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ON THE C^*-ALGEBRA OF TOEPLITZ OPERATORS ON THE QUARTER-PLANE

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Abstract. Using the device of the tensor product of C^*-algebras, we study the C^*-algebra generated by the Toeplitz operators on the quarter-plane. We obtain necessary and sufficient conditions for such an operator to be Fredholm, but show in this case that not all such operators are invertible.

During the last fifteen years the properties of a certain class of operators on Hilbert space have been much studied (cf. [11]). This class can be described in various ways: the operators defined by Toeplitz matrices, that is, matrices constant on the diagonals; the compressions of multiplication operators to the Hardy space for the disk; or the operators defined by the classical integral equation of Wiener and Hopf on the half-line or its discrete analogue on the nonnegative integers. Moreover, the study of these operators is important in various areas of physics and probability (cf. [18], [17], [15]) and more recently, in examining the convergence of certain difference schemes for solving partial differential equations (cf. [20]). Despite the importance of the corresponding classes of operators for several variables, results on these classes are rather fragmented and incomplete.

One generalization to several variables, which we shall refer to as the "half-plane case," has been studied by Gohberg and Gol'denšteĭn [13], [14] and more recently by Coburn, Douglas, Schaeffer and Singer [7], [8]. In this case it can be shown that the operators can be represented as ordinary Toeplitz operators involving a parameter. A significantly different situation is encountered in the "quarter-plane case." Previous results on this case have been obtained by Simonenko [23], Osher [21], and Malyšev [19]. (We shall describe these authors' results directly.)

Our object in this note is to show how the study of certain questions concerning the quarter-plane case can be reduced to the classical case using the device of the tensor product of C^*-algebras. In particular, with this technique we are able to prove that a Toeplitz operator on the quarter-plane is a Fredholm operator if and only if its symbol does not vanish and has topological index zero. Moreover, while

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the analytical index of such an operator is always zero, it is *not* the case that it must be invertible. We are, however, able to show that the set of operators which are invertible forms a dense open set in the collection of Toeplitz operators on the the quarter-plane which are Fredholm operators.

Our results tend to reinforce what has been found to be true in much of the recent work on the subject; namely that it is fruitful to study an algebra of Toeplitz operators even if one seeks results about just one operator.

This note is not meant to exhaust the possible applications of our methods to the study of Toeplitz operators but rather only to indicate their scope and power. In particular, our techniques are not restricted to the case of two variables but can be used to reduce questions about the n-variable case to questions concerning the $(n-1)$-variable case. (We warn the reader that the results described above, however, are not valid as stated for the n-variable case.) We shall briefly sketch at the end of this note some applications of our techniques to the case of more than two variables.

Before proceeding we want to make a couple of comments. Firstly, our techniques might be viewed as being overly abstract and indeed the paper could have been written without any mention of tensor products or exact sequences. Our choice of context or language is justified, we feel, by the fact that our arguments easily extend to more general situations and, in fact, suggest in many instances the appropriate questions, which leads us to our second remark. Most of our efforts in this note are directed to the question of determining when a Toeplitz operator on the quarter-plane is a Fredholm operator and the relation between this and the operators' being invertible. It is not entirely clear from many points of view that this is the appropriate question to ask. We will have more to say about this in our remarks at the end.

Since we reduce the quarter-plane case to the classical case we begin by recalling the definition and results about Toeplitz operators on the circle which we will need. Let $\boldsymbol{T}$ be the circle group, $L^2(\boldsymbol{T})$ be the complex Lebesgue space with respect to the normalized Lebesgue measure on $\boldsymbol{T}$, $H^2(\boldsymbol{Z}_+)$ be the corresponding Hardy space of functions in $L^2(\boldsymbol{T})$ with Fourier transform (series) supported on the semigroup $\boldsymbol{Z}_+$ of nonnegative integers, and P be the projection of $L^2(\boldsymbol{T})$ onto $H^2(\boldsymbol{Z}_+)$. For φ a continuous function on $\boldsymbol{T}$ we define the Toeplitz operator T_φ with symbol φ on $H^2(\boldsymbol{Z}_+)$ such that $T_\varphi f = P(\varphi f)$ for f in $H^2(\boldsymbol{Z}_+)$. If we let $\mathscr{A}(\boldsymbol{Z}_+)$ denote the C^*-algebra generated by the collection of Toeplitz operators with continuous symbol, then $\mathscr{A}(\boldsymbol{Z}_+)$ contains the algebra $\mathscr{L}\mathscr{C}(\boldsymbol{Z}_+)$ $(=\mathscr{L}\mathscr{C}(H^2(\boldsymbol{Z}_+)))$ of all compact operators on $H^2(\boldsymbol{Z}_+)$ as its commutator ideal and the map $\varphi \to T_\varphi + \mathscr{L}\mathscr{C}(\boldsymbol{Z}_+)$ defines a *-isometrical isomorphism of $C(\boldsymbol{T})$ onto $\mathscr{A}(\boldsymbol{Z}_+)/\mathscr{L}\mathscr{C}(\boldsymbol{Z}_+)$ [4], [5]. Thus, if j denotes the inclusion of $\mathscr{L}\mathscr{C}(\boldsymbol{Z}_+)$ in $\mathscr{A}(\boldsymbol{Z}_+)$, then there exists a *-homomorphism π from $\mathscr{A}(\boldsymbol{Z}_+)$ onto $C(\boldsymbol{T})$ such that

$$0 \longrightarrow \mathscr{L}\mathscr{C}(\boldsymbol{Z}_+) \xrightarrow{j} \mathscr{A}(\boldsymbol{Z}_+) \xrightarrow{\pi} C(\boldsymbol{T}) \longrightarrow 0$$

is a short exact sequence. (These facts can be used to study the question of which T_φ are invertible [10].)

We now consider Toeplitz operators on the quarter-plane. Let $\boldsymbol{T}^2$ denote the torus group and $L^2(\boldsymbol{T}^2)$ be the complex Lebesgue space with respect to the normalized Lebesgue measure on $\boldsymbol{T}^2$. The Fourier transform of a function in $L^2(\boldsymbol{T}^2)$ is a function on the character group $\boldsymbol{Z}^2$ of $\boldsymbol{T}^2$. We let $H^2(\boldsymbol{Z}^2)$ denote the subspace of functions in $L^2(\boldsymbol{T}^2)$ with Fourier transform supported on the semigroup $\boldsymbol{Z}_+^2$ of pairs of nonnegative integers. (This subspace can be identified with the Hardy space of analytic functions on the polydisk (cf. [22]) but we make no use of that in this note.) If P_2 denotes the projection of $L^2(\boldsymbol{T}^2)$ onto $H^2(\boldsymbol{Z}_+^2)$, then the Toeplitz operator W_ψ on $H^2(\boldsymbol{T}^2)$ with symbol ψ in $C(\boldsymbol{T}^2)$ is defined by $W_\psi f = P_2(\psi f)$ for f in $H^2(\boldsymbol{Z}_+^2)$.

