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# NONCOMMUTATIVE JORDAN ALGEBRAS OF CAPACITY TWO(1)

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**Abstract.** Let $J$ be a noncommutative Jordan algebra with 1. If $J$ has two orthogonal idempotents $e$ and $f$ such that $1=e+f$ and such that the Peirce 1-spaces of each are Jordan division rings, then $J$ is said to have capacity two. We prove that a simple noncommutative Jordan algebra of capacity two is either a Jordan matrix algebra, a quasi-associative algebra, or a type of quadratic algebra whose plus algebra is a Jordan algebra determined by a nondegenerate symmetric bilinear form.

We refer the reader to [1] for all the basic concepts used in this paper as well as for all definitions of terms with the exception of quasi-associative algebra which may be found in [3].

**Introduction.** Recently much work has been done on the structure of simple noncommutative Jordan algebras $J$ over a field $\Phi$ of characteristic not 2 which satisfy the following axioms:

(i) $J$ has an identity element 1.

(ii) $1=e_1+e_2+\cdots+e_n$, where the $e_i$'s are mutually orthogonal idempotents.

(iii) The Peirce 1-space with respect to $e_i$, $J_1(e_i)$, is a Jordan division ring for $i=1, 2, \ldots, n$.

An algebra satisfying axioms (i)–(iii) is said to have *capacity n*.

N. Jacobson [2] proved that those simple commutative Jordan algebras of capacity $n$, where $n \geqq 3$, are Jordan matrix algebras. K. McCrimmon and R. D. Schafer in [5] proved that the simple, not commutative Jordan algebras of capacity $n$ ($n \geqq 3$) are all quasi-associative. In [6] J. M. Osborn found that a simple Jordan algebra of capacity two is either a $2 \times 2$ Jordan matrix algebra or a Jordan algebra determined by a nondegenerate symmetric bilinear form. It is the goal of this paper to characterize the simple noncommutative Jordan algebras of capacity two.

For any idempotent $e$ in a noncommutative Jordan algebra $J$ we have the Peirce decomposition of $J$ with respect to $e$; namely $J=J_0(e)+J_{1/2}(e)+J_1(e)$ where $J_i=\{x_i \mid \frac{1}{2}(ex_i+x_ie)=ix_i\}$. Throughout this work $J$ will denote a noncommutative

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Jordan algebra of capacity two and $e$ and $f$ will be the two orthogonal idempotents in $J$ such that $1=e+f$ and $J_0$ (with respect to $e$) and $J_1$ are Jordan division rings. If $J_{1/2}\neq\{0\}$, which will be the case if $J$ is simple, then it is not hard to show that any Jordan ring of capacity two may be considered an algebra of capacity two over an appropriate field.

1. **Osborn's results.** If $J=J_1+J_{1/2}+J_0$ is simple then $J^+=J_1^++J_{1/2}^++J_0^+$ will be a simple Jordan algebra of capacity two [5, p. 2]. From Osborn's work [6] we know that if $J^+$ is not a Jordan algebra determined by a symmetric bilinear form, then $J^+$ is a Jordan matrix algebra $H(E_2,\gamma)$, the set of symmetric $2\times 2$ matrices under the canonical involution determined by the invertible diagonal matrix $\gamma$ and where $E$ is the associative algebra with involution * generated by

$$S = \{2\bar{R}_{x_1}^+ \mid x_1\in J_1^+ \text{ and } \bar{R}_{x_1}^+\in \mathrm{Hom}_\Phi(J_{1/2},J_{1/2}) \text{ defined by } \bar{R}_{x_1}^+(y_{1/2}) = \tfrac{1}{2}(x_1y_{1/2}+y_{1/2}x_1)\}.$$

The nonzero generators of $E$ are symmetric and invertible and moreover every symmetric element of $E$ is a generator. $E$ is one of the following two types:

(a) An associative division algebra with involution which is not commutative and which is not a quaternion algebra over its center with the standard involution.

(b) A direct sum of a division algebra (which is not commutative) with its anti-isomorphic copy and the involution switches components.

Finally $\gamma=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}$ where each $\gamma_i$ is a symmetric, invertible element of $E$.

