuous on $B \setminus C$ we can find a δ such that $0<\delta<\rho$ and if $Q, Q' \in B \setminus C$ with $QQ'<\delta$ then $\|\psi(Q)-\psi(Q')\|<\varepsilon(\rho/2)^3$.

Divide $B \setminus C$ into a finite number of cubes such that the diameter of each cube is less than δ and adjacent cubes will have only faces in common. Let

$$\begin{aligned} I &= \int_{\partial C} g(x,p)(n\psi)(x)\,ds \\ &= \int_{\partial C} g(x,p)n(x)(\psi(x)-\psi(p))\,ds + \int_{\partial C} g(x,p)n(x)\psi(p)\,ds. \end{aligned}$$

By Lemma 4 the second integral on the right has the value $4\pi\psi(p)$. Hence

$$I-4\pi\psi(p) = \int_{\partial C} g(x,p)n(x)(\psi(x)-\psi(p))\,ds.$$

Since $\|g(x,p)\| = \|x-p\|^{-2}$ we can conclude that, for each $x \in \partial C$, $\|g(x,p)\| \leqq 4\rho^{-2}$. Hence

$$\|I-4\pi\psi(p)\| \leqq 4\rho^{-2}\varepsilon\cdot 6\rho^2 = 24\varepsilon. \tag{3.2}$$

Let B_m denote a cube in the subdivision of $B \setminus C$ and let Q denote the center of B_m. If M denotes any point on ∂B_m we have

$$(MP)^2 = (PQ)^2+(MQ)^2-2(PQ)(MQ)\cos\alpha$$

where α is the angle between MQ and PQ. Therefore

$$\begin{aligned} \left(\frac{MP}{PQ}\right)^2 &= 1+\frac{MQ}{PQ}\left(\frac{MQ}{PQ}-2\cos\alpha\right) \\ &= 1+k\frac{MQ}{PQ}, \quad \text{say.} \end{aligned}$$

Then $|k|<3$. Now $(MP/PQ)^3=(1+k(MQ/PQ))^{3/2}$ and since by the Mean-Value Theorem

$$\left(1+k\frac{MQ}{PQ}\right)^{3/2} = 1+\frac{3}{2}\left(1+k\theta\frac{MQ}{PQ}\right)^{1/2}k\frac{MQ}{PQ} \qquad (0<\theta<1)$$

we have that

$$\left(\frac{MP}{PQ}\right)^3 = 1+\frac{3}{2}\left(1+k\theta\frac{MQ}{PQ}\right)^{1/2}k\frac{MQ}{PQ}. \tag{3.3}$$

We will assume without loss of generality that P is $(0, 0, 0)$. Let M denote the point $x=(x_1, x_2, x_3)$ and let Q denote the point $y=(y_1, y_2, y_3)$. Then

$$\begin{aligned} g(x,p) &= (MP)^{-3}\begin{pmatrix} x_3 & x_1-ix_2 \\ x_1+ix_2 & -x_3 \end{pmatrix} \\ &= (MP)^{-3}x \\ &= g(y,p)-\left(\frac{MP}{PQ}\right)^3(MP)^{-3}y+(MP)^{-3}x. \end{aligned}$$

Then on using (3.3) and simplifying we get that

$$(3.4)\quad g(x,p) = g(y,p)+(MP)^{-3}(x-y)-\frac{3}{2}\left(1+k\theta\frac{MQ}{PQ}\right)^{1/2}k\frac{MQ}{PQ}y(MP)^{-3}.$$

Let $I_m=\int_{\partial B_m} g(x,p)(n\psi)(x)\,ds$.

$$\therefore\ I_m = \int_{\partial B_m} g(x,p)n(x)(\psi(x)-\psi(y))\,ds+\int_{\partial B_m} g(x,p)n(x)\psi(y)\,ds.$$

The second integral on the right side is 0 by Lemma 4. So, by (3.4) we have that

$$I_m = \int_{\partial B_m} g(y,p)n(x)(\psi(x)-\psi(y))\,ds+\int_{\partial B_m}(x-y)n(x)\frac{\psi(x)-\psi(y)}{(MP)^3}\,ds$$
$$-\frac{3}{2}\int_{\partial B_m}\left(1+k\theta\frac{MQ}{PQ}\right)^{1/2}k\frac{MQ}{PQ}yn(x)\frac{\psi(x)-\psi(y)}{(MP)^3}\,ds.$$

