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DIFFERENTIABLE MONOTONE MAPS ON MANIFOLDS. II[1]

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Abstract. Let M^n and N^n be closed manifolds, and let G be any (nonzero) module. (1) If $f: M^3 \to N^3$ is C^3 G-acyclic, then there is a closed C^3 3-manifold K^3 such that $N^3 \# K^3$ is diffeomorphic to M^3, and $f^{-1}(y)$ is cellular for all but at most r points $y \in N^3$, where r is the number of nontrivial G-cohomology 3-spheres in the prime decomposition of K^3. (2) If $f: M^3 \to M^3$ or $f: S^3 \to M^3$ is G-acyclic, then f is cellular. In case G is Z or Z_p (p prime), results analogous to (1) and (2) in the topological category have been proved by Alden Wright. (3) If $f: M^n \to M^n$ or $f: S^n \to M^n$ is real analytic monotone onto, then f is a homeomorphism.

1. **Conventions and definitions.** Let G be a module, and let $A \subset M^n$ be compact. Then A is *acyclic* (*k-acyclic*; *cellular*) is the reduced Čech cohomology module $\tilde{H}^*(A; G)=0$ ($H^k(A;G)=0$; there are n-cells $A_k \subset M^n$ such that $A_{k+1} \subset \operatorname{int} A_k$ and $\bigcap_k A_k = A$). A proper onto map $f: M^n \to N^n$ is *monotone* (*acyclic*; *cellular*) if, for each $y \in N^n$, $f^{-1}(y)$ is connected (acyclic; cellular). The *branch set* B_f is the set of points in M^n at which f fails to be a local homeomorphism.

Standing hypotheses. Unless otherwise specified, all manifolds are connected, separable, and without boundary. Whenever the statement of a theorem refers to a map f without specifying its domain and range, it is understood that $f: M^n \to N^n$ is *proper* and *onto*. Whenever coefficients of cohomology are not specified, *any* (nonzero) module G may be used.

Other conventions and definitions are as in [6, pp. 185–186]. For other work on monotone, acyclic, and cellular maps $f: M^n \to N^n$, see bibliographies of this paper, [1], [15], and (vast) [30].

2. **Real analytic monotone maps.**

2.1. Definition. Let G be a module, and let

$$C_f = C_f(G) = f^{-1}(\operatorname{Cl}\{y \in N^n : \tilde{H}^*(f^{-1}(y); G) \neq 0\}).$$

2.2. Lemma. *Let L be a principal ideal domain, let M^n be orientable over L, and let* $\dim (f(C_f)) \leq 0$.

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(1) *If M^n and N^n are closed, then there is a k* ($k=0, 1, \ldots$) *such that $f(C_f)$ has exactly k points and*

$$k \leqq \sum_{r>0} (\textit{number of generators of } H^r(M^n));$$

in fact, if L is a field,

$$k \leqq \sum_{r>0} (\dim (H^r(M^n)) - \dim (H^r(N^n))).$$

(2) *If $B^n \subset N^n$ is a closed n-cell with $\partial B^n \cap f(C_f) = \varnothing$, then inclusion induces an isomorphism*

$$H^r(f^{-1}(B^n)) \approx \sum \{H^r(f^{-1}(y)) : y \in B^n\}$$

for every $r>0$.

Proof. From [13, p. 639, Theorem 3, (2)]

$$0 \to H_c^r(N^n) \to H_c^r(M^n) \to \sum \{H^r(f^{-1}(y)) : y \in N^n\} \to 0$$

is split exact for $r>0$, and (1) follows. Note that the numbers of generators of $H^r(M^n)$ and $H^r(N^n)$ are finite [27, p. 342, (11)].