If $\gamma_1\neq 1$ we change the involution in $E$ to be $\bar{\alpha}=\gamma_1^{-1}\alpha^*\gamma_1$ where * is the old involution on $E$ and at the same time we change $\gamma$ to be $\begin{pmatrix}1 & 0\\ 0 & \gamma_1^{-1}\gamma_2\end{pmatrix}$. Under the new involution, $\gamma_1^{-1}\gamma_2$ is symmetric and invertible. An easy calculation shows that using the new involution on $E$ and the new involution on $E_2$ determined by the new $\gamma$, the set $H(E_2,\gamma)$ remains the same as before. So we may assume that $\gamma$ is of the form $\begin{pmatrix}1 & 0\\ 0 & \gamma_2\end{pmatrix}$ in $H(E_2,\gamma)$ and this we will do for the remainder of this work. We note that if $(E, *)$ is of type (a) or (b) then $(E, {}^-)$ will be also.

Products in $J$ as well as those in $E$ will be denoted by juxtaposition.

2. **Preliminary results.** We define $\bar{R}_{x_1}\in \mathrm{Hom}_\Phi(J_{1/2},J_{1/2})$ by

$$\bar{R}_{x_1}(y_{1/2}) = [y_{1/2}x_1]_{1/2},$$

i.e. $\bar{R}_{x_1}(y_{1/2})$ is the component of $y_{1/2}x_1$ in $J_{1/2}$. (Likewise we define $\bar{L}_{x_1}(y_{1/2}) =[x_1y_{1/2}]_{1/2}$.) Then $2\bar{R}_{x_1}^+=\bar{R}_{x_1}+\bar{L}_{x_1}$ and $E$ is generated by $\{(\bar{R}_{x_1}+\bar{L}_{x_1}) \mid x_1\in J_1\}$. Note that the identity 1 of $E$ is $2\bar{R}_e^+=\bar{R}_e+\bar{L}_e$.

$J_{1/2}$ is a left $E$-module by

$$(\bar{R}_{x_1}+\bar{L}_{x_1})a_{1/2} = a_{1/2}x_1+x_1a_{1/2}$$

where $a_{1/2}\in J_{1/2}$. $J_{1/2}$ as an $E$-module is isomorphic to $E$ as an $E$-module. Since $\mathrm{Hom}_E(E,E)$ is anti-isomorphic to $E$, so is $\mathrm{Hom}_E(J_{1/2},J_{1/2})$. We know that $E_xF_e=F_eE_x$ if $x\in J_0+J_1$ and $E, F=R, L$ [4, p. 188] and so $\bar{R}_e$ and $\bar{L}_e$ are in the center of $\mathrm{Hom}_E(J_{1/2},J_{1/2})$ and can be regarded as elements in the center of $E$.

K. McCrimmon [4, p. 189] has shown that

$$\text{(1)}\qquad \begin{aligned} \bar{L}_e(\bar{L}_{y_1}+\bar{R}_{y_1}) &= \bar{L}_{y_1}, \\ \bar{R}_e(\bar{L}_{y_1}+\bar{R}_{y_1}) &= \bar{R}_{y_1}; \end{aligned}$$

$$\text{(2)}\qquad \begin{aligned} \bar{L}_{x_1y_1} &= \bar{L}_{x_1}\bar{L}_{y_1}+\bar{L}_{y_1}\bar{R}_{x_1}, \\ \bar{R}_{x_1y_1} &= \bar{R}_{x_1}\bar{L}_{y_1}+\bar{R}_{y_1}\bar{R}_{x_1}. \end{aligned}$$

The equations of (1) show that $\bar{L}_{y_1}$ and $\bar{R}_{y_1}$ are in $E$ for every $y_1 \in J_1$.

Define $\nu: J_1 \to S$ by $\nu(x_1)=\bar{R}_{x_1}+\bar{L}_{x_1}$. $\nu$ is onto and one-to-one. Using (1) and (2) we have

$$\begin{aligned} \bar{L}_e\nu(x_1)\nu(y_1)+\bar{R}_e\nu(y_1)\nu(x_1) &= \bar{L}_e(\bar{R}_{x_1}+\bar{L}_{x_1})(\bar{R}_{y_1}+\bar{L}_{y_1})+\bar{R}_e(\bar{R}_{y_1}+\bar{L}_{y_1})(\bar{R}_{x_1}+\bar{L}_{x_1}) \\ &= (\bar{R}_{x_1}+\bar{L}_{x_1})\bar{L}_e(\bar{R}_{y_1}+\bar{L}_{y_1})+(\bar{R}_{y_1}+\bar{L}_{y_1})\bar{R}_e(\bar{R}_{x_1}+\bar{L}_{x_1}) \\ &= (\bar{R}_{x_1}+\bar{L}_{x_1})\bar{L}_{y_1}+(\bar{R}_{y_1}+\bar{L}_{y_1})\bar{R}_{x_1} \\ &= \bar{L}_{x_1y_1}+\bar{R}_{x_1y_1} = \nu(x_1y_1). \end{aligned}$$