Denote the three integrals on the right side by J_1, J_2, J_3 respectively. Then $I_m=J_1+J_2-(3/2)J_3$. Now

$$\begin{aligned} J_1 &= \int_{\partial B_m} g(y,p)n(x)(\psi(x)-\psi(y))\,ds \\ &= g(y,p)\int_{\partial B_m} n(x)(\psi(x)-\psi(y))\,ds \\ &= g(y,p)0 = 0 \quad \text{(by Theorem 1).} \end{aligned}$$

Also

$$\begin{aligned} \|J_2\| &= \left\|\int_{\partial B_m}(x-y)n(x)\frac{\psi(x)-\psi(y)}{(MP)^3}\,ds\right\| \\ &\leqq \int_{\partial B_m} MQ\frac{\|\psi(x)-\psi(y)\|}{(MP)^3}\,ds. \end{aligned}$$

If l_m denotes the length of the side of B_m

$$\|J_2\| \leqq 3^{1/2}/2l_m(2/\rho)^3\varepsilon(\rho/2)^3 6l_m^2 = 3\cdot 3^{1/2}V_m\varepsilon$$

where $V_m=l_m^3$ is the volume of B_m. Also

$$\begin{aligned} \|J_3\| &= \left\|\int_{\partial B_m}\left(1+k\theta\frac{MQ}{PQ}\right)^{1/2}k\frac{MQ}{PQ}yn(x)\frac{\psi(x)-\psi(y)}{(MP)^3}\,ds\right\| \\ &\leqq \int_{\partial B_m}(1+|k|)^{1/2}|k|\frac{MQ}{PQ}\frac{PQ}{(MP)^3}\|\psi(x)-\psi(y)\|\,ds. \end{aligned}$$

Since $|k|<3$ we have

$$\|J_3\| \leqq 6\cdot 3^{1/2}/2l_m\cdot(2/\rho)^3\varepsilon(\rho/2)^3 6l_m^2 = 18\cdot 3^{1/2}V_m\varepsilon.$$

Hence

$$(3.5)\quad \begin{aligned} \|I_m\| &\leqq \|J_1\|+\|J_2\|+(3/2)\|J_3\| \\ &\leqq 0+3\cdot 3^{1/2}V_m\varepsilon+27\cdot 3^{1/2}V_m\varepsilon = 30\cdot 3^{1/2}V_m\varepsilon. \end{aligned}$$

By adding all the I_m's as B_m runs through all the cubes of subdivision we can conclude that

$$\sum_m I_m = \int_{\partial B} g(x,p)(n\psi)(x)\,ds - \int_{\partial C} g(x,p)(n\psi)(x)\,ds$$
$$= \varphi(p) - I.$$

Now

$$\|\varphi(p) - 4\pi\psi(p)\| \leqq \|\varphi(p) - I\| + \|I - 4\pi\psi(p)\|,$$

i.e.

$$\|\varphi(p) - 4\pi\psi(p)\| \leqq \sum_m \|I_m\| + \|I - 4\pi\psi(p)\|,$$

so by using (3.5) and (3.2) we conclude that

$$\|\varphi(p) - 4\pi\psi(p)\| \leqq \left(30\cdot 3^{1/2}\sum_m V_m + 24\right)\varepsilon \leqq (30\cdot 3^{1/2}V + 24)\varepsilon,$$

where V is the volume of the box B.

Since ε is arbitrary we conclude that $\|\varphi(p) - 4\pi\psi(p)\| = 0$, and from this it follows that

$$\psi(p) = \frac{1}{4\pi}\int_{\partial B} g(x,p)(n\psi)(x)\,ds. \quad \text{(Q.E.D.)}$$

THEOREM 3 (MORERA'S THEOREM). *Let ψ be continuous and let $\int_{\partial B} n\psi\,ds = 0$ for every box $B \subset D$, n being the outward unit normal to ∂B. Then ψ has continuous first partial derivatives in D and $(\nabla\psi)(x) = 0$ for each $x \in D$—i.e., ψ is meta-analytic in D.*