If $\mathscr{A}(\boldsymbol{Z}_+^2)$ denotes the C^*-algebra generated by the Toeplitz operators on $H^2(\boldsymbol{Z}_+^2)$ with continuous symbol, and $\mathscr{C}(\boldsymbol{Z}_+^2)$ denotes the commutator ideal of $\mathscr{A}(\boldsymbol{Z}_+^2)$, then as in the case of the circle the map $\psi \to W_\psi + \mathscr{C}(\boldsymbol{Z}_+^2)$ defines a $*$-isometrical isomorphism of $C(\boldsymbol{T}^2)$ onto the quotient algebra $\mathscr{A}(\boldsymbol{Z}_+^2)/\mathscr{C}(\boldsymbol{Z}_+^2)$ (cf. [7]). This map is not very useful since $\mathscr{C}(\boldsymbol{Z}_+^2)$ contains many noncompact operators. Thus to proceed in this case we need to do something else. We shall identify the algebra $\mathscr{A}(\boldsymbol{Z}_+^2)$ as the tensor product of simpler algebras.

We begin by first writing $H^2(\boldsymbol{Z}_+^2)$ as the tensor product of two copies of $H^2(\boldsymbol{Z}_+)$. Recall that the Fourier transform takes $H^2(\boldsymbol{Z}_+^2)$ onto $l^2(\boldsymbol{Z}_+^2)$ and $H^2(\boldsymbol{Z}_+)$ onto $l^2(\boldsymbol{Z}_+)$; the correspondence $\boldsymbol{Z}_+^2 = \boldsymbol{Z}_+ \times \boldsymbol{Z}_+$ defines an isomorphism $l^2(\boldsymbol{Z}_+^2) = l^2(\boldsymbol{Z}_+) \otimes l^2(\boldsymbol{Z}_+)$ and hence $H^2(\boldsymbol{Z}_+^2) = H^2(\boldsymbol{Z}_+) \otimes H^2(\boldsymbol{Z}_+)$. If the variables on $\boldsymbol{T}^2$ are denoted by z and w, then $W_z = T_z \otimes I$ and $W_w = I \otimes T_w$ and since $\mathscr{A}(\boldsymbol{Z}_+^2)$ is the C^*-algebra generated by the operators W_z and W_w, then $\mathscr{A}(\boldsymbol{Z}_+^2) = \mathscr{A}(\boldsymbol{Z}_+) \otimes \mathscr{A}(\boldsymbol{Z}_+)$. We record this in the following

PROPOSITION 1. *Corresponding to the canonical identification $H^2(\boldsymbol{Z}_+^2) = H^2(\boldsymbol{Z}_+) \otimes H^2(\boldsymbol{Z}_+)$ is the identification $\mathscr{A}(\boldsymbol{Z}_+^2) = \mathscr{A}(\boldsymbol{Z}_+) \otimes \mathscr{A}(\boldsymbol{Z}_+)$.*

(Recall that if $\mathscr{A}$ and $\mathscr{B}$ are type I C^*-algebras, there is a unique pre-C^*-norm (cf. [24], [25]) on the algebraic tensor product of $\mathscr{A}$ and $\mathscr{B}$; we will always denote by $\mathscr{A} \otimes \mathscr{B}$ the C^*-algebra obtained by completing the algebraic tensor product with respect to this norm.)

The following proposition is well known in homological algebra (cf. [3]) but we have been unable to find a reference for its extension to C^*-algebras.

PROPOSITION 2. *If $0 \to \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \to 0$ is a short exact sequence of C^*-algebras such that β has a continuous cross-section and $\mathscr{D}$ is a C^*-algebra, then the sequence*

$$0 \longrightarrow \mathscr{A} \otimes \mathscr{D} \xrightarrow{\alpha \otimes 1} \mathscr{B} \otimes \mathscr{D} \xrightarrow{\beta \otimes 1} \mathscr{C} \otimes \mathscr{D} \longrightarrow 0$$

is short exact.

Proof. If T is an operator in $\mathscr{A}\otimes\mathscr{D}$ of the form $T=\sum_{i=1}^n A_i\otimes D_i$, where $A_1, A_2, \ldots, A_n$ and $D_1, D_2, \ldots, D_n$ are operators in $\mathscr{A}$ and $\mathscr{D}$ respectively, then $(\beta\otimes 1)(\alpha\otimes 1)(T)=\sum_{i=1}^n \beta\alpha(A_i)\otimes D_i=0$ since $\beta\alpha=0$. Since such operators are dense in $\mathscr{A}\otimes\mathscr{D}$ and $(\beta\otimes 1)(\alpha\otimes 1)$ is continuous we have $\operatorname{ran}(\alpha\otimes 1)\subset\ker(\beta\otimes 1)$.

Suppose ρ is a continuous cross-section of β, that is, ρ is a bounded linear transformation from $\mathscr{C}$ to $\mathscr{B}$ such that $\beta\rho=1$. We want to show that if T in $\mathscr{B}\otimes\mathscr{D}$ is in the kernel of $\beta\otimes 1$, then T is also in the range of $\alpha\otimes 1$. Since $\alpha\otimes 1$ is an isometry, the range of $\alpha\otimes 1$ is a closed subspace of $\mathscr{B}\otimes\mathscr{D}$ and hence it is sufficient to exhibit an operator S in $\mathscr{A}\otimes\mathscr{D}$ such that $\|T-(\alpha\otimes 1)(S)\|$ is small. To this end let ε be positive and choose operators $B_1, B_2, \ldots, B_n$ in $\mathscr{B}$ and $D_1, D_2, \ldots, D_n$ in $\mathscr{D}$ such that $\|T-\sum_{i=1}^n B_i\otimes D_i\|<\varepsilon$. If we write