Thus,

$$\text{(3)}\qquad \nu(x_1y_1) = \bar{L}_e\nu(x_1)\nu(y_1)+\bar{R}_e\nu(y_1)\nu(x_1).$$

We are now assuming that $J^+$ is isomorphic to $H(E_2,\gamma)$ and so we can consider $J_0, J_{1/2}, J_1$ as sets of matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$, $\begin{pmatrix} 0 & \delta \\ \gamma_2^{-1}\delta^* & 0 \end{pmatrix}$, $\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}$, respectively, where $\alpha, \delta \in E$ with $\alpha$ having the property that $\gamma_2^{-1}\alpha^*\gamma_2=\alpha$ and where

$$\beta \in S = \{\beta \in E \mid \beta^* = \beta\} = \{(\bar{R}_{x_1}+\bar{L}_{x_1}) \mid x_1 \in J_1\}.$$

We are ready to describe the multiplication in $J_1$. For notation we will let $\alpha e$ denote the element $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ in $J_1$. If $x_1=\alpha e$ and $y_1=\beta e$, then (3) implies $(\alpha e)(\beta e) =(\bar{L}_e\alpha\beta+\bar{R}_e\beta\alpha)e$ where $\bar{L}_e\alpha\beta+\bar{R}_e\beta\alpha \in S$. We have therefore proved

LEMMA 1. *If $\alpha e, \beta e \in J_1$, then $(\alpha e)(\beta e)=(\bar{L}_e\alpha\beta+\bar{R}_e\beta\alpha)e$.*

LEMMA 2. *The involution * on E has the property that $\bar{L}^*_{x_1}=\bar{R}_{x_1}$ for every $x_1 \in J_1$.*

**Proof.** From Lemma 1 we know that

$$\text{(4)}\qquad (\bar{L}_e\alpha\beta+\bar{R}_e\beta\alpha)^* = \bar{L}_e\alpha\beta+\bar{R}_e\beta\alpha.$$

Since $\bar{L}_e+\bar{R}_e=1$, (4) implies

$$\alpha\beta-\beta\alpha = (\bar{L}^*_e+\bar{L}_e)(\alpha\beta-\beta\alpha)$$

or

$$[1-(\bar{L}^*_e+\bar{L}_e)](\alpha\beta-\beta\alpha) = 0,$$

for every $\alpha, \beta \in S$. Since $E$ is not commutative there exists $\alpha, \beta \in S$ such that $\alpha\beta-\beta\alpha \neq 0$ and since $1-(\bar{L}^*_e+\bar{L}_e) \in S$ it is either 0 or invertible. So from the above equation we conclude that $1-(\bar{L}^*_e+\bar{L}_e)=0$ and this means $\bar{L}^*_e+\bar{L}_e=1$ and hence $\bar{L}^*_e=\bar{R}_e$. Using the equations of (1) we get $\bar{L}^*_{x_1}=\bar{R}_{x_1}$ for every $x_1 \in J_1$.

Let "$\cdot$" be the Jordan product in $J^+$. We define a new multiplication "$\times$" on the set $J$ by $a\times b = a\cdot b - ab$ where $ab$ is the multiplication in $J$. Then $a\cdot b = \frac{1}{2}(ab+ba)$ implies

$$a\times b = -b\times a \tag{5}$$

and using the fact that $J$ satisfies the flexible law we obtain

$$(a\times b)\cdot a + (a\cdot b)\times a = 0. \tag{6}$$

Linearization of (6) gives

$$(c\times b)\cdot a + (a\times b)\cdot c + (c\cdot b)\times a + (a\cdot b)\times c = 0. \tag{7}$$

By Lemma 1 $(\alpha e)(\beta e) = (\bar{L}_e\alpha\beta + \bar{R}_e\beta\alpha)e$, so, from the definition of the $\times$-multiplication, $\alpha e\times\beta e = m(\alpha\beta - \beta\alpha)e$ where $m = \frac{1}{2} - \bar{L}_e$.

LEMMA 3. *If* $\alpha e\in J_1$ *and* $\beta f\in J_0$, *then* $(\alpha e)(\beta f) = 0$.