From the exactness of

$$0 \to H_c^r(\text{int } B^n) \to H_c^r(f^{-1}(\text{int } B^n)) \to \sum \{H^r(f^{-1}(y)) : y \in \text{int } B^n\} \to 0$$

for $r>0$ it follows that $H_c^r(f^{-1}(\text{int } B^n)) \approx \sum H^r(f^{-1}(y))$ for $1 \leqq r \leqq n-1$ and $\sum H^n(f^{-1}(y)) = 0$. From the Vietoris Mapping Theorem [27, p. 346, (18)] $H^r(f^{-1}(\partial B^n)) \approx H^r(\partial B^n)$, and from the cohomology sequence

$$\cdots \leftarrow H^r(f^{-1}(\partial B^n)) \leftarrow H^r(f^{-1}(B^n)) \leftarrow H_c^r(f^{-1}(\text{int } B^n)) \leftarrow H^{r-1}(f^{-1}(\partial B^n)) \leftarrow \cdots$$

(2) follows.

2.3. Lemma. *Let L be a principal ideal domain, let M^n be orientable over L, and let f be C^n monotone but not acyclic. Then* $\dim C_f > \dim f(C_f)$.

Proof. Suppose $\dim f(C_f) = 0$. If $\dim C_f \leqq 0$, then f is acyclic; thus $\dim C_f > 0$.

Thus we may suppose that $\dim f(C_f) = q > 0$. By [11, (3.7), (3.5), and (3.8)] there are a C^∞ closed manifold J^m, a C^∞ map $h: J^m \times R^{n-m} \to S^m \times R^{n-m}$ with $h(J^m \times \{t\}) \subset S^m \times \{t\}$ ($m=0, 1, \ldots, n-q$; thus $n-m \geqq q \geqq 1$), a C^∞ embedding γ of $J^m \times R^{n-m}$ in M^n, and a C^∞ embedding δ of $S^m \times R^{n-m}$ in N^n such that

(a) $\delta^{-1} h \gamma$ is the restriction of f,

(b) $f(C_f) \cap \text{imag } \delta \neq \varnothing$, and

(c) $\dim (f(B_f) \cap \delta(S^m \times \{t\})) \leqq 0$ for every $t \in R^{n-m}$.

Let $h_t: J^m \times \{t\} \to S^m \times \{t\}$ be the restriction of h; clearly $\bigcup_t C(h_t) \subset C_h$, so that (from (c)) each $\dim h_t(C(h_t)) \leqq 0$ and 2.2 (1) applies to h_t yielding an integer $k(t) \geqq 0$. There is a t such that $k(t)$ is maximal for $t \in R^{n-m}$, and we may suppose that $t=0$. Let y_j $(j=1, 2, \ldots, s)$ $(s=k(0))$ be exactly those points in S^m such that $H^r(h^{-1}(y_j, 0)) \neq 0$ for some $r>0$.

The proof now follows that given in [6, pp. 201–202], beginning with the third paragraph, with the following changes: h, m, L, C_h, and C_e in place of f, p, Z, B_f, and B_e, respectively; S^m in place of B^p, int B^p, and Σ; references to (1) should be omitted, and (2) should be interpreted as 2.2 (2). At the end we observe that $\dim C_f \geqq \dim C_h \geqq n-m+i > n-m \geqq q = \dim f(C_f)$.

2.4. Corollary. *Let f be real analytic monotone, but not a homeomorphism. Then* $\dim B_f > \dim f(B_f)$.

Proof. Use Z_2 coefficients. The map $g\colon M^n - C_f \to N^n - f(C_f)$ defined by restriction of f is acyclic, and thus a homeomorphism ([10, p. 33, (6.1)]; the conclusion should be "is a homeomorphism"). Then $C_f = B_f$, and the conclusion results from (2.3).

2.5. Lemma. *Let f be real analytic monotone, but not necessarily proper. Then B_f is a C-analytic subset of M^n.*

The proof follows from the first five paragraphs of the proof of [10, p. 31, (5.4)]; in particular, from the first sentence of the sixth paragraph.