Proof. The proof of Theorem 3 is almost similar to the proof of Theorem 2. We easily see that, under the hypothesis of Theorem 3, (3.2) is still valid. So, on adopting the same notation as in the proof of Theorem 2, we see that

$$\|I - 4\pi\psi(p)\| \leqq 24\varepsilon. \tag{3.2}$$

Still imitating the proof of Theorem 2 we see that the estimates for J_2 and J_3 are still valid under the hypothesis of Theorem 3. The only place in the proof of Theorem 2 where we used the fact that ψ is meta-analytic was in showing that $J_1 = 0$. But

$$J_1 = g(y,p)\int_{\partial B_m} n(x)(\psi(x) - \psi(y))\,ds$$
$$= g(y,p)\left[\int_{\partial B_m} n(x)\psi(x)\,ds - \int_{\partial B_m} n(x)\psi(y)\,ds\right].$$

The first integral on the right side is equal to 0 by the hypothesis and the second is equal to 0 by Theorem 1 since $\psi(y)$ is constant on ∂B_m. Hence we can repeat the proof of Theorem 2 and conclude that

$$\psi(p) = \frac{1}{4\pi}\int_{\partial B} g(x,p)(n\psi)(x)\,ds, \tag{3.6}$$

where B is a box in D such that $p \in \operatorname{Int} B$. Since

$$g(x,p) = \frac{1}{\|x-p\|^3}\begin{pmatrix} x_3-p_3 & (x_1-p_1)-i(x_2-p_2) \\ (x_1-p_1)+i(x_2-p_2) & -(x_3-p_3) \end{pmatrix}$$

we can conclude that for any fixed $x \neq p$, $g(x,p)$ as a function of p has continuous first partial derivatives at p. Hence from (3.6) we can conclude that ψ has continuous partial derivatives at p. Since p is an arbitrary point in D we can conclude that ψ has continuous first partial derivatives in D.

Next we have to show that $(\nabla\psi)(x)=0$ for all $x \in D$. Suppose that the desired conclusion is false. Then $\exists\, y \in D$ such that $(\nabla\psi)(y) \neq 0$. Then using the continuity of the partial derivatives of ψ we can find a box $B \subset D$ such that $y \in \operatorname{Int} B$ and $\int_{\operatorname{Int} B} \nabla\psi\, dv \neq 0$. But by the Divergence Theorem

$$\int_{\partial B} n\psi\, ds = \int_{\operatorname{Int} B} \nabla\psi\, dv \neq 0$$

which contradicts the hypothesis that $\int_{\partial B} n\psi\, ds = 0$ for every box B in D. So $(\nabla\psi)(x)=0$ for all $x \in D$ and the proof of Theorem 3 is complete.

COROLLARY 1. *Since, for any fixed $x \neq p$, $g(x,p)$ as a function of p has partial derivatives of all orders at p we can conclude from* (3.6) *that ψ has partial derivatives of all orders at p and so ψ has derivatives of all orders at p. Since p is an arbitrary point of D, ψ has derivatives of all orders in D.*

COROLLARY 2 (TO THEOREMS 2 AND 3). *Let ψ be meta-analytic in D and let $p \in D$. Let S be a closed ball with center p and radius r (>0) such that $S \subset D$. Then*

$$\psi(p) = \frac{1}{4\pi r^2}\int_{\partial S} \psi(x)\, ds.$$

Proof. Let B be a box such that $p \in \operatorname{Int} B$ and $B \subset \operatorname{Int} S$. Let R denote the region enclosed between ∂B and ∂S.

By Theorem 3, ψ has continuous first partial derivatives and so it follows from Lemma 3 that

$$\int_{\partial S} g(x,p)(n\psi)(x)\, ds - \int_{\partial B} g(x,p)(n\psi)(x)\, ds = \int_R g(x,p)(\nabla\psi)(x)\, dv.$$

But $(\nabla\psi)(x)=0$ for all $x \in R$ by the hypothesis. Hence

$$\int_{\partial S} g(x,p)(n\psi)(x)\, ds = \int_{\partial B} g(x,p)(n\psi)(x)\, ds = 4\pi\psi(p) \quad \text{(by Theorem 2).}$$

But on ∂S, $g(x,p)n(x)=r^{-2}$. Hence

$$\psi(p) = \frac{1}{4\pi r^2}\int_{\partial S} \psi(x)\, ds. \quad \text{(Q.E.D.)}$$