$$\sum_{i=1}^n B_i\otimes D_i = \sum_{i=1}^n (B_i-\rho\beta B_i)\otimes D_i+\sum_{i=1}^n \rho\beta B_i\otimes D_i,$$

then there exists A_i in $\mathscr{A}$ such that $\alpha(A_i)=B_i-\rho\beta B_i$ and

$$\begin{aligned}\left\|\sum_{i=1}^n \rho\beta B_i\otimes D_i\right\| &\leqq \|\rho\|\left\|\sum_{i=1}^n \beta B_i\otimes D_i\right\|\\ &\leqq \|\rho\|\left\|(\beta\otimes 1)\left\{T-\sum_{i=1}^n \alpha(A_i)\otimes D_i-\sum_{i=1}^n \rho\beta B_i\otimes D_i\right\}\right\|\\ &\leqq \|\rho\|\left\|T-\sum_{i=1}^n B_i\otimes D_i\right\| \leqq \|\rho\|\varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left\|T-(\alpha\otimes 1)\left(\sum_{i=1}^n A_i\otimes D_i\right)\right\| &= \left\|T-\sum_{i=1}^n B_i\otimes D_i+\sum_{i=1}^n \rho\pi B_i\otimes D_i\right\|\\ &\leqq \varepsilon+\|\rho\|\varepsilon = (1+\|\rho\|)\varepsilon\end{aligned}$$

which completes the proof by our previous remarks.

We need one more standard result (cf. [8]) concerning tensor products.

PROPOSITION 3. *If X is a compact Hausdorff space and $\mathscr{B}$ is a C^*-algebra, then $C(X)\otimes\mathscr{B}$ is naturally isomorphic to the C^*-algebra $C(X,\mathscr{B})$ of continuous functions from X to $\mathscr{B}$. Moreover, if $\sum_{i=1}^n f_i\otimes B_i$ in $C(X)\otimes\mathscr{B}$ corresponds to F in $C(X,\mathscr{B})$, then $F(x)=\sum_{i=1}^n f(x)B_i$.*

We consider now the application of these results to our problem. In the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{LC}(Z_+) \otimes \mathscr{LC}(Z_+) \arrow[r, "j \otimes 1"] \arrow[d, "1 \otimes j"'] & \mathscr{A}(Z_+) \otimes \mathscr{LC}(Z_+) \arrow[r, "\pi \otimes 1"] \arrow[d, "1 \otimes j"'] & C(T) \otimes \mathscr{LC}(Z_+) \arrow[r] \arrow[d, "1 \otimes j"'] & 0 \\
0 \arrow[r] & \mathscr{LC}(Z_+) \otimes \mathscr{A}(Z_+) \arrow[r, "j \otimes 1"] \arrow[d, "1 \otimes \pi"'] & \mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+) \arrow[r, "\pi \otimes 1"] \arrow[d, "1 \otimes \pi"'] & C(T) \otimes \mathscr{A}(Z_+) \arrow[r] \arrow[d, "1 \otimes \pi"'] & 0 \\
0 \arrow[r] & \mathscr{LC}(Z_+) \otimes C(T) \arrow[r, "j \otimes 1"] \arrow[d] & \mathscr{A}(Z_+) \otimes C(T) \arrow[r, "\pi \otimes 1"] \arrow[d] & C(T) \otimes C(T) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

all of the rows and columns are exact by our previous comments along with Proposition 2.

Consideration of the diagonal maps shows that the commutator ideal of $\mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+)$ is the subspace spanned by $\mathscr{A}(Z_+) \otimes \mathscr{LC}(Z_+)$ and $\mathscr{LC}(Z_+) \otimes \mathscr{A}(Z_+)$ and that the corresponding quotient algebra is $C(T) \otimes C(T)$ which is naturally isomorphic to $C(T^2)$. We are primarily interested, however, in the dual sequence

$$\begin{aligned} 0 \longrightarrow \mathscr{LC}(Z_+) \otimes \mathscr{LC}(Z_+) &\xrightarrow{\alpha} \mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+) \\ &\xrightarrow{\pi \otimes 1 \oplus 1 \otimes \pi} C(T) \otimes \mathscr{A}(Z_+) \oplus \mathscr{A}(Z_+) \otimes C(T) \end{aligned}$$

since $\mathscr{LC}(Z_+) \otimes \mathscr{LC}(Z_+)$ is naturally isomorphic to the algebra $\mathscr{LC}(Z_+^2)$ of all compact operators on $H^2(Z_+^2)$. After we have shown this sequence to be exact at $\mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+)$, the desired results on which Toeplitz operators are Fredholm will follow.

PROPOSITION 4. *The sequence*

$$\begin{aligned} \mathscr{LC}(Z_+) \otimes \mathscr{LC}(Z_+) &\xrightarrow{\alpha} \mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+) \\ &\xrightarrow{\pi \otimes 1 \oplus 1 \otimes \pi} C(T) \otimes \mathscr{A}(Z_+) \oplus \mathscr{A}(Z_+) \otimes C(T) \end{aligned}$$

is exact at its middle term.

Proof. The proof involves routine diagram chasing. If the operator T in $\mathscr{A}(Z_+) \otimes \mathscr{A}(Z_+)$ is in the kernel of $\pi \otimes 1 \oplus 1 \otimes \pi$, then T is in the kernel of both $\pi \otimes 1$ and $1 \otimes \pi$. Thus there exists an operator S in $\mathscr{LC}(Z_+) \otimes \mathscr{A}(Z_+)$ such that $(j \otimes 1)(S) = T$ and since $(j \otimes 1)(1 \otimes \pi)(S) = (1 \otimes \pi)(j \otimes 1)(S) = (1 \otimes \pi)(T) = 0$, it follows that there exists R in $\mathscr{LC}(Z_+) \otimes \mathscr{LC}(Z_+)$ such that $(1 \otimes j)(R) = S$ and hence $T = \alpha(R)$.

Before we can make use of this sequence we need to know more about the map $\pi \otimes 1 \oplus 1 \otimes \pi$.

