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CYCLIC VECTORS AND IRREDUCIBILITY FOR PRINCIPAL SERIES REPRESENTATIONS

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Abstract. Canonical sets of cyclic vectors for principal series representations of semisimple Lie groups having faithful representations are found. These cyclic vectors are used to obtain estimates for the number of irreducible subrepresentations of a principal series representations. The results are used to prove irreducibility for the full principal series of complex semisimple Lie groups and for $SL(2n+1, R)$, $n \geqq 1$.

1. **Introduction.** Let G be a real semisimple Lie group which is connected and has a faithful finite-dimensional representation. Let K be a maximal compact subgroup of G and let $S=AN$ be an Iwasawa subgroup of G. N is the nil-radical of S and A is the split part of S. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ be respectively the Lie algebras of G, K, A and N. Let $\mathfrak{a}'$ be the real dual of $\mathfrak{a}$. Let $\rho \in \mathfrak{a}'$ be defined by $\rho(x) =(1/2) \operatorname{tr} (\operatorname{ad} x|_{\mathfrak{n}})$ for $x \in \mathfrak{a}$. Let M be the centralizer of A in K. Let (ξ, H_ξ) be an irreducible unitary representation of M (ξ is the action, H_ξ is the Hilbert space), and let $\nu \in \mathfrak{a}'$. We now define a unitary representation $(\pi_{\xi,\nu}, H^{\xi,\nu})$ of G. $H^{\xi,\nu}$ is the space of all measurable functions $f\colon G \to H_\xi$ such that

(i) $f(gman)=e^{-(\rho+i\nu)(\log a)}\xi(m)^{-1}f(g)$ for $g \in G$, $m \in M$, $a \in A$, $n \in N$ (here $\log\colon A \to \mathfrak{a}$ is the inverse map to $\exp\colon \mathfrak{a} \to A$).

(ii) $\int_{K/M} \|f(x)\|^2\, dx<\infty$. (Here $\|\cdots\|$ is the Hilbert space norm on H_ξ, dx is K-invariant measure on K/M and $\|f(kM)\| = \|f(k)\|$ is well defined for f satisfying (i) and $k \in K$.)

The action $\pi_{\xi,\nu}$ of G on $H^{\xi,\nu}$ is given by $(\pi_{\xi,\nu}(g_0)\cdot f)(x)=f(g_0^{-1}x)$ for $g_0, x \in G$.

In this paper we give a new proof, using a result of Kostant [7], of a result of Bruhat [2] that says that $H^{\xi,\nu}=H_1 \oplus \cdots \oplus H_p$, H_i irreducible, closed, G-invariant subspaces of $H^{\xi,\nu}$, with H_i, H_j orthogonal for $i\neq j$. Using Kostant's result, we derive an upper bound (Theorem 3.3) for p which is in many cases better than that of Bruhat [2]. It proves in particular that every element of the full principal series $\{(\pi_{\xi,\nu}, H^{\xi,\nu})\}$ is irreducible for the following classes of groups:

(1) Complex semisimple Lie groups.

(2) $SL(2n+1, R)$, $n \geqq 1$.

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Želobenko [10] seems to also have a proof of (1). We discovered our proof before we were aware of Želobenko's. Gel'fand and Graev [3] assert (2). However their technique and their theorem indicate the irreducibility for $SL(2n, R)$, $n>1$. This is not true (see §5 for details on $SL(n, R)$). There seems therefore to be a gap in their proof of (2).

We thank Professor G. Schiffman for suggesting the use of the Stone-Weierstrass theorem to prove Theorem 2.1. We also thank Professor Garth Warner for allowing us access to the manuscript of his forthcoming book on representation theory.

2. **Extendible representations.** We retain the notation and assumptions of §1. Let $\bar{N}$ be the unipotent subgroup of G corresponding to the negative restricted root spaces of $\mathfrak{a}$ assuming N corresponds to the positive restricted root spaces of $\mathfrak{a}$. Let (ξ, H_ξ) be a unitary irreducible representation of M. We say that (ξ, H_ξ) is extendible if there is an irreducible, finite-dimensional, complex G module, V, so that the M-module, $V^{\bar{N}}=\{v \in V \mid n\cdot v=v \text{ for all } n \in \bar{N}\}$ is equivalent with (ξ, H_ξ). We call V an extension of ξ.

