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ARCWISE CONNECTEDNESS OF SEMIAPOSYNDETIC PLANE CONTINUA

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Abstract. In a recent paper, the author proved that if a compact plane continuum M contains a finite point set F such that, for each point x in $M-F$, M is semi-locally-connected and not aposyndetic at x, then M is arcwise connected. The primary purpose of this paper is to generalize that theorem. Semiaposyndesis, a generalization of semi-local-connectedness, is defined and arcwise connectedness is established for certain semiaposyndetic plane continua.

A compact plane continuum which is semi-locally-connected may fail to be arcwise connected [9]. F. Burton Jones has conjectured that although the arcwise connectedness theorem for locally connected continua does not extend to semi-locally-connected spaces, there are general arcwise connectedness theorems for nonlocally connected plane continua [7]. According to Jones, a continuum M is said to be *aposyndetic* at a point p of M with respect to a point q of $M-\{p\}$ if there exist an open set U and a continuum H in M such that $p \in U \subset H \subset M-\{q\}$. A continuum M is said to be *aposyndetic* at a point p if for each point q in $M-\{p\}$, M is aposyndetic at p with respect to q. Recent results indicate that if a compact plane continuum M contains a finite point set F such that, for each point x in $M-F$, M is semi-locally-connected and not aposyndetic at x, then M is arcwise connected [1, Theorem 9]. In fact, plane continua with these properties are hereditarily arcwise connected [3, Theorem 1]. Note that since M is semi-locally-connected at each point of $M-F$ and is not aposyndetic at any point of $M-F$, for each point x of $M-F$ there exists a point y of F such that M is not aposyndetic at x with respect to y [6, Theorem 0].

There are many non-locally-connected arcwise connected plane continua which do not have the properties stated in the hypothesis of this theorem. For example the Cantor cone (the upper semicontinuous decomposition of the topological product of the unit interval [0, 1] and the Cantor discontinuum C in which the set $0 \times C$ is a point) is arcwise connected and fails to have the "semi-local-connectedness at all but a finite number of points" property.

DEFINITION. Let p be a point of a continuum M. M is said to be *semiaposyndetic* at p if for each point q of $M-\{p\}$, M is aposyndetic at p with respect to q or M is

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aposyndetic at q with respect to p. A continuum is said to be *semiaposyndetic* if it is semiaposyndetic at each of its points.

In this paper the following theorem is established. If a compact plane continuum M contains a finite point set F such that, for each point x in $M-F$, M is semiaposyndetic at x and is not aposyndetic at x with respect to some point of F, then M is arcwise connected. Since a continuum M is semi-locally-connected at a point p of M only if M is semiaposyndetic at p, this is a generalization of the preceding theorem. Note that the Cantor cone has the properties stated in this more general theorem.

Throughout this paper S is the set of points of a simple closed surface (that is, a 2-sphere). For definitions of unfamiliar terms and phrases see [8] and [10].

THEOREM 1. *Suppose that M is a continuum in S and the set of points of M at which M is not semiaposyndetic is countable. Then if x and y are distinct points of M and M is not aposyndetic at x with respect to y, there exists an arc J in M with extremities x and y such that, for each point z of $J-\{y\}$, M is not aposyndetic at z with respect to y.*

Proof. Since M is not aposyndetic at x with respect to y, the set $S-M$ is folded [5, Theorem 2]. In fact, $S-M$ is folded around x with respect to y [2]. It follows that there exist two monotone descending sequences of circular regions $U_1, U_2, U_3, \dots$ and $V_1, V_2, V_3, \dots$ in S centered on and converging to x and y respectively, such that $\mathrm{Cl}\, U_1 \cap \mathrm{Cl}\, V_1 = \varnothing$ ($\mathrm{Cl}\, U_1$ is the closure of U_1), and there exists a sequence of mutually exclusive sets $X_1, X_2, X_3, \dots$ in $S-M$ having the following properties. For each positive integer i, the set X_i is the union of two intersecting arc-segments (open arcs) I_i and T_i such that (1) $I_i \cap T_i$ is connected, (2) I_i is contained in $\mathrm{Bd}\, U_i$ ($\mathrm{Bd}\, U_i$ is the boundary of U_i) and has endpoints a_i and b_i in M, (3) T_i is contained in $S-\mathrm{Cl}\,(V_i \cup U_{i+1})$ and has two distinct endpoints in $\mathrm{Bd}\, V_i - M$, and (4) $T_i \cup \mathrm{Bd}\, V_i$ contains a simple closed curve S_i which separates a_i from the set $\{b_i, x, y\}$ in S. Note that for each positive integer i, $\mathrm{Cl}\, T_i$ separates the a_i-component of $M-V_i$ from the x-component of $M-V_i$ in $S-V_i$.