LEMMA. *There exist *-homomorphisms γ_z and γ_w from $\mathscr{A}(\boldsymbol{Z}_+^2)$ onto $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$ such that $\gamma_z(W_\psi)=F$ and $\gamma_w(W_\psi)=G$, where $F(z)=T_{\psi(z,\cdot)}$, $G(w)=T_{\psi(\cdot,w)}$ and ψ is a continuous function on $\boldsymbol{T}^2$. Moreover, under the natural identification of $\mathscr{A}(\boldsymbol{Z}_+^2)$ with $\mathscr{A}(\boldsymbol{Z}_+) \otimes \mathscr{A}(\boldsymbol{Z}_+)$, $C(\boldsymbol{T}) \otimes \mathscr{A}(\boldsymbol{Z}_+)$ with $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$ and $\mathscr{A}(\boldsymbol{Z}_+) \otimes C(\boldsymbol{T})$ with $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$, then $\pi \otimes 1 = \gamma_z$ and $1 \otimes \pi = \gamma_w$.*

Proof. Since $\pi \otimes 1$ is a *-homomorphism from $\mathscr{A}(\boldsymbol{Z}_+) \otimes \mathscr{A}(\boldsymbol{Z}_+)$ onto $C(\boldsymbol{T}) \otimes \mathscr{A}(\boldsymbol{Z}_+)$ it induces a *-homomorphism γ_z from $\mathscr{A}(\boldsymbol{Z}_+^2)$ onto $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$ and we need only compute the action of γ_z on W_ψ for ψ in $C(\boldsymbol{T}^2)$. If $\varphi_1, \varphi_2, \ldots, \varphi_{2n}$ are continuous functions on $\boldsymbol{T}$, then $\psi(z, w)=\sum_{i=1}^n \varphi_i(z)\varphi_{n+i}(w)$ is a continuous function on $\boldsymbol{T}^2$. Moreover, since W_ψ corresponds to $\sum_{i=1}^n T_{\varphi_i} \otimes T_{\varphi_{i+n}}$ we have $\pi \otimes 1(\sum_{i=1}^n T_{\varphi_i} \otimes T_{\varphi_{i+n}}) = \sum_{i=n}^n \varphi_i \otimes T_{\varphi_{i+n}}$ and if we set $\gamma_z(W_\psi)=F$, then $F(z)=\sum_{i=1}^n \varphi_i(z)T_{\varphi_{i+n}}$. Lastly, since the functions of the form ψ are dense in $C(\boldsymbol{T}^2)$, the result follows.

We state the following reformulation of the preceding proposition as

THEOREM 1. *There exists a *-isometrical isomorphism α' from $\mathscr{LC}(\boldsymbol{Z}_+^2)$ into $\mathscr{A}(\boldsymbol{Z}_+^2)$ such that the sequence*

$$\mathscr{LC}(\boldsymbol{Z}_+^2) \xrightarrow{\alpha'} \mathscr{A}(\boldsymbol{Z}_+^2) \xrightarrow{\gamma_z \oplus \gamma_w} C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+)) \oplus C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$$

is exact at $\mathscr{A}(\boldsymbol{Z}_+^2)$.

COROLLARY. *If ψ is a continuous function on $\boldsymbol{T}^2$, then the Toeplitz operator W_ψ on $H^2(\boldsymbol{Z}_+^2)$ is a Fredholm operator if and only if ψ does not vanish and is homotopic to a constant. Moreover, the analytical index of such an operator is* 0(2).

Proof. An operator T_ψ is a Fredholm operator if and only if T_ψ is invertible in the quotient algebra $\mathscr{L}(H^2(\boldsymbol{Z}_+^2))/\mathscr{LC}(H^2(\boldsymbol{Z}_+^2))$. Since T_ψ is contained in the C^*-algebra $\mathscr{A}(\boldsymbol{Z}_+^2)$, then T_ψ is a Fredholm operator if and only if it is invertible in $\mathscr{A}(\boldsymbol{Z}_+^2)/\mathscr{LC}(\boldsymbol{Z}_+^2)$ and, therefore, if and only if $(\gamma_z \oplus \gamma_w)(T_\psi)$ is invertible in the range of $\gamma_z \oplus \gamma_w$. Since range $(\gamma_z \oplus \gamma_w)$ is a C^*-subalgebra of $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+)) \oplus C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$ it follows that T_ψ is a Fredholm operator if and only if both $\gamma_z(T_\psi)$ and $\gamma_w(T_\psi)$ are invertible in $C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+))$. This latter requirement is equivalent in view of the results on the classical case to asking that the functions $F_z(w)=\psi(z, w)$ and $G_w(z)=\psi(z, w)$ defined on $\boldsymbol{T}$ for z and w in $\boldsymbol{T}$ be nonvanishing and have winding number zero about the origin—that ψ be nonvanishing and homotopic to a constant.

(2) If the Fourier series of ψ is absolutely convergent, then Strang (Bull. Amer. Math. Soc. **76** (1970), 1303–1307) has obtained an explicit operator which is an inverse for W_ψ modulo the ideal of compact operators.

Since the collection of continuous functions on $\boldsymbol{T}^2$ which do not vanish and are homotopic to a constant form a connected set (e.g., they are the set of exponentials) and the analytical index is continuous, we see that the index of all such T_ψ must be 0.

Before continuing let us compare our result to that of Simonenko. In [23] Simonenko studies convolution operators on cones and as an application of his results is able to prove results similar to ours for the operators obtained from the Wiener-Hopf equations on the quarter-plane. We point out, however, that whereas in the classical case this class of operators essentially coincides with the class of Toeplitz operators on the circle, in the case of several variables the operators obtained from the Wiener-Hopf equation are a special case of the Toeplitz operators on the n-torus and, in particular, the symbols of the former are all homotopic to a constant. We will have more to say about this in our remarks.

In [19] Malyšev states the above result for Toeplitz operators but goes on to claim that such operators are always invertible. We shall see directly that this is not the case even when the symbol is a trigonometric polynomial.

Lastly, Osher has shown in [21] that for a function ψ on $\boldsymbol{T}^2$ of the form $\psi(z, w) = \varphi_1(z)\varphi_2(w)+\varphi_3(z)$, the operator W_ψ is invertible if and only if ψ is nonvanishing and homotopic to a constant.

Before proceeding we point out in the following Corollary that our techniques can be used to determine when an arbitrary operator in $\mathscr{A}(\boldsymbol{Z}_+^2)$ is a Fredholm operator. The usefulness of this result depends largely on one's ability to determine the invertibility of operators in $\mathscr{A}(\boldsymbol{Z}_+)$.