Let $\hat{M}$ be the set of all equivalence classes of irreducible unitary representations of M. We denote the class of (ξ, H_ξ) by $[\xi]$. Let M^* be the normalizer of A in K. We set $W(A)=M^*/M$. Then $W(A)$ has canonical actions on $\hat{M}$ and $\mathfrak{a}'$. If $[\xi] \in \hat{M}$, $\sigma \in W(A)$, define $[\xi]^\sigma$ to be the class of the representation $(\xi \circ \operatorname{Ad}(m^{*-1}), H_\xi)$ where $m^* \in \sigma$. It is clear that the action is well defined. Let $W(A)$ act in $\mathfrak{a}'$ by $\nu^\sigma=\nu \circ \operatorname{Ad}(m^{*-1})$ for $m^* \in \sigma$.

We say that $\gamma \in \hat{M}$ is extendible if there is $(\xi, H_\xi) \in \gamma$ that is extendible.

Let V be an irreducible finite-dimensional complex G-module. We analyze the M-module $V^{\bar{N}}$. The contragradient module, V^*, to V is defined to be the complex dual space V^* of V with G-acting by

$$(g\lambda)(v) = \lambda(g^{-1}v) \quad \text{for } \lambda \in V^*,\ v \in V,\ g \in G.$$

We define $V^{*N}=\{\lambda \in V^* \mid n\cdot\lambda=\lambda \text{ for all } n \in N\}$.

Lemma 2.1. *Let V and V^* be as above.*

(1) *$V^{\bar{N}}$ is irreducible as an M-module.*

(2) *As an MA-module, $V^{\bar{N}}$ is equivalent with $(V^{*N})^*$.*

Proof. Let i: $V^{\bar{N}} \to (V^{*N})^*$ be defined by $i(v)(\lambda)=\lambda(v)$ for all $v \in V^{\bar{N}}$, $\lambda \in V^{*N}$. If $v \in V^{\bar{N}}$, $\lambda \in V^{*N}$, $g \in MA$ then $i(gv)(\lambda)=\lambda(gv)=(g^{-1}\cdot\lambda)(v)=(g\cdot i(v))(\lambda)$. Thus i is an MA-module homomorphism. The proof proceeds from here in exactly the same way as the proof of Proposition 3.3 of Wallach [9], to see that $V^{\bar{N}}$ is irreducible as an M-module and i is an MA-module isomorphism.

Theorem 2.1. *Let $\gamma \in \hat{M}$. Then there is $\sigma \in W(A)$ such that γ^σ is extendible.*

Proof. Let E_0 be the set of all extendible elements of $\hat{M}$. Let

$$E = \{\gamma^\sigma \mid \gamma \in E_0,\ \sigma \in W(A)\}.$$

We first show that if $\gamma \in E_0$ and if γ^* is the class of contragradient representation to γ then $\gamma^* \in E$. Indeed let V be an extension of $\gamma \in E_0$. Let $\sigma \in W(A)$ be such that if $m^* \in \sigma$ then $\mathrm{Ad}\,(m^*)N = \bar{N}$. Let $\tilde{\gamma}$ be the class of $V^{*\bar{N}}$. We assert that $\tilde{\gamma}^{\sigma^{-1}} = \gamma^*$. Indeed the action of $\tilde{\gamma}^{\sigma^{-1}}$ on $V^{*\bar{N}}$ is $\Pi(m)\cdot v = m^* m m^{*-1}\cdot v = m^*(mm^{*-1}v)$. Now $m^{*-1}V^{*\bar{N}} = V^{*N}$. Thus the action of $\gamma^{\sigma^{-1}}$ on $V^{*\bar{N}}$ is equivalent to the action of M on V^{*N}. But V^{*N} is equivalent with $(V^{\bar{N}})^*$ by Lemma 2.1.