For each positive integer i, let A_i be the a_i-component of $M-V_i$. Let A be the limiting set of $A_1, A_2, A_3, \dots$. The set A is a subcontinuum of M which contains x and y [8, Theorem 58, p. 23]. There exists a subsequence of $A_1, A_2, A_3, \dots$ which converges to a continuum containing x and y in M [8, Theorem 59, p. 24]. Assume without loss of generality that $A_1, A_2, A_3, \dots$ converges to A. Note that for each point z of $A-\{y\}$, M is not aposyndetic at z with respect to y. For each positive integer i, define Y_i to be the x-component of $A-V_i$. Let Y be the limiting set of $Y_1, Y_2, Y_3, \dots$. The set Y is a subcontinuum of A which contains x and y.

Assume that Y is not locally connected. It follows that Y is not connected im kleinen at some point v in $Y-\{x, y\}$. There exist two circular regions T and W in S centered on v such that (1) $T \supset \mathrm{Cl}\, W$, (2) $\{x, y\} \cap \mathrm{Cl}\, T = \varnothing$, and (3) there exists a sequence of mutually exclusive continua $H_1, H_2, H_3, \ldots$ in $Y \cap (\mathrm{Cl}\, T - W)$ such that each continuum meets both Bd T and Bd W and the sequence converges to a continuum Z [8, Theorem 66 (Proof), p. 124]. It is not necessary here to insist that $H_1, H_2, H_3, \ldots$ be in different components of $Y \cap (\mathrm{Cl}\, T - W)$, since Y is a continuum of convergence of M. Let q be a point of Z which is in $T - \mathrm{Cl}\, W$. Let $q_1, q_2, q_3, \ldots$ be a sequence of points which converges to q such that, for each positive integer n, $q_n \in H_n \cap (T - \mathrm{Cl}\, W)$. Let $Q_1, Q_2, Q_3, \ldots$ be a sequence of mutually exclusive circular regions in S which converges to q such that, for each positive integer n, the region Q_n is centered on q_n and Cl Q_n is contained in $T - \mathrm{Cl}\, W$. For each positive integer n, there exists an integer i such that $Y_i \cap Q_n \neq \varnothing$. It follows that there exists a sequence of mutally exclusive continua $J_1, J_2, J_3, \ldots$ in $Y \cap (\mathrm{Cl}\, T - W)$ which converges to a continuum I containing q in Y such that, for each positive integer n, J_n meets Bd $(T - W)$ and there exists an integer i such that $J_n \subset Y_i$. Note that I meets Bd $(T - W)$. Since M is semiaposyndetic at all except at most countably many of its points, there exists a point p and two circular regions R and E $(R \supset \mathrm{Cl}\, E)$ centered on q in $T - W$ such that (1) the point p is in $(R - \mathrm{Cl}\, E) \cap I$, (2) M is semiaposyndetic at p, and (3) there exists a sequence of mutually exclusive continua $F_1, F_2, F_3, \ldots$ such that, for each positive integer n, there exists an integer i such that F_n is in $J_i \cap (\mathrm{Cl}\, R - E)$, F_n meets both Bd R and Bd E, and the sequence $F_1, F_2, F_3, \ldots$ converges to a continuum in Y which contains p.