COROLLARY. *An operator T in $\mathscr{A}(\boldsymbol{Z}_+^2)$ is a Fredholm operator if and only if the operators $[\gamma_z T](z)$ and $[\gamma_w(T)](z)$ are invertible for z in $\boldsymbol{T}$. Moreover, there exist operators T in $\mathscr{A}(\boldsymbol{Z}_+^2)$ which are Fredholm of all analytical indices.*

Proof. Only the last statement needs further proof and for this an example will suffice. Consider the operator $Y = W_z W_z^* + \frac{1}{2} W_w^*$ in $\mathscr{A}(\boldsymbol{Z}_+^2)$. Each of the subspaces $\mathscr{M}_n$ reduces T, where $\mathscr{M}_n$ is the subspace of $H^2(\boldsymbol{Z}_+^2)$ of functions whose Fourier transform vanishes off the vertical line of abscissa n. Further, $T|\mathscr{M}_0$ is unitarily equivalent to $\frac{1}{2}T_z^*$ and $T|\mathscr{M}_n$ is unitarily equivalent to $I+\frac{1}{2}T_z^*$ for $n>0$. Thus T is a Fredholm operator of index one.

We now consider the Toeplitz operator $W_{z\bar{w}}$ on $H^2(\boldsymbol{Z}_+^2)$. For k in $\boldsymbol{Z}_+$ let $\mathscr{H}_k$ be the $(k+1)$-dimensional subspace of $H^2(\boldsymbol{Z}_+^2)$ of functions whose Fourier transform is supported on $\{(m, n) \in \boldsymbol{Z}_+^2 = m+n=k\}$. A moment's thought shows that each $\mathscr{H}_k$ reduces $W_{z\bar{w}}$ and that $W_{z\bar{w}}|\mathscr{H}_k$ is the cyclic nilpotent operator of order $k-1$. Thus the tensor product of the unilateral shift with its adjoint is a standard example (cf. [16, p. 243]) in operator theory.

Now let φ be a continuous function on $\boldsymbol{T}$ and ψ the function on $\boldsymbol{T}^2$ defined by $\psi(z, w) = \varphi(z\bar{w})$. Again each $\mathscr{H}_k$ reduces W_ψ and $W_\psi|\mathscr{H}_k$ is the operator defined by the $(k+1)\times(k+1)$ section of the Toeplitz matrix for T_φ, that is, the matrix for $W_\psi|\mathscr{H}_k$

with respect to the obvious orthonormal basis for $\mathscr{H}_k$ is

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_k \\ c_{-1} & c_0 & c_1 & & \cdot \\ c_{-2} & c_{-1} & c_0 & & \cdot \\ \cdot & & \cdots & & \cdot \\ \cdot & & & & \\ \cdot & & & c_0 & c_1 \\ c_{-k} & & \cdots & c_{-1} & c_0 \end{bmatrix}$$

where $\{c_i\}_{i=-\infty}$ are the Fourier coefficients of φ. If φ is assumed to be nonvanishing and have winding number zero about the origin, then ψ is nonvanishing and homotopic to a constant. Hence, W_ψ is a Fredholm operator of analytical index 0 and is invertible if and only if each finite section of T_φ is invertible. (That W_ψ is a Fredholm operator could be shown directly using the known behavior of the finite sections of T_φ.) Since it is well known that there exist functions φ for which T_φ is invertible, but not all of the finite sections are, we see that there exists Toeplitz operators W_ψ on the quarter-plane which are Fredholm operators of index zero but which are not invertible. To show just how nice the function ψ can be, we explicitly compute one example.

Let ψ be the trigonometric polynomial defined by $\psi(z, w)=8z^2\bar{w}^2-18z\bar{w}+27w\bar{z}$. Since $\psi(z, w)=\varphi(z\bar{w})$, where $\varphi(z)=8z^2-18z+27\bar{z}=(8/z)(z+\frac{3}{4})(z-\frac{3}{2})^2$, we see by our previous remarks that W_ψ is a Fredholm operator of index zero. A trivial computation shows, however, that $W_\psi 1=0$ and hence W_ψ is not invertible.

We remark that Malyšev claims in [19] that if ψ is a trigonometric polynomial involving only the monomials $1, z, w, \bar{z}, \bar{w}, zw, z\bar{w}, \bar{z}w$, and $\bar{z}\bar{w}$ and ψ is nonvanishing on $\boldsymbol{T}^2$ and homotopic to a constant, then W_ψ is invertible. The above example shows this result is, in a sense, "best possible." Osher also obtains invertibility for his special class of functions. The problem of identifying the class of functions for which W_ψ is invertible or, more generally, to determine the dimension of the kernel of W_ψ for ψ an exponential in $C(\boldsymbol{T}^2)$ would seem to be very difficult. In particular, it contains the problem of determining necessary and sufficient conditions that all the finite sections of a Toeplitz operator be invertible simultaneously. General principles, however, show that the dimension of the kernel is an upper semi-continuous function. Our next result shows that the "generic" W_ψ is invertible.

THEOREM 2. *The collection of invertible Toeplitz operators on $H^2(\boldsymbol{Z}_+^2)$ is a proper dense open subset of the collection of Fredholm Toeplitz operators.*

Proof. We begin by introducing some additional notation. For $i=1, 2, 3, 4$ let Σ_i be the ith closed quarter-plane in $\boldsymbol{Z}^2$ and let $\mathscr{P}_i$ be the trigonometric polynomials on $\boldsymbol{T}^2$ whose Fourier transform vanishes outside of Σ_i and which have a continuous logarithm. Let $H^2(\boldsymbol{Z}\times\boldsymbol{Z}_+)$ denote the subspace of functions in $L^2(\boldsymbol{T}^2)$ whose

Fourier transforms vanish outside of the closed upper half-plane and let P_U denote the projection of $L^2(\boldsymbol{T}^2)$ onto $H^2(\boldsymbol{Z}\times\boldsymbol{Z}_+)$. For φ in $C(\boldsymbol{T}^2)$ let M_φ denote the operator on $H^2(\boldsymbol{Z}\times\boldsymbol{Z}_+)$ defined by $M_\varphi f=P_U(\varphi f)$, that is, the Toeplitz operator on the half-space $H^2(\boldsymbol{Z}\times\boldsymbol{Z}_+)$.