Now let P be the algebra generated over C by the matrix elements of the elements of E. By the above, if $f \in P$ then the complex conjugate $\bar{f}$ of f is in P. Thus to prove the theorem we need only show that P separates the points of M. If P did not separate the points of M there would be $g \in M$, $g \neq e$ so that if V is any irreducible, complex, finite-dimensional, G-module and if $m^* \in M^*$ then $\mathrm{Ad}\,(m^*)g|_{V^{\bar{N}}}$ is the identity.

Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$, the Lie algebra of G. Since G has a finite-dimensional faithful representation we may assume that $G \subset G_C$, G_C a complex connected Lie group with Lie algebra $\mathfrak{g}^C$, and that G is the connected subgroup of G^C with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{u} = \mathfrak{k} \oplus (-1)^{1/2}\mathfrak{p}$ in $\mathfrak{g}^C$. Then $\mathfrak{u}$ is a compact form of g^C. Let U be the corresponding connected subgroup of G_C. We note that $\mathfrak{a} \subset \mathfrak{p}$. It is not hard to prove (see Matsumoto [8]) that if $Z = \exp\,((-1)^{1/2}\mathfrak{a}) \cap K$ and if M_e is the identity component of M then $M_e Z = M$. Let $\mathfrak{h}_*^+$ be an abelian subalgebra of $\mathfrak{k}$, maximal subject to the constraint $\mathfrak{h}_*^+ + \mathfrak{a}$ is abelian. Let $T_0 = \exp\,(\mathfrak{h}_+^*)$ and let $T_1 = \exp\,((-1)^{1/2}\mathfrak{a})$. Then $T = T_0 T_1$ is a maximal torus of U and T_0 is a maximal torus of M_e. Now up to conjugacy by an element of M^* we may assume that the element $g \in M$ above is in $T_0 Z \subset T$. The theorem of the highest weight applied to irreducible finite-dimensional holomorphic G_C modules now implies such a $g \in T_0 Z \subset T$, $g \neq e$ cannot exist.

Now let V be an irreducible, finite-dimensional nonzero complex G-module. We identify $V^{\bar{N}}$ with $(V^{*N})^*$. For each $v \in V$, we define a C^∞-map, $\alpha(v)\colon G \to V^{\bar{N}} = (V^{*N})^*$, as follows: if $g \in G$, $\lambda \in V^{*N}$ we set $\alpha(v)(g)(\lambda) = (g\cdot\lambda)(v) = \lambda(g^{-1}v)$.

LEMMA 2.2. (1) *If* $v \in V$, g_0, $g \in G$ *then* $\alpha(g_0 v)(g) = \alpha(v)(g_0^{-1}g)$.

(2) *If* $v \in V$, $g \in G$, $m \in MA$ *then* $\alpha(v)(gm) = m^{-1}\cdot(\alpha(v)(g))$.

(3) *If* $\bar{n} \in \bar{N}$, $n \in N$, $g \in G$, $v \in V^{\bar{N}}$ *then* $\alpha(v)(\bar{n}gn) = \alpha(v)(g)$.

Proof. (1) Let $\lambda \in V^{*N}$, then

$$\alpha(g_0 v)(g)(\lambda) = (g\cdot\lambda)(g_0 v) = (g_0^{-1}g\lambda)(v) = \alpha(v)(g_0^{-1}g)(\lambda).$$

This proves (1).

(2) Let $\lambda \in V^{*N}$, $v \in V$, $g \in G$, $m \in MA$. Then

$$\alpha(v)(gm)(\lambda) = (gm\lambda)(v) = (g\cdot(m\lambda))(v) = \alpha(v)(g)(m\lambda) = (m^{-1}\cdot\alpha(v)(g))(\lambda).$$

This proves (2).

(3) Let $\bar{n} \in \bar{N}$, $n \in N$, $g \in G$, $v \in V^{\bar{N}}$, $\lambda \in V^{*N}$. Then

$$\alpha(v)(\bar{n}gn)(\lambda) = (\bar{n}gn)\lambda(v) = (\bar{n}g\lambda)(v) = (g\lambda)(\bar{n}^{-1}v) = (g\lambda)(v) = \alpha(v)(g)(\lambda).$$

This proves (3).