2.6. Lemma. *Let M^n and N^n be closed, and let f be real analytic monotone with* $\dim B_f = q \geqq 0$. *Then*

$$\dim (H^q(N^n; Z_2)) < \dim (H^q(M^n; Z_2)).$$

Proof. Suppose the contrary. By 2.4 $q>0$, by 2.5 and [3] $H^q(B_f) \neq 0$, and by 2.4 $H^q(f(B_f))=0$. Since f maps $M^n - B_f$ homeomorphically onto $N^n - f(B_f)$, it induces [27, p. 318, (5)] $H^k(N^n, f(B_f)) \approx H^k(M^n, B_f)$ and [6, p. 192, (2.13)] a monomorphism of $H^k(N^n)$ into $H^k(M^n)$ $(k=0, 1, \ldots)$. Since $H^k(M^n)$ and $H^k(N^n)$ are finite dimensional [27, p. 342, (11)], it follows from the contrary assumption that the monomorphism is an isomorphism (onto) if $k=q$; a contradiction results from the Five Lemma [14, p. 14, (3.3)(ii)] applied to the cohomology sequences of (M^n, B_f) and $(N^n, f(B_f))$.

2.7. Corollary. *If $M^n = S^n$ and f is real analytic monotone, then f is a homeomorphism.*

Proof. Since f is continuous, N^n is compact, i.e. closed.

2.8. Corollary. *If M^n is closed, a monotone real analytic map $f\colon M^n \to M^n$ is a homeomorphism.*

2.9. Corollary. *If M^n and N^n are closed, and $f\colon M^n \to N^n$ and $g\colon N^n \to M^n$ are real analytic and monotone, then f and g are homeomorphisms.*

Statements corresponding to 2.7, 2.8, and 2.9 for C^∞ maps are false, in fact f need not be acyclic [6, p. 192, (2.12)]. Results somewhat analogous for $n=3$ to 2.8 and 2.9 are given in [18], and to 2.7 in [2].

2.10. REMARK. The author does not know whether, for M^n and N^n closed, a real analytic monotone (onto) map $f: M^n \to N^n$ is necessarily a homeomorphism (see [10, §6]). Suppose that there is such a counterexample, and that n is the smallest dimension for which it occurs. Then $n \geqq 2$, and $f^{-1}(y)$ is a single point except for s points $y_i \in N$, where

$$s \leqq \dim (H^*(M^n; Z_2)) - \dim (H^*(N^n; Z_2)).$$

In fact, M^n is homeomorphic to the connected sum of N^n and K_i $(i=1, 2, \ldots, r)$ where no K_i is homeomorphic to S^n.

Proof. If $n=1$, then $M^1=S^1$. If for some $y \in N^1$, the C-analytic set $f^{-1}(y)$ is not a single point, then $f^{-1}(y)=S^1$, contradicting the ontoness of f. Thus $n \geqq 2$.

Let y_1 and y_2 be two distinct points of $f(B_f)$, let $\alpha: R^n \to N^n$ be a real analytic embedding with $\alpha(0)=y_1$ and $y_2 \notin \operatorname{imag} \alpha$, and let $g: f^{-1}(\alpha(R^n)) \to R^n$ be the restriction of $\alpha^{-1} \circ f$. There is a C^n embedding $\mu: S^{n-1} \to R^n$ such that $d(\mu, \mathrm{id}) < \frac{1}{2}$ in the (coarse-fine) C^n topology [20, p. 25] and g is transverse regular on $\mu(S^{n-1})$ ([29, p. 26] and [8, p. 376, (2.6)]), and there is a δ, $0<\delta<\frac{1}{2}$, such that if $\nu: S^{n-1} \to R^n$ is a C^n embedding with $d(\mu, \nu)<\delta$, then f is transverse regular on $\nu(S^{n-1})$ ([29, p. 27] and [8, p. 376, (2.6)]). By [31, p. 654, Theorem 2] there is a real analytic embedding $\nu: S^{n-1} \to R^n$ such that $d(\mu, \nu)<\delta$. For $r>0$ let ν_r be the embedding defined by $\nu_r(x)=r \cdot \nu(x)$; for r in some open interval J about 1, $d(\mu, \nu_r)<\delta$ also. Thus $g^{-1}(\nu_r(S^{n-1}))=L^{n-1}$ is a real analytic $(n-1)$-manifold also [29, p. 23], and by the minimality of n, $g|L^{n-1}: L^{n-1} \to \nu_r(S^{n-1})$ is a homeomorphism for all $r \in J$. The open set $V=\bigcup \{\nu_r(S^{n-1}) : r \in J\}$ has compact closure and separates R^n with 0 in a bounded component of $R^n - V$. It follows that y_1 and y_2 are in different components of $f(B_f)$. Since y_1 and y_2 were arbitrary, $f(B_f)$ is a closed 0-dimensional set.