Choose q_i in $\mathscr{P}_i$ for $i=1, 2, 3, 4$ having the continuous logarithm θ_i; the Fourier transform of θ_i vanishes outside of Σ_i. (This is clear for $\Sigma_i=\boldsymbol{Z}_+^2$ and the other cases follow by symmetry.) If we set $\psi=\exp -(\theta_1+\theta_2+\theta_3+\theta_4)$, then the operator M_ψ can be seen to be invertible with $M_\psi^{-1}=M_{\exp(-\theta_1-\theta_2)}M_{\exp(-\theta_3-\theta_4)}=M_{q_1q_2}M_{q_3q_4}$. Let N be an integer greater than four times the orders of q_1, q_2, q_3 and q_4 and let $\mathscr{M}_N$ denote the closed subspace of functions in $H^2(\boldsymbol{Z}_+^2)$ whose Fourier transforms vanish at (i,j) in $\boldsymbol{Z}_+^2$ for $i\leqq N$. Since M_ψ is invertible, for f in $\mathscr{M}$ there exists g in $H^2(\boldsymbol{Z}\times\boldsymbol{Z}_+)$ such that $M_\psi g=f$. Moreover, since $g=P_U(q_1q_2P_U(q_3q_4f))$, we see that g is in $H^2(\boldsymbol{Z}_+^2)$ and hence $W_\psi g=f$. Therefore, $\mathscr{M}_N$ is contained in the range of W_ψ. A parallel argument using the right half-plane shows that the subspace $\mathscr{M}_N$ of functions in $H^2(\boldsymbol{Z}_+^2)$, whose Fourier transforms vanish at (i,j) in $\boldsymbol{Z}_+^2$ for $j\leqq N$, is also contained in the range of W_ψ. Thus the kernel of W_ψ^* is contained in the subspace $\mathscr{K}_N$ of $H^2(\boldsymbol{Z}_+^2)$ consisting of the functions whose Fourier transform vanishes at (i,j) in $\boldsymbol{Z}_+^2$ for $i>N$ or $j>N$. Hence for ψ of the prescribed form, the operator ψ is invertible on $H^2(\boldsymbol{Z}_+^2)$ if the determinant of the compression of W_ψ to $\mathscr{K}_N$ is not zero for a sufficiently large integer N. We now want to show that for N sufficiently large the determinant vanishes on a third set and that the collection of functions of the prescribed form is dense.

For N an integer and $i=1, 2, 3, 4$, let $\mathscr{P}_i^N$ denote the trigonometric polynomials in $\mathscr{P}_i$, whose Fourier transforms vanish outside of $\{(j,k)\in\boldsymbol{Z}^2 : |j|\leqq N, |k|\leqq N\}$; then $\mathscr{P}_i^N\subset\mathscr{P}_i^{N+1}\subset\mathscr{P}_i$ and $\bigcup_{N=1}^\infty \mathscr{P}_i^N=\mathscr{P}_i$. Viewing $\mathscr{P}_i^N$ as a subset of the complex space $\boldsymbol{C}^{(N+1)^2}$ in the obvious way, $\mathscr{P}_i^N$ is an open set. Define the function Φ_N at the point (q_1, q_2, q_3, q_4) of the open set $\mathscr{Q}^N=\mathscr{P}_1^N\times\mathscr{P}_2^N\times\mathscr{P}_3^N\times\mathscr{P}_4^N$ of $\boldsymbol{C}^{4(N+1)^2}$ to be the determinant of the compression of W_ψ to $\mathscr{K}_{4N}$, where $\psi=(q_1\cdot q_2\cdot q_3\cdot q_4)^{-1}$. From the previous paragraph we see that W_ψ is invertible if $\Phi_M(q_1, q_2, q_3, q_4)\neq 0$ for $M\geqq N$. Since the matrix elements of the compression of W_ψ to $\mathscr{K}_{4M}$ are Fourier coefficients of ψ, it follows that Φ_M is an analytic function on $\mathscr{Q}^M$. If M is chosen sufficiently large, then the points (q_1, q_2, q_3, q_4) and $(1, 1, 1, 1)$ will lie in the same component σ of $\mathscr{Q}^M$. Since $\Phi_M(1, 1, 1, 1)=1$, it follows that Φ_M vanishes on a thin subset of σ and hence there exist points (p_1, p_2, p_3, p_4) arbitrarily near (q_1, q_2, q_3, q_4) at which Φ_M is not zero.

Finally, if φ is a continuous function on $\boldsymbol{T}^2$ with the continuous logarithm θ and $\varepsilon>0$, we proceed as follows: choose $\delta>0$ such that $e^\delta-1\leqq\min\{\varepsilon/3, \|\varphi\|\}$ and let θ_0 be a trigonometric polynomial such that $\|\theta-\theta_0\|<\delta$. Then we obtain $\|\varphi-\exp\theta_0\|<\varepsilon/3$ and $\|\exp(\theta_0)\|\leqq 2\|\varphi\|$. Set $\theta_0=\theta_1+\theta_2+\theta_3+\theta_4$ (3), where θ_i is a trigonometric polynomial whose Fourier transform vanishes outside of Σ_i and set

(3) Such a decomposition is not unique since the Σ_i are not disjoint.

$L=\max\{\|\exp(\pm\theta_i)\| : i=1, 2, 3, 4\}$. There exists q_i in $\mathscr{P}_i$ such that

$$\|\exp(-\theta_i)-q_i\| < \min\left\{\frac{1}{3L}, \frac{\varepsilon}{700\|\varphi\|L^7}\right\}$$

and (q_1, q_2, q_3, q_4) is in $\mathscr{Q}^N$ for N greater than the orders of q_1, q_2, q_3, and q_4. By the previous paragraph there exists (p_1, p_2, p_3, p_4) in $\mathscr{Q}^M$ for M sufficiently large such that W_ψ is invertible for $\psi=(p_1p_2p_3p_4)^{-1}$ and

$$\|\exp(-\theta_i)-p_i\| < \min\left\{L, \frac{1}{3L}, \frac{\varepsilon}{700\|\varphi\|L^7}\right\}.$$

Repeated use of the previous estimates yield $\|\psi-\varphi\|<\varepsilon$. Thus the invertible Toeplitz operators on $H^2(\boldsymbol{Z}_+^2)$ are dense in the collection of Toeplitz operators which are Fredholm operators. Since the invertible operators are an open set of all operators, the result follows.