Now let V, $V^{\bar{N}}$ be as above. Put an M-invariant unitary structure on $V^{\bar{N}}$. Let the corresponding unitary representation of M be denoted (ξ, H_ξ) $(H_\xi = V^{\bar{N}})$. Let λ be the action of A on $V^{\bar{N}}$. That is if $a \in A$, $v \in V^{\bar{N}}$ then $a \cdot v = e^{\lambda(\log a)}v$. Let $\nu \in \mathfrak{a}'$. Let $h: G \to \mathfrak{a}$ be defined by $h(kan) = \log a$ for $k \in K$, $a \in A$, $n \in N$. We define a map

$$E_{V,\nu}: V \to H^{\xi,\nu}$$

by setting

$$E_{V,\nu}(v)(g) = e^{(\lambda - (\rho + i\nu))(h(g))}\alpha(v)(g), \quad \text{for } v \in V, g \in G.$$

LEMMA 2.3. *$E_{V,\nu}: V \to H^{\xi,\nu}$ is a K-module injection.*

Proof. We first show that if $v \in V$ then $E_{V,\nu}(v) \in H^{\xi,\nu}$. Indeed let $g \in G$, $m \in M$, $a \in A$, $n \in N$, then

$$\begin{aligned} E_{V,\nu}(v)(gman) &= e^{(\lambda - (\rho + i\nu))(\log a)}e^{-\lambda(\log a)}\xi(m)^{-1}E_{V,\nu}(g) \\ &= e^{-(\rho + i\nu)(\log a)}\xi(m)^{-1}E_{V,\nu}(g). \end{aligned}$$

Since $E_{V,\nu}(v)|_K$ is continuous we see that $E_{V,\nu}(v) \in H^{\xi,\nu}$.

We now show that $E_{V,\nu}: V \to H^{\xi,\nu}$ is a K-module homomorphism. Let $k, k_0 \in K$, $a \in A$, $n \in N$, $v \in V$. Then

$$\begin{aligned} E_{V,\nu}(k_0 \cdot v)(kan) &= e^{(\lambda - (\rho + i\nu))(\log a)}\alpha(k_0 v)(kan) \\ &= e^{(\lambda - (\rho + i\nu))(\log a)}\alpha(v)(k_0^{-1}kan) \\ &= E_{V,\nu}(v)(k_0^{-1}kan). \end{aligned}$$

Thus $E_{V,\nu}$ is indeed a K-module homomorphism.

To complete the proof of the lemma we need only show that $E_{V,\nu}$ is injective. But this is obvious since if $E_{V,\nu}(v) = 0$ then $e^{(\lambda - (\rho + i\nu))(h(g))}\alpha(v)(g) = 0$ for all $g \in G$. Hence $\alpha(v)(g) = 0$ for all $v \in V$. But $\alpha(v) = 0$ implies $v = 0$, since, by Lemma 2.2, Ker α is G-invariant and $\alpha \not\equiv 0$.