By [3], if $f^{-1}(y)$ is Z_2-acyclic, it is a single point; the second conclusion follows from 2.2(1).

For the third conclusion, there are (as above) mutually disjoint real analytic n-balls $B_i \subset N^n$ such that $y_i \in \operatorname{int} B_i$ and f is transverse regular on ∂B_i; thus $f^{-1}(\partial B_i)$ is homeomorphic to S^{n-1}, and the conclusion results from 2.2(1).

3. **Acyclic maps.** In this section the cohomology coefficient module $(G \neq 0)$ is arbitrary, unless otherwise stated.

3.1. LEMMA. *Let K^m be a closed m-manifold* $(m=2$ *or* $3)$, *and let* $h: K^m \to S^m$ *be acyclic. Then K^m is orientable, and K^2 is homeomorphic to S^2.*

Proof. By the Vietoris Mapping Theorem [27, p. 346, (18)]

$$h^*: \tilde{H}^j(S^m; G) \approx \tilde{H}^j(K^m; G),$$

and K is a finite polyhedron.

Suppose that K^m is nonorientable. Then $H^m(K^m; Z) \approx Z_2$, and from the naturality of the Universal Coefficient Theorem [27, p. 236, (1)], some homomorphism

$$\alpha \otimes 1 : Z \otimes G \to Z_2 \otimes G$$

is an isomorphism; thus, for every $g \in G$, $2g = 0$. From [27, p. 206, E.2] $H_{m-1}(K^m; Z)$ has a summand of Z_2, and hence Hom $(H_{m-1}(K^m; Z), G) \neq 0$. Since $H^{m-1}(K^m; G) = 0$, a contradiction results [27, p. 243, (3)].

Since $H^1(K^2; G) = 0$, the second conclusion is immediate from the classification of closed orientable 2-manifolds [27, p. 148–149].

3.2. Remark. For any closed 3-manifold M^3 there is a smallest natural number $p(M^3)$ such that, if M^3 is homeomorphic to the connected sum of closed 3-manifolds K_i $(i = 1, 2, \ldots, p(M^3)+1)$, then at least one of the K_i is homeomorphic to S^3 [12, p. 253]. In case M^3 is orientable, M^3 actually has a unique decomposition into primes [19].

3.3. Proposition. *Let f be a C^3 proper monotone map, and let $Y \subset N^n$ be the smallest closed subset such that $f^{-1}(y)$ is cellular for each $y \in N^n - Y$.*

(a) *Then* dim $Y \leqq n-2$.

(b) *If $H^1(f^{-1}(y)) = 0$ except for a discrete set of $y \in N^n$, then*

(i) dim $Y \leqq n-3$, *and*

(ii) *for every open set $U \subset N^n$ with $\pi_1(U) = 0$, $\pi_1(U - Y) = 0$.*

(c) *If, in addition, $n = 3$, then Y is discrete.*

(d) *If, in addition, M^3 is closed, then there is a C^3 closed 3-manifold K^3 such that M^3 is diffeomorphic to $N^3 \# K^3$ and the number of points $\nu(Y) \leqq p(K^3) \leqq p(M^3)$.*

(If M^n is orientable, then $p(K^3) = p(M^3) - p(N^3)$ (Remark 3.2).)

Proof. Conclusion (a) results from [6, p. 189, (2.6)] and [9, Proposition 4].

Suppose that (b) is false. Then [11, p. 80, (3.7)] there is a C^3 embedding $\lambda : S^m \times R^{n-m} \to N^n$ $(m = 0, 1, \text{ or } 2)$ such that f is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \varnothing$ for each $t \in R^{n-m}$. By [11, p. 80, (3.5)] there are $\varepsilon > 0$, a C^3 closed m-manifold L^m, and a diffeomorphism σ of $L^m \times S(0, \varepsilon)$ onto $f^{-1}(\lambda(S^m \times S(0, \varepsilon)))$ such that $\lambda^{-1} f \sigma(L^m \times \{t\}) \subset S^m \times \{t\}$ for each $t \in S(0, \varepsilon)$. From [6, (2.6)], $m = 2$. Let $g = \lambda^{-1} f \sigma$.