**Proof.** In (6) with $a=\alpha e$, $b=\beta f$ we get $(\alpha e\times\beta f)\cdot\alpha e + (\alpha e\cdot\beta f)\times\alpha e = 0$. So $(\alpha e\times\beta f)\cdot\alpha e = 0$ and if $\alpha e\times\beta f = \xi e + \delta_{12} + \varepsilon f$ where $\delta_{12} = \begin{pmatrix} 0 & \delta \\ \gamma_2^{-1}\delta^* & 0\end{pmatrix}$, then

$$\begin{aligned} 0 &= (\alpha e\times\beta b)\cdot\alpha e = (\xi e + \delta_{12} + \varepsilon f)\cdot\alpha e \\ &= \tfrac{1}{2}(\alpha\xi + \xi\alpha)e + \tfrac{1}{2}(\alpha\delta)_{12}. \end{aligned}$$

So $\alpha\xi + \xi\alpha = 0$ and $\alpha\delta = 0$. But $\alpha\in S$ and every nonzero element of $S$ is invertible so $\delta = 0$ and $\alpha e\times\beta f$ has zero component in $J_{1/2}$. Letting $\alpha = 1$ shows $e\times\beta f$ has zero component in $J_1$.

Linearization of $(\alpha e\times\beta f)\cdot\alpha e = 0$ gives $(\alpha e\times\beta f)\cdot\eta e + (\eta e\times\beta f)\cdot\alpha e = 0$. Letting $\eta = 1$ we get

$$(\alpha e\times\beta f)\cdot e + (e\times\beta f)\cdot\alpha e = 0 \tag{8}$$

and since the second term of (8) has zero component in $J_1$ ($e\times\beta f$ has zero component in $J_1$) we conclude that $\alpha e\times\beta f$ does also.

With $a=\beta f$ and $b=\alpha e$ in (6) we get $(\beta f\times\alpha e)\cdot\beta f = 0$. We recall that $\beta$ has the property that $\gamma_2^{-1}\beta^*\gamma_2 = \beta$. Hence $\beta^*\gamma_2 = \gamma_2\beta = (\beta^*\gamma_2)^*$ and so $\gamma_2\beta\in S$. This means $\gamma_2\beta$ is invertible if $\beta\neq 0$ and we conclude that $\beta$ is invertible also. So now we can repeat the argument used in the first part of the proof of this lemma to get that the $J_0$ and $J_{1/2}$ components of $\beta f\times\alpha e = -\alpha e\times\beta f$ are zero. Hence $\alpha e\times\beta f = 0$. But $\alpha e\times\beta f = \alpha e\cdot\beta f - (\alpha e)(\beta f)$, so $(\alpha e)(\beta f) = 0$ and the proof of the lemma is complete.

For notation let $\lambda = \bar{L}_e$ and let $\delta_{12} = \begin{pmatrix} 0 & \delta \\ \gamma_2^{-1}\delta^* & 0\end{pmatrix}$. So $\lambda^* = \bar{R}_e$ and elements of $J_{1/2}$ now have the form $\delta_{12}$ where $\delta\in E$.

LEMMA 4. *For* $\alpha e\in J_1$ *and* $\delta_{12}\in J_{1/2}$, $(\alpha e)\delta_{12} = (\lambda\alpha\delta)_{12}$.

**Proof.** Since $\bar{L}_{x_1}, \bar{R}_{x_1}\in E$ we have that $J_1J_{1/2}\subset J_{1/2}$ and $J_{1/2}J_1\subset J_{1/2}$, so $\alpha e\times\delta_{12}\in J_{1/2}$ for every $\alpha\in S$, $\delta\in E$. In particular $e\times 1_{12}\in J_{1/2}$ so $e\times 1_{12} = n_{12}$ for some $n\in E$.

Substitution in (7) of $\alpha e$, $\beta e$, $\delta_{12}$ for $a$, $b$, $c$ respectively gives

$$\alpha e\cdot(\beta e\times\delta_{12})+\tfrac{1}{2}(m(\beta\alpha-\alpha\beta)\delta)_{12}+\tfrac{1}{2}(\alpha e)\times(\beta\delta)_{12} = \tfrac{1}{2}((\alpha\beta+\beta\alpha)e)\times\delta_{12}. \tag{9}$$

Letting $\alpha=1$ in (9) gives

$$\tfrac{1}{2}(\beta e\times\delta_{12})+\tfrac{1}{2}e\times(\beta\delta)_{12} = \beta e\times\delta_{12}, \qquad \beta\in S,\ \delta\in E,$$

or

$$\beta e\times\delta_{12} = e\times(\beta\delta)_{12}, \qquad \beta\in S,\ \delta\in E. \tag{10}$$