Assume without loss of generality that the sequence $F_1, F_2, F_3, \ldots$ is such that for each positive integer n, there exist arc-segments R_n and E_n such that (1) $R_n \subset \mathrm{Bd}\, R$, (2) $E_n \subset \mathrm{Bd}\, E$, and (3) each arc-segment meets $F_1, F_2, F_3, \ldots$ only in F_{2n} and has one endpoint in F_{2n-1} and the other endpoint in F_{2n+1}. Let $p_1, p_2, p_3, \ldots$ be a sequence of points which converges to p such that, for each positive integer n, the point p_n is in $F_{2n} \cap (R - \mathrm{Cl}\, E)$. Let $P_1, P_2, P_3, \ldots$ be a sequence of circular regions in S such that, for each positive integer n, the region P_n is centered on P_n and Cl P_n does not meet $F_{2n-1} \cup F_{2n+1} \cup R_n \cup E_n$. Note that the regions of the sequence $P_1, P_2, P_3, \ldots$ are mutually exclusive, since $F_{2n-1} \cup F_{2n+1} \cup R_n \cup E_n$ separates Cl P_n from Cl P_i, for $i \neq n$ [8, Theorem 28, p. 156], and converge to p.

There exist subsequences $V_{n_1}, V_{n_2}, V_{n_3}, \ldots$ of $V_1, V_2, V_3, \ldots$ and $A_{n_1}, A_{n_2}, A_{n_3}, \ldots$ of $A_1, A_2, A_3, \ldots$ such that Cl $V_{n_1} \cap \mathrm{Cl}\, R = \varnothing$ and for each positive integer k, the set $F_{2k-1} \cup F_{2k} \cup F_{2k+1} \cup F_{2k+2}$ is in the x-component of $Y - V_{n_k}$ and A_{n_k} meets P_k and P_{k+1}.

For each positive integer k, let e_k be a point of $A_{n_k} \cap \mathrm{Bd}\, P_k$ and let g_k be a point of $A_{n_k} \cap \mathrm{Bd}\, P_{k+1}$. For each positive integer k, there exists an arc-segment B_k belonging to $\{E_k, R_k\}$ such that $B_k \cup$ (x-component of $Y - V_{n_k}$) separates e_k from g_k in S [8, Theorem 20, p. 173]. Let s be a limit point of the sequence of arc-segments $B_1, B_2, B_3, \ldots$.

M is not aposyndetic at s with respect to p. To see this assume M is aposyndetic at s with respect to p. It follows that there exist mutually exclusive circular regions

U, V in S and a continuum H in M such that p is in U and $s \in M \cap V \subset H \subset S - \mathrm{Cl}\, U$. There exists a positive integer k such that $P_k \cup P_{k+1} \subset U$ and $\mathrm{Cl}\, B_k \subset V$. Let K denote the x-component of $Y - V_{n_k}$. Define D to be the complementary domain of $K \cup B_k$ which contains e_k. Since $\mathrm{Cl}\, T_{n_k}$ separates A_{n_k} from K in $S - V_{n_k}$, T_{n_k} meets both $\mathrm{Bd}\, P_k \cap D$ and $\mathrm{Bd}\, P_{k+1} \cap (S-(K \cup D))$. Let z be a point of $D \cap \mathrm{Bd}\, P_k \cap T_{n_k}$. The boundary of the component of $D \cap U$ which contains z is contained in $K \cup \mathrm{Bd}\, U$. It follows that T_{n_k} meets $D \cap \mathrm{Bd}\, U$. Let w be a point of $(S-D) \cap \mathrm{Bd}\, P_{k+1} \cap T_{n_k}$. The boundary of the w-component of $U \cap (S-(K \cup B_k))$ is contained in $K \cup \mathrm{Bd}\, U$. Therefore T_{n_k} must also meet $(S-(K \cup D)) \cap \mathrm{Bd}\, U$. Assume without loss of generality that T_{n_k} meets $D \cap \mathrm{Bd}\, U$ before it meets $(S-(K \cup D)) \cap \mathrm{Bd}\, U$ with respect to the linear order on T_{n_k}. Let f be the first point of $T_{n_k} \cap (S-(K \cup D)) \cap \mathrm{Bd}\, U$ and let h be the last point of $T_{n_k} \cap D \cap \mathrm{Bd}\, U$ which precedes f with respect to the order on T_{n_k}. Let X denote the subarc of T_{n_k} which has endpoints h and f. Note that $X \cap \mathrm{Cl}\, U = \{h, f\}$. Let C_1 and C_2 be the mutually exclusive arc-segments in $\mathrm{Bd}\, U$ which have endpoints h and f. Since $K \cup B_k$ separates h from f in S, the set $(K \cup B_k) - U$ contains a continuum C which separates h from f in $S - U$ [8, Theorem 27, p. 177]. For $i=1$ and 2, there exists a point c_i in $C \cap C_i$. The points c_1 and c_2 are contained in distinct components of $C - B_k$ [8, Theorem 28, p. 156]. For $i=1$ and 2, let d_i be a point of $\mathrm{Cl}\, B_k \cap \mathrm{Cl}\,(c_i\text{-component of } C - B_k)$. The set ($\theta$-curve) $X \cup \mathrm{Bd}\, U$ separates d_1 from d_2 in S [8, Theorem 28, p. 156]. $X \cup \mathrm{Bd}\, U$ is contained in $S - H$ and the set $\{d_1, d_2\}$ is contained in H. Since H is connected, this is a contradiction.