3. **Implications of a theorem of Kostant.** We need some notation to state the result of Kostant that we will need. Let $G = KAN$ be as in §§1 and 2. Let M be, as usual, the centralizer of A in K. Let $h: G \to \mathfrak{a}$ be defined by $h(kan) = \log a$ for $k \in K$, $a \in A$, $n \in N$. Let $\mathfrak{n}$ be the lie algebra of N, then $\mathfrak{a}$ acts on $\mathfrak{n}$ via the adjoint action, relative to this action $\mathfrak{n} = \sum \mathfrak{n}_\alpha$, $\alpha \in \mathfrak{a}'$, $\mathfrak{n}_\alpha = \{x \in \mathfrak{n} \mid [h, x] = \alpha(h)x$ for all $h \in \mathfrak{a}\}$. Let $\Lambda^+ = \{\alpha \in \mathfrak{a}' \mid \mathfrak{n}_\alpha \neq \{0\}\}$. We extend the killing form $\langle\ ,\ \rangle$, of $\mathfrak{a}$ to $\mathfrak{a}'$ in the canonical fashion. Let $\mathfrak{a}'_C$ be the space of all complex valued (real) linear forms on $\mathfrak{a}$. We define the function $1_\lambda: G \to C$ by $1_\lambda(g) = e^{-\lambda(h(g))}$. 1_λ is in particular a C^∞ function on G. Let for $x \in \mathfrak{g}$, $f \in C^\infty(G)$ (the C^∞ complex valued functions on G), $(x \cdot f)(g) = (d/dt)f(\exp(-tx)g)|_{t=0}$. The action of $\mathfrak{g}$ on $C^\infty(G)$ extends to $U(\mathfrak{g})$ the complexified universal enveloping algebra of $\mathfrak{g}$. Let $X^\lambda = U(\mathfrak{g}) \cdot 1_\lambda|_K$. A function $f \in C^\infty(K)$ is said to be K-finite if the space spanned by all the functions $(k_0 f)(k) = f(k_0^{-1}k)$, $k_0 \in K$, is finite dimensional.

THEOREM 3.1 (KOSTANT [7, THEOREM 8]). *Let* $\lambda \in \mathfrak{a}'_C$ *be so that* $\lambda-\rho=\mu_1+(-1)^{1/2}\mu_2$, $\mu_i \in \mathfrak{a}'$, $i=1, 2$, *and* $\langle \mu_1, \alpha\rangle \geqq 0$ *for each* $\alpha \in \Lambda^+$. *Then* X^λ *is the space of all K-finite functions on* K, f, *such that* $f(km)=f(k)$ *for* $k \in K$, $m \in M$.

Using this deep result of Kostant's we prove

THEOREM 3.2. *Let* (ξ, H_ξ) *be an irreducible unitary representation of M. Let V be an extension of* ξ. *Let* $\nu \in \mathfrak{a}'$. *If* $f \in E_{V,\nu}(H_\xi), f\neq 0$, *then f is a cyclic vector for* $(\pi_{\xi,\nu}, H^{\xi,\nu})$.

Proof. Let $\lambda \otimes \xi$ be the action of $A \times M$ on $V^{\bar{N}}$. By the theorem of the highest weight, $\langle\lambda, \alpha\rangle \leqq 0$ for α a positive restricted root. Set $\gamma=-\lambda+\rho+i\nu$. Then Kostant's theorem implies $U(\mathfrak{g})\cdot 1_\gamma|_K$ is the space of all K-finite functions on K constant on the left cosets of M.

Now if $f \in E_{V,\nu}(H_\xi)$, $f\neq 0$, then $f=E_{V,\nu}(v)$, $v \in V^{\bar{N}}$, $v\neq 0$. Thus $f=1_\gamma\cdot\alpha(v)$. Now if $k \in K$, $k\cdot 1_\gamma=1_\gamma$. Thus since $G=\bar{N}AK$ we see that $G\cdot 1_\gamma=\bar{N}A1_\gamma$. On the other hand $\bar{N}A\alpha(v)\subset R^*(\alpha(v))$ ($R^*=R-\{0\}$). Thus if $H_e^{\xi,\nu}$ is the smallest closed G-invariant subspace of $H^{\xi,\nu}$ containing f then $(G\cdot 1_\gamma)\alpha(v)\subset H_e^{\xi,\nu}$. Thus by taking derivatives $(U(\mathfrak{g})\cdot 1_\gamma)\alpha(v)\subset H_e^{\xi,\nu}$. Taking the uniform closure of $(U(\mathfrak{g})1_\gamma)\alpha(v)|_K$ we have $C^\infty(K/M)\cdot\alpha(v)|_K\subset H_e^{\xi,\nu}|_K$ by the Peter-Weyl theorem (here we have identified $C^\infty(K/M)$ with the space of all complex valued C^∞ functions on K which are constant on the left cosets of M). Let $C^\infty(K; \xi)=\{f: K\to H_\xi \mid f(km)=\xi(m)^{-1}f(h)$ for $k \in K$, $m \in M$ and such that f is a C^∞ map of $K\to H_\xi\}$. A partition of unity argument shows that the linear span of $C^\infty(K/M)\cdot K\alpha(v)|_K=C^\infty(K, \xi)$. Thus $H_e^{\xi,\nu}$ is dense in $H^{\xi,\nu}$. Hence $H_e^{\xi,\nu}=H^{\xi,\nu}$. Q.E.D.