Recall that for Toeplitz operators defined on the circle by a matrix function, the operator is a Fredholm operator if and only if the determined function does not vanish and that the index of the operator is equal to the negative of the winding number of this function about the origin. It is no longer true, however, that "index" zero implies invertibility as it does in the scalar case. Kreĭn and Gohberg have shown in [12], however, that in this case the invertible operators also form a dense open subset of the operators having index zero. A proof of this can be given modeled along the lines of the proof of the preceding theorem. In fact, if one generalizes the notion of Toeplitz operator to the context of a subsemigroup of a (possibly nonabelian) group acting isometrically on L^2 of a measure space, then a close relationship between these two results becomes apparent.

Before proceeding, let us observe that the simplicity of the one-variable case can be deceiving. In particular, although it is easy to decide when a Toeplitz operator with continuous symbol is invertible, neither the inverse nor even its norm are readily obtainable. Rather the easy thing to obtain is a "pseudo-inverse" and its norm. In fact, in all cases an inverse modulo the commutator ideal is at hand and in the one-variable case the commutator ideal coincides with the algebra of compact operators.

We now want to make a few further remarks concerning the problem of determining when a W_ψ which is a Fredholm operator of index zero is invertible. Firstly, as we mentioned earlier there are the results of Osher and Malyšev which we do not repeat here. Secondly, one can show without too much difficulty that if the Fourier transform $\hat{\psi}$ of ψ vanishes on either the open second *or* fourth quadrants, or if the support of $\log\hat{\psi}$ is contained in the union of a sector at the origin of opening less than π with either the closed first or third quadrants, then W_ψ is invertible. (We shall not give the details.) Lastly, by combining our result (yet to be stated for $n>1$) relating invertibility in the n-variable case to "Fredholmness" in $(n+1)$-variables with Simonenko's result [23] that the operators arising from Wiener-Hopf equations are

Fredholm operators if only their symbol does not vanish, we reach the rather puzzling conclusion that operators arising from Wiener-Hopf equations are *invertible* if their symbol does not vanish. We have been unable to prove this result directly.

We now consider the significance of the "topological index" in cases where it is not zero. The topological index of a nonvanishing function ψ in $C(\boldsymbol{T}^2)$ is a pair of integers (n, m) which can be computed as follows: n and m are the winding numbers of the curves $\psi(z, 1)$ and $\psi(1, w)$, respectively, about the origin. Consideration of the one-variable case and a few examples shows that the integers n and m should "measure" the size of the kernel and cokernel of W_ψ in some sense. That this is not entirely the case is shown by the following example.

Consider again the operator $W_{z\bar{w}}$; if λ is a complex number of modulus less than one, then the topological index of the function $z\bar{w}+\lambda$ is $(1, -1)$, while the corresponding operator $W_{z\bar{w}+\lambda}$ has both kernel and cokernel equal to $\{0\}$. The latter statement is easily verified since $W_{z\bar{w}+\lambda}$ is invertible on each of the reducing subspaces $\mathscr{H}_k$.

Thus the relationship between topological index and some form of analytical index (not associated with Fredholm operators) is more complex in the two-variable case. It seems quite reasonable to expect, however, that if one considers representations of the C^*-algebra $\mathscr{A}(\boldsymbol{Z}_+^2)$ in a manner analogous to that done in [8], then the preceding difficulty will disappear. In this note we do not pursue these thoughts and conclude this discussion with the following

THEOREM 3. *Consider the collection of Toeplitz operators on $H^2(\boldsymbol{Z}_+^2)$ having a nonvanishing symbol whose topological index is the form (n, m) for fixed integers n and m.*

If $n, m \geqq 0$ and $n+m>0$, then the range of W_ψ is closed, W_ψ has finite-dimensional kernel and an infinite-dimensional cokernel.

If $n, m \leqq 0$ and $n+m<0$, then the range of W_ψ is closed, W_ψ has an infinite-dimensional kernel and finite-dimensional cokernel.

If n and m have opposite signs, then the subset of such operators having both an infinite-dimensional kernel and cokernel contains a dense set.

Proof. If ψ is a nonvanishing continuous function with index (n, m), $n, m \geqq 0$, then there exists a continuous function ψ_0 homotopic to a constant such that $\psi=\psi_0 z^n w^m$. Moreover, we have $W_\psi = W_{\psi_0} W_z^n W_w^m$, where W_{ψ_0} is a Fredholm operator of index 0 and W_z and W_w are isometries with infinite-dimensional cokernels. Thus, there exists a finite rank operator F such that $W_{\psi_0}+F$ is invertible. The first statement is proved by considering the identity $W_\psi = (W_{\psi_0}+F)W_z^n W_w^m - FW_z^n W_w^m$, and the second statement follows by taking adjoints.

For the third statement, we note that if φ is one of the dense sets of functions of index $(0, 0)$ whose inverse is a product of polynomials with supports in one of the four quadrants, as in Theorem 1, then the image of H under $W_\varphi W_z^n$

(respectively $W_\varphi W_w^m$) contains all functions with support above some horizontal line (respectively to the right of some vertical line), and in particular, intersects ker $W_{\bar{w}}^m$ (respectively $W_{\bar{z}}^n$) in an infinite-dimensional space.

The above operators have also the nice property that their range is closed. The example given earlier, $W_{z\bar{w}+\lambda}$ for small λ, shows that the property of having a kernel, or a closed range is highly unstable and is not preserved under small perturbations. We showed above the operators having these nice properties are dense. We suspect, however, that the set of operators W_ψ for ψ of index (n, m), where $nm<0$, which fails to have these properties, is also dense.

Thus far we have been considering the scalar case. After introducing the necessary notation we indicate how our techniques can be applied to the matrix case.

For N a positive integer let $L_N^2(\boldsymbol{T}^2)$ denote the space of norm square integrable measurable functions from $\boldsymbol{T}^2$ to the N-dimensional Hilbert space $\boldsymbol{C}^N$. The Fourier transform of a function in $L_N^2(\boldsymbol{T}^2)$ is a $\boldsymbol{C}^N$-valued function on $\boldsymbol{Z}^2$. Again we let $H_N^2(\boldsymbol{Z}_+^2)$ denote the subspace of functions in $L_N^2(\boldsymbol{T}^2)$ with Fourier transform supported on $\boldsymbol{Z}_+^2$ and P_2 the projection of $L_N^2(\boldsymbol{T}^2)$ onto $H_N^2(\boldsymbol{Z}_+^2)$. If Ψ is a continuous function from $\boldsymbol{T}^2$ to the algebra M_N of $N\times N$ complex matrices, then the Toeplitz operator W_Ψ is defined $W_\Psi f=P_2(\Psi f)$ for f in $H_N^2(\boldsymbol{Z}_+^2)$.