Suppose that M is aposyndetic at p with respect to s. There exist mutually exclusive circular regions Q and O in S and a continuum F in M such that s is in Q and $p \in M \cap O \subset F \subset S - Q$. There exists a positive integer j such that $P_j \cup P_{j+1} \subset O$ and $B_j \subset Q$. Let L be the set $B_j \cup$ (x-component of $Y - V_{n_j}$). Either the complementary domain G of L which contains e_j does not meet $\mathrm{Cl}\, V_{n_j}$ or the complementary domain of L which contains g_j does not meet $\mathrm{Cl}\, V_{n_j}$ since V_{n_j} is a connected subset of $S - L$. Assume without loss of generality that $G \cap \mathrm{Cl}\, V_{n_j} = \varnothing$. Let r be a limit point of the e_j-component of ($\mathrm{Bd}\, P_j \cap G$) which is not in G. Since r is contained in the x-component of $Y - V_{n_j}$, the simple closed curve S_{n_j} separates e_j from r in S. Furthermore $B_j \cup (T_{n_j} \cap G)$ separates r from e_j in S. This follows from the contrapositive of [8, Theorem 20, p. 173] and the observations that, if B is an arc in B_j that contains $\mathrm{Cl}\, S_{n_j} \cap \mathrm{Bd}\, G$, then (1) $B \cup (S_{n_j} \cap G)$ and $B \cup (S_{n_j} \cap (S-G))$ are closed and compact point sets with a connected intersection and their union separates r from e_j in S, (2) $B \cup (S_{n_j} \cap (S-G))$ does not separate r from e_j in S, and (3) $B_j \cup (T_{n_j} \cap G) \supset B \cup (S_{n_j} \cap G)$. The continuum F contains the set $\{r, e_j\}$ and does not meet $B_j \cup T_{n_j}$. This is clearly impossible. Hence M is not aposyndetic. at p with respect to s. But this contradicts the fact that M is semiaposyndetic at p. It follows that Y is a locally connected subcontinuum of A and there exists an arc J lying in Y with endpoints x and y [8, Theorem 13, p. 91]. Since $J \subset Y \subset A$, for each point z of $J - \{y\}$, M is not aposyndetic at z with respect to y.

THEOREM 2. *If a compact plane continuum M contains a finite point set F such that, for each point x in $M-F$, M is semiaposyndetic at x and there exists a point y of F such that M is not aposyndetic at x with respect to y, then M is arcwise connected.*

Proof. For each point y of F, let L_y be the set consisting of y and all points x of $M-\{y\}$ such that M is not aposyndetic at x with respect to y. For each point y of F, the set L_y is a continuum [6, Theorem 3] and is (according to Theorem 1) arcwise connected. $M=\bigcup_{y\in F} L_y$. The continuum M being the union of a finite number of arcwise connected continua is arcwise connected.

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