With $\beta=1$ in (9) we get

$$\alpha e\cdot(e\times\delta_{12})+\tfrac{1}{2}\alpha e\times\delta_{12} = \alpha e\times\delta_{12}, \qquad \alpha\in S,\ \delta\in E,$$

or

$$2\alpha e\cdot(e\times\delta_{12}) = \alpha e\times\delta_{12}, \qquad \alpha\in S,\ \delta\in E. \tag{11}$$

So for $\delta=1$, (10) and (11) imply

$$\begin{aligned}\alpha e\times 1_{12} &= e\times\alpha_{12} = 2\alpha e\cdot(e\times 1_{12})\\ &= 2\alpha e\cdot n_{12} = (\alpha n)_{12}, \qquad \alpha\in S.\end{aligned} \tag{12}$$

Using (10), (11), and (12) we have for $\alpha,\beta\in S$

$$\alpha e\times\beta_{12} = e\times(\alpha\beta)_{12} = 2\alpha e\cdot(e\times\beta_{12}) = 2\alpha e\cdot(\beta n)_{12} = (\alpha\beta n)_{12}. \tag{13}$$

Now we assume that for any $k$ elements $\alpha_1,\alpha_2,\ldots,\alpha_k\in S$ the following is valid:

$$\alpha_k e\times(\alpha_{k-1}\cdots\alpha_1)_{12} = (\alpha_k\alpha_{k-1}\cdots\alpha_1 n)_{12}. \tag{14}$$

Then for $\alpha_{k+1},\alpha_k,\ldots,\alpha_1\in S$ we have

$$\begin{aligned}\alpha_{k+1}e\times(\alpha_k\alpha_{k-1}\cdots\alpha_1)_{12} &= e\times(\alpha_{k+1}\alpha_k\cdots\alpha_1)_{12} = 2\alpha_{k+1}e\cdot(e\times(\alpha_k\cdots\alpha_1)_{12})\\ &= 2\alpha_{k+1}e\cdot(\alpha_k\cdots\alpha_1 n)_{12} = (\alpha_{k+1}\cdots\alpha_1 n)_{12}.\end{aligned} \tag{15}$$

So since $S$ generates $E$, (15) implies that

$$\alpha e\times\delta_{12} = (\alpha\delta n)_{12}, \qquad \alpha\in S,\ \delta\in E. \tag{16}$$

Finally, using (16) in (9) and remembering that $E$ is not commutative will show that $m=n=\frac{1}{2}-\bar{L}_e=\frac{1}{2}-\lambda$.

The $\times$-multiplication relating $J_1$ with $J_{1/2}$ is now known so

$$\begin{aligned}(\alpha e)(\delta_{12}) &= (\alpha e)\cdot(\delta_{12})-(\alpha e)\times(\delta_{12})\\ &= \tfrac{1}{2}(\alpha\delta)_{12}-(\alpha\delta(\tfrac{1}{2}-\lambda))_{12}\\ &= \tfrac{1}{2}(\alpha\delta)_{12}-\tfrac{1}{2}(\alpha\delta)_{12}+(\alpha\delta\lambda)_{12} = (\lambda\alpha\delta)_{12}.\end{aligned}$$

LEMMA 5. *If $\delta_{12},\eta_{12}\in J_{1/2}$, then the $J_0$-component of $\delta_{12}\eta_{12}$ is*

$$(\lambda\gamma_2^{-1}\delta^*\eta+\lambda^*\gamma_2^{-1}\eta^*\delta)f.$$

**Proof.** In (7) let $a=e$, $b=\eta_{12}$ and $c=\delta_{12}$ to get

$$(\delta_{12}\times\eta_{12})\cdot e+(e\times\eta_{12})\cdot\delta_{12}+(\delta_{12}\cdot\eta_{12})\times e+(e\cdot\eta_{12})\times\delta_{12} = 0. \tag{17}$$

Since $\delta_{12}\cdot\eta_{12}\in J_1+J_0$ the third term of (17) is zero using Lemmas 1 and 3. Also, $e\times\eta_{12}=e\cdot\eta_{12}-e\eta_{12}=\frac{1}{2}\eta_{12}-\lambda\eta_{12}=(\frac{1}{2}-\lambda)\eta_{12}$ using Lemma 4. So (17) reduces to

$$(\delta_{12}\times\eta_{12})\cdot e+((\tfrac{1}{2}-\lambda)\eta)_{12}\cdot\delta_{12}+\tfrac{1}{2}\eta_{12}\times\delta_{12} = 0. \tag{18}$$