If $\mathscr{A}_N(\boldsymbol{Z}_+^2)$ denotes the C^*-algebra generated by the Toeplitz operators with a continuous $N\times N$-matrix valued symbol, then $\mathscr{A}_N(\boldsymbol{Z}_+^2)$ is naturally isomorphic to the tensor product $\mathscr{A}(\boldsymbol{Z}_+^2)\otimes M_N$. (The isomorphism is the linear extension of the map which identifies $W_\psi\otimes A$ with W_Ψ, where ψ is in $C(\boldsymbol{T}^2)$, A is in M_N and $\Psi(z, w)=\psi(z, w)A$.) If we tensor the diagram preceding Proposition 4 with M_N, then we obtain

THEOREM 4. *There exists a *-isometrical isomorphism α'' from $\mathscr{L}\mathscr{C}(H_N(\boldsymbol{Z}_+^2))$ into $\mathscr{A}_N(\boldsymbol{Z}_+^2)$ such that the sequence*

$$\mathscr{L}\mathscr{C}(H_N(\boldsymbol{Z}_+^2)) \xrightarrow{\alpha''} \mathscr{A}_N(\boldsymbol{Z}_+^2) \xrightarrow{\gamma_z'\oplus\gamma_w'} C(\boldsymbol{T}, \mathscr{A}_N(\boldsymbol{Z}_+))\oplus C(\boldsymbol{T}, \mathscr{A}_N(\boldsymbol{Z}_+))$$

is exact at $\mathscr{A}_N(\boldsymbol{Z}_+^2)$.

COROLLARY. *If Ψ is a continuous matrix valued function on $\boldsymbol{T}^2$, then the Toeplitz operator W_Ψ on $H_N^2(\boldsymbol{Z}_+^2)$ is a Fredholm operator if and only if the operators $T_\Psi(\cdot, w)$ and $T_\Psi(z, \cdot)$ on $H_N^2(\boldsymbol{Z}_+)$ are invertible for each z and w in $\boldsymbol{T}$.*

Thus in the matrix quarter-plane case not every operator whose symbol is homotopic to the identity is a Fredholm operator. We presume that a refinement of the argument which we gave for Theorem 2 would show that the generic case is a Fredholm operator and in all probability even invertible. We do not consider this question further in this note.

We now briefly consider the application of our techniques to the case of n-variables. We do not give precise definitions of the notation we use but extend the previous notation in the obvious manner. For the case of three variables one considers a three-dimensional analogue of the commutative diagram obtained in the

proof of Proposition 4 and is enabled to exhibit a *-isometrical isomorphism from $\mathscr{L}\mathscr{C}(H^2(\boldsymbol{Z}_+^3))$ into $\mathscr{A}(\boldsymbol{Z}_+^3)$ such that the sequence

$$\mathscr{L}\mathscr{C}(H^2(\boldsymbol{Z}_+^3)) \longrightarrow \mathscr{A}(\boldsymbol{Z}_+^3)$$
$$\xrightarrow{\gamma_z \oplus \gamma_w \oplus \gamma_v} C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+^2)) \oplus C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+^2)) \oplus C(\boldsymbol{T}, \mathscr{A}(\boldsymbol{Z}_+^2))$$

is exact at $\mathscr{A}(\boldsymbol{Z}_+^3)$. Hence, as a corollary one obtains that W_ψ is a Fredholm operator if and only if the operators $W_{\psi(\cdot,\cdot,v)}$, $W_{\psi(\cdot,w,\cdot)}$, and $W_{\psi(z,\cdot,\cdot)}$ on $H^2(\boldsymbol{Z}_+^2)$ are invertible for each z, w and v in $\boldsymbol{T}$. A similar result obtains for the matrix case in n-variables. Thus, the question of an operator in $\mathscr{A}_N(\boldsymbol{Z}_+^n)$ being a Fredholm operator is equivalent to an n-parameter family of operators in $\mathscr{A}_N(\boldsymbol{Z}_+^{n-1})$ being *invertible*.

If, as our proof of Theorem 2 suggests, the set of noninvertible Toeplitz operators with symbol homotopic to a constant is of topological codimension two, then it would seem likely that, for three variables, the generic case is Fredholm, but that for $n \geqq 5$, the set of operators which are Fredholm is no longer even dense. Analogous limitations hold in the matrix case.

We now consider briefly two further directions for possible generalization. Instead of the quarter-plane, it would be possible to consider other sub-semigroups of $\boldsymbol{Z}^2$. Certain basic questions for such a study have been considered in [7]. If the semigroup is a certain class of sectors of opening less than π determined by two lines of rational slope, then this case is completely equivalent to that of the quarter-plane. (If a two-by-two matrix with integer coefficients and determinant one exists taking the quarter-plane semigroup onto such a semigroup, then this defines an isometrical isomorphism between the H^2 spaces and induces an isometrical isomorphism between the corresponding C^*-algebras.) For other sectors determined by lines with rational slope, problems of multiplicity would have to be considered. If the semigroup, however, is a sector having at least one line of irrational slope then the situation is rather different and the techniques of this note do not apply. In higher dimensions there are a plethora of possible semigroups determined by cones. Our techniques might apply to those determined by hyperplanes of "rational slope."

In a different direction one could consider the generalization of the study in [6] to the quarter-plane in $\boldsymbol{R}^2$. The results of [8] can be applied along with our techniques to obtain a representation of $\mathscr{A}(\boldsymbol{R}_+^2)$ into a II_∞-factor in such a manner that a Toeplitz operator is a generalized Fredholm operator W_φ in the sense of Breuer [1], [2] if and only if the function φ is invertible and homotopic to a constant in the space of continuous functions on the Bohr compactification of $\boldsymbol{R}^2$. A similar example should show that not all such operators are invertible but that a dense open set is. In the latter proof use would be made of the determinant function on a II_1-factor.

We conclude by remarking that our results are successful only in the lower-dimensional cases largely due to the nature of the questions asked. For example if

one were to consider operators invertible modulo $C(T, \mathscr{L}\mathscr{C})$ as being "nice," then we would reduce the questions of "niceness" in n-variables to the question of being Fredholm in $(n-1)$-variables or even invertibility in $(n-2)$-variables. Rather than speculate beyond this, we believe further work on these matters should await the formulation of more precise questions to be answered. At this point it is not clear what such questions should be.

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