THEOREM 3.3. *Let* (ξ, H_ξ) *be an irreducible unitary representation of M. Let V be an extension of* ξ. *Let m be the multiplicity of* ξ *in V as an M-module. If* $\nu \in \mathfrak{a}'$ *then* $H^{\xi,\nu}$ *is a direct sum* $H_1 \oplus\cdots\oplus H_r$ *of irreducible unitary subrepresentations of* $H^{\xi,\nu}$ *and* $r\leqq m$.

Proof. Let $E_{V,\nu}: V\to H^{\xi,\nu}$ be defined as in §2. Suppose that $H^{\xi,\nu}$ is not irreducible. Let $U\subset H^{\xi,\nu}$ be a proper nonzero invariant, closed subspace of $H^{\xi,\nu}$. Let $P: H^{\xi,\nu}\to U$ be the corresponding projection. If $P(E_{V,\nu}(V^{\bar{N}}))=0$ then $E_{V,\nu}(V^{\bar{N}})\subset U^\perp$ (the orthogonal complement of U in $H^{\xi,\nu}$). Theorem 3.2 implies that $U^\perp=H^{\xi,\nu}$; this would imply $U=(0)$. Thus $P(E_{V,\nu}(V^{\bar{N}}))\neq 0$. Similarly $(I-P)(E_{V,\nu}(V^{\bar{N}}))\neq 0$. Furthermore if $f \in P(E_{V,\nu}(V^{\bar{N}}))$, $f\neq 0$ (resp. $f \in (I-P)(E_{V,\nu}(V^{\bar{N}}))$, $f\neq 0$), then f is a cyclic vector for U (resp. f is a cyclic vector for $U^\perp$). Continuing this process we find, after (say) k steps, $H^{\xi,\nu}=U_1\oplus\cdots\oplus U_p$, U_i invariant, closed, proper and now zero and U_i, U_j orthogonal for $i\neq j$. Furthermore $P_i(E_{V,\nu}(V^{\bar{N}}))\neq 0$ ($P_i: H^{\xi,\nu}\to U_i$ the corresponding projection) and if $f \in P_i(E_{V,\nu}(V^{\bar{N}}))$, $f\neq 0$, then f is a cyclic vector for U_i. Now Frobenius reciprocity implies that the dimension of the space of all K-homomorphisms of $V|_K\to H^{\xi,\nu}|_K$ is equal to m. Since

$$A_i = P_i\circ E_{V,\nu}: V\to H^{\xi,\nu}|_K$$

is a K-homomorphism and $A_1, \ldots, A_p$ are linearly independent we see that $p \leqq m$. This proves the result.

We indicate that Theorem 3.3 is generally applicable to connected semisimple Lie groups with finite-dimensional faithful representations. This follows from Theorem 2.1 and the following result of Bruhat [2].

THEOREM 3.4. *Let* (ξ, H_ξ) *be an irreducible unitary representation of* M. *Let* $\nu \in a'$. *Then* $(\pi_{\xi,\nu}, H^{\xi,\nu})$ *is unitarily equivalent with* $(\pi_{\xi^\sigma,\nu^\sigma}, H^{\xi^\sigma,\nu^\sigma})$ *for* $\sigma \in W(A)$.

4. **Complex semisimple Lie groups.** We assume in this section that G is a connected complex semisimple Lie group. Then K is a compact form of G, M is a maximal torus of K, MA is a Cartan subgroup of G. $W(A)$ acts on MA as the Weyl group of G relative to MA. The irreducible unitary representations of M are just the characters of M.