4. This section is devoted to some applications of the results of the previous section.

We begin with the following:

THEOREM 4 (WEIERSTRASS). *Let $\{\psi_k : k=1, 2, 3, \ldots\}$ be a sequence of functions mapping an open subset D of $\mathfrak{E}_3$ into H_2 and suppose that each ψ_k is meta-analytic in D. Let ψ: $D \to H_2$ and let $\psi_k \to \psi$ uniformly on every compact subset of D. Then ψ is meta-analytic in D and further $\psi_k' \to \psi'$ uniformly on every compact subset of D.*

REMARK. The proof of the corresponding result for functions mapping a region in the plane into $\boldsymbol{C}$ relies on Morera's theorem. Since an analog of Morera's theorem, namely Theorem 3, is valid in our present context we can imitate the proof of the Weierstrass theorem for ordinary analytic functions and obtain a proof for Theorem 4. (For a proof of the Weierstrass theorem for ordinary analytic functions, see e.g. p. 174 of [1].)

In view of the Remark we omit the proof of Theorem 4.

LEMMA 5. *Let D be a region in $\mathfrak{E}_3$ and let ψ be meta-analytic in D. Suppose $\|\psi(x)\|$ is constant in D. Then ψ is constant in D.*

Proof. Let

$$\psi(x) = \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} P(x)+iQ(x) \\ R(x)+iS(x) \end{pmatrix},$$

where P, Q, R, S are real-valued functions on D.

Let $\|\psi(x)\| = c$, where $x=(x_1, x_2, x_3) \in D$ and c is a constant. Then, by the definition of $\|\psi(x)\|$ it follows that

$$P^2+Q^2+R^2+S^2 = c^2. \tag{4.1}$$

Since ψ is meta-analytic in D, $(\nabla\psi)(x)=0$ for all $x \in D$ and so $(\Delta\psi)(x)=\nabla(\nabla\psi)(x) =0$ for all $x \in D$. Hence $\Delta U=0$ and $\Delta V=0$ and so U and V are harmonic functions in D.

On differentiating (4.1) partially with respect to x_1, x_2, x_3 we get that

$$\begin{aligned} PP_1+QQ_1+RR_1+SS_1 &= 0, \\ PP_2+QQ_2+RR_2+SS_2 &= 0, \\ PP_3+QQ_3+RR_3+SS_3 &= 0, \end{aligned}$$

where P_j denotes the partial derivative of P with respect to x_j $(j=1, 2, 3)$ etc.

Now on partially differentiating the first of the above three equations w.r.t. x_1, the second w.r.t. x_2 and the third w.r.t. x_3 we get that

$$\begin{aligned} P_1^2+PP_{11}+Q_1^2+QQ_{11}+R_1^2+RR_{11}+S_1^2+SS_{11} &= 0, \\ P_2^2+PP_{22}+Q_2^2+QQ_{22}+R_2^2+RR_{22}+S_2^2+SS_{22} &= 0, \\ P_3^2+PP_{33}+Q_3^2+QQ_{33}+R_3^2+RR_{33}+S_3^2+SS_{33} &= 0, \end{aligned}$$

where, for $j=1, 2, 3$,

$$P_{jj} = \frac{\partial^2 P}{\partial x_j^2}, \qquad Q_{jj} = \frac{\partial^2 Q}{\partial x_j^2}, \qquad R_{jj} = \frac{\partial^2 R}{\partial x_j^2}, \quad \text{and} \quad S_{jj} = \frac{\partial^2 S}{\partial x_j^2}.$$

Adding these three results and using the fact that $\Delta P = \Delta Q = \Delta R = \Delta S = 0$, we conclude that

$$\sum_{j=1}^{3} (P_j^2 + Q_j^2 + R_j^2 + S_j^2) = 0.$$

Hence for all $x \in D$ and for $j=1, 2, 3$

$$P_j(x) = Q_j(x) = R_j(x) = S_j(x) = 0.$$

$\therefore$ P, Q, R, S are constant functions in D and consequently ψ is a constant function in D. (Q.E.D.)