$(\delta_{12}\times\eta_{12})\cdot e$ has zero component in $J_0$ so (18) says

$$[\delta_{12}\times\eta_{12}]_0 = [\eta_{12}\cdot\delta_{12}]_0-[(2\lambda\eta)_{12}\cdot\delta_{12}]_0.$$

But

$$\begin{aligned}[\delta_{12}\eta_{12}]_0 &= [\delta_{12}\cdot\eta_{12}]_0-[\delta_{12}\times\eta_{12}]_0 = [(2\lambda\eta)_{12}\cdot\delta_{12}]_0\\ &= (\gamma_2^{-1}\delta^*\lambda\eta+\gamma_2^{-1}(\lambda\eta)^*\delta)f = (\lambda\gamma_2^{-1}\delta^*\eta+\lambda^*\gamma_2^{-1}\eta^*\delta)f.\end{aligned}$$

LEMMA 6. *If* $\delta_{12}$, $\eta_{12}\in J_{1/2}$, *then the* $J_{1/2}$-*component of* $\delta_{12}\eta_{12}$ *is* 0.

**Proof.** In (7) let $a=\alpha e$, $b=\eta_{12}$, $c=\delta_{12}$, and comparing components in $J_{1/2}$ will give

$$2[(\delta_{12}\times\eta_{12})\cdot\alpha e]_{1/2} = [\delta_{12}\times(\alpha\eta)_{12}]_{1/2}, \qquad \alpha\in S,\ \delta,\ \eta\in E. \tag{19}$$

Again in (7) let $c=\delta_{12}$, $b=\alpha e$, $a=\eta_{12}$ and get

$$(\alpha\delta)_{12}\times\eta_{12} = \delta_{12}\times(\alpha\eta)_{12}, \qquad \alpha\in S,\ \delta,\ \eta\in E. \tag{20}$$

For $\alpha\in S$ we have from (19) that $[1_{12}\times\alpha_{12}]_{1/2}=2[(1_{12}\times 1_{12})\cdot\alpha e]_{1/2}=0$. For $\alpha,\beta\in S$, (20) and (19) say $[\beta_{12}\times\alpha_{12}]_{1/2}=[1_{12}\times(\beta\alpha)_{12}]_{1/2}=2[(1_{12}\times\alpha_{12})\cdot\beta e]_{1/2}=0$. Using an inductive process as in Lemma 4 and recalling that $S$ generates $E$ will give $[\delta_{12}\times\eta_{12}]_{1/2}=0$ for every $\delta,\eta\in E$. This implies $[\delta_{12}\eta_{12}]_{1/2}=0$.

3. **The first structure theorem.** With the aid of the lemmas it is now quite easy to prove our first theorem.

THEOREM 1. *Let $J$ be a simple noncommutative Jordan algebra of capacity two such that $J^+$ is not a Jordan algebra determined by a symmetric bilinear form. Then $J$ is either commutative Jordan or quasi-associative.*

**Proof.** Let "$\circ$" denote ordinary associative matrix multiplication in $E_2$. If $x_1, y_1\in J_1$, Lemma 1 says $x_1y_1=\lambda\circ x_1\circ y_1+(1-\lambda)\circ y_1\circ x_1$. If $x_1\in J_1$, $y_{1/2}\in J_{1/2}$, Lemma 4 says $x_1y_{1/2}=\lambda\circ x_1\circ y_{1/2}+(1-\lambda)\circ y_{1/2}\circ x_1$.

Since $f\delta_{12}=(1-e)\delta_{12}=\delta_{12}-e\delta_{12}=\delta_{12}-(\lambda\delta)_{12}=((1-\lambda)\delta)_{12}=\delta_{12}e$ we have that $\bar{L}_f=(1-\lambda)=\lambda^*=\bar{R}_e$. Symmetry of the arguments used in the lemmas with respect to $f$ will give corresponding multiplications with $J_0$ and the analogue of Lemma 5 together with Lemma 6 will show that $\delta_{12}\eta_{12}=\lambda\circ\delta_{12}\circ\eta_{12}+(1-\lambda)\circ\eta_{12}\circ\delta_{12}$. If $\lambda=\lambda^*$, $J$ is commutative Jordan and if $\lambda\neq\lambda^*$, $J$ is quasi-associative.