THEOREM 4.1. *Let G be a connected complex semisimple Lie group. Then every member of the full principal series of G is irreducible.*

Proof. Let ξ be a character of M and let $\nu \in a'$. Then if $\sigma \in W(A)$, $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is equivalent with $(\pi_{\xi^\sigma,\nu^\sigma}, H^{\xi^\sigma,\nu^\sigma})$. We will therefore have proved the theorem if we prove that $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible for ξ in the negative Weyl chamber of A. Let V be the holomorphic, finite-dimensional, irreducible representation of G with lowest weight ξ. Then V is an extension of ξ. Furthermore the theorem of the highest weight implies that the multiplicity of ξ in V as an M-module is 1. Theorem 3.3 now implies that $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible.

5. *$SL(n, R)$*. Let $G = SL(n, R)$ and $K = SO(n)$. We take A to be the set of all diagonal matrices of G with positive entries. M is the group of all diagonal elements of G with entries ± 1 on the diagonal. N is the group of all upper triangular matrices with ones on the diagonal. $W(A)$ acts on M by permuting the entries along the diagonal. If $m = \operatorname{diag}(m_1, \ldots, m_n)$ is in M. Then set $\varepsilon_0(m) = 1$, $\varepsilon_i(m) = m_i$, $i = 1, \ldots, n-1$. Then every nontrivial unitary character of M is of the form $\varepsilon_{i_1}, \ldots, \varepsilon_{i_r}$, $1 \leqq i_1 < \cdots < i_r \leqq n-1$. Set $\varepsilon_n = \varepsilon_1 \cdots \varepsilon_{n-1}$. Then $W(A)$ acts by permuting $\varepsilon_1, \ldots, \varepsilon_n$. We consider the representations V^i, $i = 0, \ldots, n-1$, where V^0 is the trivial representation of G. V^1 is the standard (matrix) action of $SL(n, R)$ on C^n. $\Lambda^i V^1 = V^i$ is the ith Grassman product of the representation V^1. Let $e_1, \ldots, e_n$ be the standard basis of C^n. Then if $m \in M$, $m \cdot e_i = \varepsilon_i(m) e_i$. Thus in general as an M-module V^k splits into a direct sum

$$
\begin{aligned}
&V^0 = 1, \\
(*) \qquad &V^k = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n-1} \varepsilon_{i_1} \cdots \varepsilon_{i_k} \\
&\qquad + \sum_{1 \leqq j_1 < \cdots < j_{n-k} \leqq n-1} \varepsilon_{j_1} \cdots \varepsilon_{j_{n-k}} \quad \text{for } n-1 \geqq k > 0.
\end{aligned}
$$

We note that every representation of M appears exactly once except in the case $n = 2k$, $V = V^k$. In this case every representation of M appears exactly twice.

THEOREM 5.1. (1) *If n is odd, every element of the full principal series for $SL(n, R)$ is irreducible.*

(2) *If n is even then if $\xi=\varepsilon_{i_1}\cdots\varepsilon_{i_j}$, $1\leqq i_1<\cdots<i_j\leqq n-1$ and $j\neq n/2$ and if $\nu\in\mathfrak{a}'$ then $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible. If $j=n/2$, and if $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is reducible then $H^{\xi,\nu}=H_1\oplus H_2$, unitary direct sum, H_i irreducible unitary representations of G.*

Proof. Up to the action of the Weyl group $W(A)$ we may assume (using Theorem 3.4) that $\xi=\xi_r=\varepsilon_0\varepsilon_1\cdots\varepsilon_r$, $r=0,\ldots,n-1$. If $r=0$ then ξ_0 is the action of M on $(V^0)^{\bar N}$ if $r\neq 0$ then ξ_r is the action of M on $(V^{n-r})^{\bar N}$. (1), (2) now follow from (*) above and Theorem 3.3.

Stein and Knapp [6] have shown that if $n=2k$, $\xi=\varepsilon_1\varepsilon_3\cdots\varepsilon_{n-1}$ and $\sigma=(12)(34)\cdots(n-1, n)\in W(A)$ then if $\nu\in\mathfrak{a}'$ and $\nu^\sigma=\nu$ then $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is reducible.

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