THEOREM 5 (MAXIMUM MODULUS PRINCIPLE). *Let R be a bounded region in $\mathfrak{E}_3$ and let ψ be a nonconstant function which is continuous in $\bar{R}$ and meta-analytic in R. Then $\|\psi(x)\|$ attains its maximum value on the boundary of R.*

Proof. That $\|\psi(x)\|$ has a maximum value in $\bar{R}$ follows from the continuity of ψ. Let M denote this maximum value. Let $E = \{x \in \bar{R} : \|\psi(x)\| = M\}$. Then we have to show that $E \cap R = \varnothing$. Suppose $E \cap R \neq \varnothing$. Let $p \in E \cap R$. Since R is open we can find an open ball, say S, of radius r (>0) and center p such that $S \subset R$. For $0 < \rho < r$ let S_ρ denote the sphere of center p and radius ρ. Then by the Corollary 2 to Theorem 3

$$\psi(p) = \frac{1}{4\pi\rho^2} \int_{S_\rho} \psi(x)\, ds.$$

Hence

$$\psi(p) = \frac{1}{4\pi\rho^2} \left[\int_{S_\rho \cap E} \psi(x)\, ds + \int_{S_\rho \cap E^c} \psi(x)\, ds \right].$$

Therefore

$$M = \|\psi(p)\| \leqq \frac{1}{4\pi\rho^2} \left[\int_{S_\rho \cap E} \|\psi(x)\|\, ds + \int_{S_\rho \cap E^c} \|\psi(x)\|\, ds \right].$$

If $S_\rho \cap E^c \neq \varnothing$ then the above relation would imply that

$$M < \frac{1}{4\pi\rho^2} \int_{S_\rho} M\, ds = M,$$

a contradiction. Hence $S_\rho \subset E$. Thus for $0 < \rho < r$, $S_\rho \subset E$ and so

$$S = \bigcup \{S_\rho : 0 < \rho < r\} \subset E.$$

Thus $p \in S \subset R \cap E$ and this implies that $R \cap E$ is a relatively open subset of R. But $R \cap E$ is obviously a relatively closed subset of R—i.e., $R \cap E$ is a nonempty subset of R which is both closed and open in R. But R is connected and so we must

have that $R \cap E = R$. But this means that $R \subset E$ and so $\|\psi(x)\|$ is constant in R. Then by Lemma 5 we conclude that ψ is constant in R. But this, then, contradicts the hypothesis that ψ is nonconstant in R. Thus we must have $E \cap R = \varnothing$. (Q.E.D.)

Next we will obtain a "Taylor's formula" for meta-analytic functions. But, first let us briefly recall some facts.

Let f be a map from an open subset D of a Banach space E into another Banach space F. Suppose f is differentiable in D. Then the derivative of f at $p \in D$, denoted $f'(p)$, is an element of $\mathfrak{L}(E, F)$—the set of all continuous linear maps of E into F. If f is twice differentiable at $p \in D$ then $f''(p) \in \mathfrak{L}(E, \mathfrak{L}(E, F))$. If we write $\mathfrak{L}_2(E, F)$ for $\mathfrak{L}(E, \mathfrak{L}(E, F))$ and in general $\mathfrak{L}_k(E, F)$ for $\mathfrak{L}(E, \mathfrak{L}_{k-1}(E, F))$ then the kth derivative of f at p, when it exists, is an element of $\mathfrak{L}_k(E, F)$; in symbols $f^{(k)}(p) \in \mathfrak{L}_k(E, F)$. We also know (see e.g. 5.7.8 of [2]) that $\mathfrak{L}_k(E, F)$ can be identified, in a natural way, with the space $\mathfrak{L}(E_1 \times \cdots \times E_k, F)$, where $E = E_i$ $(1 \leqq i \leqq k)$, of k-linear continuous mappings of E into F.

Now let ψ be meta-analytic in D. Then by the first corollary to Theorem 3 we know that ψ has derivatives of all orders in D. Thus if k is any positive integer and if the segment joining p and $p+x$ is in D, we have by 8.14.3 of [2] that

$$\psi(p+x) = \psi(p) + \frac{\psi'(p)x}{1!} + \frac{\psi''(p)x^{(2)}}{2!} + \cdots + \frac{\psi^{(k-1)}(p)x^{(k-1)}}{(k-1)!} + R_k, \tag{4.2}$$

where $x^{(n)}$ stands for $(x, x, \ldots, x)$ (n times) and R_k is the remainder term.