If $E=\Delta\oplus\bar{\Delta}$, a direct sum of a division ring $\Delta$ with its anti-isomorphic copy $\bar{\Delta}$, then $J$ is an algebra over $\Phi=\{(\alpha,\bar{\alpha})\mid\alpha\in\text{center of }\Delta\}$. Any skew element $\lambda$ in

$\Delta \oplus \bar{\Delta}$ is of the form $\lambda=(\beta, -\bar{\beta})$. Hence if $\lambda$ is skew and central then $\lambda^2=(\beta, -\bar{\beta})^2 =(\beta^2, \bar{\beta}^2)$ which has a square root in $\Phi$. So by a theorem of McCrimmon [3, p. 1458], $J$ is split quasi-associative.

If $E$ is a division ring with a central skew element $\lambda$ then $\lambda^2$ does not have a square root in $\Phi$, the basefield of $E$. Hence by McCrimmon's theorem, $J$ is nonsplit quasi-associative.

4. **The second structure theorem.** For the rest of this work we will assume $J$ is a noncommutative Jordan algebra such that $J^+$ is a Jordan algebra determined by a symmetric bilinear form. Then $J^+=\Phi+V$ where $\Phi$ is a field, $V$ is a vector space over $\Phi$, and $f$: $V\times V\to \Phi$ is a symmetric bilinear form on $V$. If we denote the elements of $J^+$ by $(\alpha, v)$ where $\alpha\in\Phi$, $v\in V$ then the multiplication in $J^+$ is given by $(\alpha, v)(\beta, w)=(\alpha\beta+f(v, w), \alpha w+\beta v)$. It can easily be shown that if $f$ is nondegenerate, then $J^+$ is simple. The elements of $J$ will also be denoted by ordered pairs of elements from $\Phi$ and $V$. If $J^+$ has an idempotent $e\neq(1, 0)$, then $J^+$ has capacity two. An easy calculation will show that such an idempotent exists if and only if $f(u, u)=\frac{1}{4}$ for some vector $u\in V$.

If $f$ is not identically 0 there exists an element $w\in V$ such that $f(w, w)=\beta\neq 0$. If $\alpha=\beta^{1/2}\in\Phi$, then $f(\frac{1}{2}\alpha^{-1}w, \frac{1}{2}\alpha^{-1}w_\delta)=\frac{1}{4}$ and $J^+$ (and $J$) contains a nontrivial idempotent. If $\beta^{1/2}\notin\Phi$ we may take a quadratic extension of $\Phi$ to obtain an idempotent. So after a possible quadratic extension of the base field $\Phi$ we may assume $J$ has capacity two.

Henceforth $u$ will denote the vector in $V$ such that $f(u, u)=\frac{1}{4}$. Then $e=(\frac{1}{2}, u)$ and $f=(\frac{1}{2}, -u)$ are two orthogonal idempotents such that $1=e+f$. With respect to $e$ we have

$$J_0 = \{(\tfrac{1}{2}\alpha, -\alpha u) \mid \alpha\in\Phi\},$$
$$J_{1/2} = \{(0, w) \mid f(u, w) = 0\},$$
$$J_1 = \{(\tfrac{1}{2}\alpha, \alpha u) \mid \alpha\in\Phi\}.$$

We recall that $E$ is the associative ring generated by $\{2\bar{R}^+_{\alpha e} \mid \alpha e=(\frac{1}{2}\alpha, \alpha u)\in J_1\}$. Since $\bar{R}^+_{\alpha e}\bar{R}^+_{\beta e}=\bar{R}^+_{\beta e}\bar{R}^+_{\alpha e}$, $E$ is a field isomorphic to $\Phi$. Since $\nu(x_1)\nu(y_1)=\nu(y_1)\nu(x_1)$, equation (3) says that $\nu(x_1y_1)=\nu(x_1)\nu(y_1)$. So $J_1$ is a field isomorphic to $\Phi$.

We will have occasions to identify the element $(0, v)$ with the vector $v\in V$, so we will make this identification whenever necessary. $J$ induces a multiplication on the vectors of $V$ as follows: for $w, y\in V$ define

$$w\times y = [(0, w)(0, y)]_{1/2}, \tag{21}$$

i.e. $w\times y$ is the $J_{1/2}$-component of the product $(0, w)(0, y)$ in $J$. Clearly we have

$$w\times y = -y\times w.$$

For $a_{1/2}=(0, w)\in J_{1/2}$ we know $ea_{1/2}\in J_{1/2}$, so $e(0, w)=(\frac{1}{2}, u)(0, w)=(\frac{1}{2}, 0)(0, w) +(0, u)(0, w)=(0, \frac{1}{2}w)+(0, u)(0, w)$. Hence $(0, u)(0, w)\in J_{1/2}$, so $(0, u)(0, w)= (0, u\times w)$.