Since f is monotone onto, L^2 is connected [32, p. 138, (2.2)] and $g(L^2 \times \{t\}) = S^2 \times \{t\}$; thus for every $(w, t) \in S^2 \times S(0, \varepsilon)$, $g^{-1}(w, t) \neq L^2 \times \{t\}$, so that $H^2(g^{-1}(w, t)) = 0$. Since $H^1(f^{-1}(y)) = 0$ except for a discrete set of $y \in N^n$, we may suppose that $H^1(g^{-1}(w, t)) = 0$ for every $(w, t) \in S^2 \times S(0, \varepsilon)$; thus g is acyclic.

By 3.1 L^2 is homeomorphic to S^2. From [6, p. 191, (2.11)] each $H^1(g^{-1}(w, t); Z_2) = 0$, so that $H_1(g^{-1}(w, t); Z_2) = 0$. For each open subset U of $L^2 \times S(0, \varepsilon)$ with $g^{-1}(w, t) \subset U$, there is [24, p. 853, Lemma 2] a closed 2-cell $D \subset U \cap (K^2 \times \{t\})$ with $g^{-1}(w, t) \subset \operatorname{int} D$. Since there is a $\delta > 0$ with $D \times S(t, \delta) \subset U$, $g^{-1}(w, t)$ is cellular in $L^2 \times S(0, \varepsilon)$, and a contradiction results. Thus (b) is proved.

Now suppose, in addition, that $n = 3$. Let $\bar{y} \in Y$ and let B^3 be a C^3 3-ball with $\bar{y} \in \operatorname{int}(B^3)$; we may suppose that $H^1(f^{-1}(y)) = 0$ for each $y \in \partial B^3$ and that f is transverse regular on ∂B^3 ([29, p. 26] and [8, p. 376, (2.6)]). By [29, p. 23] $f^{-1}(\partial B^3)$ is a C^3 2-manifold, and it follows as above that $f^{-1}(\partial B)$ is homeomorphic, and thus C^3 diffeomorphic, to S^2.

Let L^3 (T^3) be the C^3 3-manifold obtained from $f^{-1}(B^3)$ (B^3) by identifying $f^{-1}(\partial B^3)$ (∂B) to a point a (b). Let $g: L^3 \to T^3$ be the map defined by $g(a)=b$ and $g|f^{-1}(\text{int } B^3)=f|f^{-1}(\text{int } B^3)$; since g is C^3 except on $g^{-1}(b)=\{a\}$, we may suppose [6, p. 189, (2.4)] that g is C^3. Thus to prove that $\bar{y}$ is an isolated point of Y, and hence prove (c), it suffices to prove (d) for g.

Thus we may suppose that M^3 is closed; hence N^3 is closed. Let y_i ($i=1, 2, \ldots, s$) be distinct points of Y; choose as above C^3 3-balls B_i^3 with $y_i \in \text{int } (B_i^3)$, f transverse regular on ∂B_i^3, and the B_i^3 mutually disjoint. From the definition of Y we may suppose that no $f^{-1}(B_i^3)$ is a 3-cell. Let K_i^3 be the C^3 manifold obtained by attaching a copy of the unit 3-ball D^3 to $f^{-1}(B_i^3)$ by a diffeomorphism of their boundaries; let P^3 be similarly obtained from $M^3 - \bigcup_i f^{-1}(\text{int } B_i^3)$ by attaching s copies of D^3. Let K_i^3 be the connected sum of the K_i^3; then M^3 is diffeomorphic to $P^3 \# K^3$. It follows from 3.2 that $s \leqq p(K^3)$, which is $p(M^3)-p(N^3)$ if M^3 (and hence N^3) is orientable.

Hence we may suppose that y_i ($i=1, 2, \ldots, s$) are all the points of Y. Define $h: P^3 \to N^3$ by $h(x)=f(x)$ for $x \in M^3 - \bigcup_i f^{-1}(\text{int } B_i^3)$, and on the balls it is the cone map. Then h is cellular, P^3 is homeomorphic to N^3 [1], and thus [21] diffeomorphic to N^3 (or simply use [6, p. 189, (2.4)] and 3.8).