We call (4.2) Taylor's formula for ψ about p.

Now we will show that each term on the right side of (4.2) except possibly the remainder term satisfies the equation $\nabla\tilde{\psi} = 0$. This is an analogy with the fact that each term in Taylor's expansion of an analytic function is a harmonic function—i.e. satisfies the Laplace equation.

LEMMA 6. *If ψ is meta-analytic in D and if k is any positive integer then*

$$\psi^{(k)}(p)x^{(k)} = (\partial_x^k \psi)p,$$

where $p \in D$ and $x \in \mathfrak{E}_3$.

Proof. The proof will be by induction on k.

The lemma is true for $k=1$ by the definition of $(\partial_x \psi)p$. Suppose that the lemma is true for the positive integer $k-1$. Then

$$\begin{aligned}\psi^{(k)}(p)x &= \left(\lim_{h\to 0} \frac{\psi^{(k-1)}(p+hx) - \psi^{(k-1)}(p)}{h}\right)x^{(k-1)} \\ &= \lim_{h\to 0}\left(\frac{\psi^{(k-1)}(p+hx)x^{(k-1)} - \psi^{(k-1)}(p)x^{(k-1)}}{h}\right).\end{aligned}$$

But by induction hypothesis the right side can be written as

$$\lim_{h\to 0} \frac{(\partial_x^{(k-1)}\psi)(p+hx) - (\partial_x^{(k-1)}\psi)p}{h}.$$

But this limit is equal to $((\partial_x^{(k-1)}\psi)'p)x$. Hence

$$\psi^{(k)}(p)x^{(k)} = ((\partial_x^{(k-1)}\psi)'p)x = \partial_x(\partial_x^{(k-1)}\psi)p = (\partial_x^{(k)}\psi)p.$$

Thus the validity of the lemma for a positive integer $k-1$ implies that the lemma is valid for k. Hence, by the principle of induction, the lemma is true for all positive integers.

LEMMA 7. *Let ψ be a function which is n times continuously differentiable, n being a positive integer. Then for $x \in \mathfrak{E}_3$*

$$\partial_x^{(n)}\nabla\psi = \nabla\,\partial_x^{(n)}\psi.$$

Proof. Let $(\zeta_1, \zeta_2, \zeta_3)$ be an o.n. basis for $\mathfrak{E}_3$. Then $\nabla = \sum_{j=1}^3 \zeta_j\,\partial_j$, where, for $j=1,2,3$, $\partial_j = \partial_{\zeta_j}$. If $x = \sum_{j=1}^3 x_j\zeta_j$ then we have that $\partial_x = \sum_{j=1}^3 x_j\,\partial_j$. Let $1 \leqq k \leqq 3$. Then

$$\partial_k\nabla\psi = \partial_k \sum_{j=1}^3 \zeta_j\,\partial_j\psi = \sum_{j=1}^3 \zeta_j\,\partial_k\,\partial_j\psi = \sum_{j=1}^3 \zeta_j\,\partial_j\,\partial_k\psi$$

$$= \left(\sum_{j=1}^3 \zeta_j\,\partial_j\right)\partial_k\psi = \nabla\,\partial_k\psi.$$

Hence

$$\partial_x\nabla\psi = \sum_{j=1}^3 x_j\,\partial_j\nabla\psi = \sum_{j=1}^3 x_j\nabla\,\partial_j\psi = \nabla\sum_{j=1}^3 x_j\,\partial_j\psi = \nabla\,\partial_x\psi.$$

Now the proof of the lemma can be easily completed by the principle of induction. (Q.E.D.)

If ψ is meta-analytic in D then for $p \in D$

$$\nabla\left(\frac{\psi^{(k)}(p)x^{(k)}}{k!}\right) = \frac{1}{k!}\nabla(\partial_x^{(k)}\psi)p \qquad \text{(by Lemma 7)}$$

$$= \frac{1}{k!}\partial_x^{(k)}(\nabla\psi)(p) \qquad \text{(by Lemma 8)}$$

$$= 0$$

since ψ is meta-analytic at p.

Thus we have shown that every term on the right side of (4.2) except possibly R_k satisfies the equation $\nabla\tilde{\psi} = 0$.

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