LEMMA 7. *If* $\alpha e \in J_1$ *and* $(0, w) \in J_{1/2}$, *then* $(\alpha e)(0, w) = (0, \frac{1}{2}\alpha w + \alpha(u \times w))$.

**Proof.** The linearized flexible law says

$$(\alpha e)[e(0, w)] + (0, w)[e(\alpha e)] = [(\alpha e)e](0, w) + [(0, w)e](\alpha e)$$

or

$$(22) \qquad (\alpha e)(0, \tfrac{1}{2}w + u \times w) + (0, w)(\alpha e) = (\alpha e)(0, w) + (0, \tfrac{1}{2}w + w \times u)(\alpha e).$$

Due to the fact that we know the multiplication in $J^+$, (22) becomes

$$(\alpha e)(0, \tfrac{1}{2}w + u \times w) + (0, \alpha w) - (\alpha e)(0, w)$$
$$= (\alpha e)(0, w) + (0, \tfrac{1}{2}\alpha w + \alpha(w \times u)) - (\alpha e)(0, \tfrac{1}{2}w + w \times u).$$

So $(\alpha e)(0, w) = (0, \frac{1}{2}\alpha w + \alpha(u \times w))$.

LEMMA 8. *Let* $v, w \in V$, *then* $(0, v)(0, w) = (f(v, w), v \times w)$.

**Proof.** The flexible law says

$$(0, v)[(0, w)(0, v)] - [(0, v)(0, w)](0, v) = 0,$$

and so we have

$$(0, v)[(0, w)(0, v)] + [(0, w)(0, v)](0, v) = 2(f(v, w), 0)(0, v).$$

Hence we know that the product in $J^+$ of $(0, v)$ with $(0, w)(0, v)$ is the same as that of $(0, v)$ with $(f(v, w), 0)$. This together with (21) implies that $(0, v)(0, w) = (f(v, w), v \times w)$.

Lemmas 7 and 8 imply the following theorem which was motivated by Theorem 1 of [7].

THEOREM 2. *If $J$ is a noncommutative Jordan algebra such that $J^+ = \Phi + V$ is a Jordan algebra determined by a nonzero symmetric bilinear form, then possibly after a quadratic extension of the field $\Phi$ the multiplication in $J$ is given by*

$$(\alpha, v)(\beta, w) = (\alpha\beta + f(v, w), \alpha w + \beta v + v \times w)$$

*where "$\times$" is an anti-commutative multiplication on the vector space $V$ and $f$ is the symmetric bilinear form of $J^+$ such that $f(v \times w, y) = f(v, w \times y)$.*

If we assume that $V$ has a basis $\{w_i\}$ over $\Phi$ such that $f(w_i, w_j) = 0$ if $i \neq j$ and $f(w_i, w_i) = \beta_i$, then we can give more information about the $\times$-multiplication on $V$. Let $w_i \times w_j = \sum_k \gamma_{ijk} w_k$ where $\gamma_{ijk} \in \Phi$. Then $w_i \times w_j = -w_j \times w_i$ means that

$$(23) \qquad \gamma_{ijk} = -\gamma_{jik}.$$

Also $f(w_i, w_j \times w_k) = f(w_i \times w_j, w_k)$ is equivalent to

$$(24) \qquad f(w_k, w_k)\gamma_{ijk} = f(w_i, w_i)\gamma_{jki} = f(w_j, w_j)\gamma_{kij}.$$

So given any set of constants $\{\gamma_{ijk}\}$ satisfying (23) and (24), $w_i \times w_j = \sum_k \gamma_{ijk} w_k$ gives a multiplication on $V$ such that $J$ is a noncommutative Jordan algebra and $J^+$ is a

Jordan algebra determined by a symmetric bilinear form. Whenever $V$ has an orthogonal basis, such a set of $\gamma_{ijk}$'s can always be found, for we can define $\gamma_{ijk}$ arbitrarily for $i<j<k$ and then use (23) and (24) to find $\gamma_{ijk}$ for the other permutations of $i, j, k$.

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