3.4. Examples. Let $f: M^3 \to S^3$ be the example of [6, (4.5)], and let 1 be the identity map on S^{n-3}; then dim $Y=n-3$ for $f \times 1$. In (b) some hypothesis is needed [6, (2.12)].

3.5. Corollary. *Let M^3 and N^3 be closed, and let f be a C^3 acyclic map. Then there is a cohomology* 3*-sphere K^3 such that $N^3 \# K^3$ is diffeomorphic to M^3 and $f^{-1}(y)$ is cellular for all but at most $p(K^3)$ points $y \in N^3$, where $p(K^3)$ is the number of nontrivial cohomology* 3*-spheres in the prime decomposition of K^3.*

Proof. In the above proof it is immediate from 3.1 and the Vietoris Mapping Theorem applied to g that K_i^3 are orientable and cohomology 3-spheres. Thus K^3 and every summand of K^3 are (orientable) cohomology 3-spheres.

3.6. Corollary. *Let M^3 and N^3 be closed, and let $f: M^3 \to N^3$ and $g: N^3 \to M^3$ be C^3 acyclic. Then f and g are cellular, and M^3 and N^3 are diffeomorphic.*

Proof. By 3.5 $M^3 \approx N^3 \# K^3$ and $N^3 \approx M^3 \# L^3$, so that $M^3 \approx M^3 \# K^3 \# L^3$. From 3.2 $K^3 \approx S^3 \approx L^3$, and the conclusion results from 3.5.

3.7. Corollary. *Let M^3 be closed, and let $f: M^3 \to M^3$ be C^3 acyclic. Then f is cellular.*

If M^3 is orientable in 3.7 and 3.8, then 1-acyclic suffices.

3.8. Corollary. *If $f: S^3 \to M^3$ is C^3 1-acyclic, then f is cellular and $M^3 \approx S^3$.*

Alden Wright [33], [34] has proved analogs of 3.3(c), (d), 3.5, 3.6, 3.7, and 3.8: he assumes no differentiability hypothesis, but instead of 1-acyclic over an *arbitrary* coefficient module, he assumes strongly 1-acyclic over Z or Z_p (p prime). In addition the techniques in the proof (of 3.3) are quite different from his.

Corollaries 3.5, 3.6, and 3.7 are somewhat analogous to results of McMillan [18], and 3.8 is analogous to [17, Theorems 2 and 3] (note that the coefficient module is arbitrary in our case). Actually the author proved 3.2, [11, p. 80, (3.7)] and 3.5 in the spring of 1966, but he has been tardy about publishing them. See also McMillan [16, pp. 134–135, Theorem 5 and Addendum 2], and [15, Corollary 3.5] (see [17]) and Armentrout [1] and [2].

Armentrout [1] has proved (essentially) the following result in the topological category, where it is quite deep. (See also [26].) We now observe that it can be easily obtained in the differential category.

3.9. PROPOSITION. *Let* $f\colon M^3 \to N^3$ *be* C^3 *cellular, and let* $\varepsilon\colon N^3 \to (0, \infty)$ *be continuous. Then there is a* C^3 *diffeomorphism* $h\colon M^3 \to N^3$ *such that* $d(f(x), h(x)) < \varepsilon(f(x))$ *for each* $x \in M^3$.

Proof. We may suppose that ε is sufficiently small that, for each $y \in N^3$, $f^{-1}(S(y, \varepsilon(y)))$ is contained in a closed 3-cell. Let T be a C^3 triangulation of N^3 of mesh less than ε; by [11, p. 78, (3.4)] we may suppose that f is transverse regular on each simplex of each dimension. As in the proof of 3.3 for each r-simplex σ^r $(r=0, 1, 2)$, $f^{-1}(\sigma^r) \approx \sigma^r$, and $f^{-1}(\partial\sigma^3) \approx \partial\sigma^3$. From the transverse regularity $f^{-1}(\partial\sigma^3)$ is bicollared, so that [4] $f^{-1}(\sigma^3) \approx \sigma^3$. Thus the $f^{-1}(\sigma^r)$ define a C^3 triangulation of M^3, and a simplicial homeomorphism $g\colon M^3 \to N^3$; h results from [21].

In general dim n, this argument, using induction, fails, since it is not clear that

$$f|f^{-1}(\operatorname{int} \sigma^r) : f^{-1}(\operatorname{int} \sigma^r) \to \operatorname{int} \sigma^r$$

is cellular for $r < n$. If the Poincaré Conjecture is true in dimensions 3 and 4, it is.

4. A differential characterization of monotone maps. The hypothesis $J \not\equiv 0$ is needed [5, p. 707, (11)].

4.1. REMARK. *Let* M^n *and* N^n *be oriented manifolds, and let* f *be* C^1, *not necessarily onto, such that the Jacobian determinant* $J \not\equiv 0$ *on every nonempty open set. Then* f *is monotone onto if and only if* $\deg f = 1$ *and* $J \geqq 0$ *or* $\deg f = -1$ *and* $J \leqq 0$.

Proof. "Only if" is immediate from Sard's Theorem [28, p. 47, Theorem 3.1] and [28, p. 127, Theorem 4.2].

Suppose $\deg f = 1$, $J \geqq 0$, and f is not monotone. Then for some $\bar{y} \in N^n$, $f^{-1}(\bar{y})$ has at least two components Y_1 and Y_2. Choose a closed n-ball U such that Y_1 and Y_2 are in different components U_1 and U_2 of $f^{-1}(U)$. Then $f(\text{bdy } U_i) \subset \text{bdy } U$, and the local degree $\mu(y, f, U_i)$ [23, §II.2] is independent of $y \in \operatorname{int} U$. Let $w \in \operatorname{int} U$ be a regular value [28, p. 47]; from the hypotheses, $w \in f(U_i)$ for at most one i—say $w \notin f(U_1)$. Thus $\mu(w, f, U_1) = 0$, so that $\mu(y, f, U_1) = 0$ for every $y \in \operatorname{int} U$. Since $J \mid U_1 \not\equiv 0$, there is a regular point $x \in \operatorname{int} U_1 - f^{-1}(f(R_{n-1}))$, so that $\mu(f(x), f, U_1) = 1$, and a contradiction results.

5. Improved hypotheses in [7].

5.1. REMARK. *The differentiability hypotheses of various results in* [7] *can be improved, viz.* C^n *can be replaced by*

(1.3) C^{n-q} ([25] or [9]),

(1.5), (1.6), (1.7), (1.8), (1.9) C^2 (use above (1.3)),

(2.1) (the structure theorem for differentiable open maps) C^3 (proof below),

(2.2), (2.3) C^3 (use above (2.1)),

(2.5) C^3 (proof below).

In addition the hypothesis "M^n is compact" in (1.8) and (2.1) can be replaced by "f is proper".

Proof. (2.1) For the first conclusion, use the original proof, but with the above (1.3). For the second conclusion, follow the original proof through the last full paragraph on p. 93. It suffices to prove that $B_f \not\subset R_{n-3}(f)$; suppose the contrary.

Given $\bar{x} \in B_f$, choose a connected open neighborhood V of x such that $f(V)$ is an open n-ball and $g=f|V: V \to f(V)$ is a proper map [11, p. 74, (1.14)]. By above (1.3) $\dim R_{n-3}(f) \leqq \dim f(R_{n-3}(f)) \leqq n-3$, and since $f(R_{n-3}(f))$ is closed $V-f(R_{n-3}(f))$ is connected. By [9, Theorem 1] $V-f(R_{n-3}(f))$ is simply connected, so that the covering map [22, p. 128] $g|(V-f^{-1}(f(R_{n-3}(f))))$ is a homeomorphism. Thus g is a homeomorphism, contradicting the choice of $\bar{x}$.

(2.5) The argument is unchanged, except for the case $R_{n-1}(f) \subset R_0(f)$. Thus $B_f \subset R_{n-3}(f)$; by above (1.3) $\dim R_{n-3}(f) \leqq n-3$, by above (1.7) f is open, and by the above (2.1) f is a local homeomorphism, yielding the desired contradiction.

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