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INVARIANT STATES(1)

BY
RICHARD H. HERMAN

Abstract. States of a C^*-algebra invariant under the action of a group of automorphisms of the C^*-algebra are considered. It is shown that "clustering" states in the same part are equal and thus the same is true of extremal invariant states under suitable conditions. The central decomposition of an invariant state is considered and it is shown that the central measure is mixing if and only if the state satisfies a strong notion of clustering. Under transitivity of the central measure and some reasonable restrictions, the central decomposition is the ergodic decomposition of the state with respect to the isotropy subgroup.

Introduction. It was shown by Glimm and Kadison [8] that if two pure states ρ and ψ of a C^*-algebra, $\mathfrak{A}$, are in the same part, i.e. $\|\rho-\psi\|<2$, then they induce unitarily equivalent representation of $\mathfrak{A}$. If we are given a group of automorphisms of $\mathfrak{A}$ and ρ and ψ are not pure, but extremal invariant, one is tempted to ask the same question. Størmer [16] has dealt successfully with this question when the group of automorphisms is large [15]. We ask the question only for a subset of the extremal invariant states but without imposing any condition on the group, save that it be amenable. The result, Theorem 1, is that two such states in the same part are equal. Størmer shows this to be true when dealing with extremal invariant factor states and a large group of automorphisms.

The method of proof of Theorem 1 is suggested by Tomiyama's proof [19] of the Glimm, Kadison result. Indeed we obtain information about the structure of the fixed point subalgebra in the second dual of $\mathfrak{A}$. In I.2 we give a condition such that our subset of states coincide with the extremal invariant states under the supposition that our system is η-asymptotically abelian [6].

The second section deals with the central decomposition of invariant states [9], [14]. In the spirit of [9] it provides a link with notions from classical ergodic theory. If ρ is an invariant state on a (now separable) C^*-algebra $\mathfrak{A}$ and μ_ρ is its "central measure", we show the equivalence of the clustering of ρ and the mixing of a transformation group with respect to the measure μ_ρ.

Under further specialization we show, Theorem 3, that the central decomposition is an ergodic decomposition with respect to a certain subgroup of our original group.

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I. 1. Let $\mathscr{G}$ be an amenable group and $\mathfrak{A}$ a C^*-algebra with unit. Suppose there exists a representation of $\mathscr{G}$ as automorphisms of $\mathfrak{A}$, i.e. there exists a homomorphism $g \to \alpha_g$ from $\mathscr{G}$ into Aut $(\mathfrak{A})$. At this point we have the option of requiring that $\mathfrak{A}$ be amenable as a discrete or topological group. Only Proposition 3 requires the notion of topological amenability. Since discrete amenability is a somewhat restrictive condition we point out that the other propositions remain true for amenable topological groups provided that the map $g \to \alpha_g$ satisfies appropriate continuity conditions (e.g. those preceding Proposition 3). Let $\mathfrak{A}^{**}$ denote the double dual of $\mathfrak{A}$. Then $\mathfrak{A}^{**}$ can be identified with a von Neumann algebra $\mathscr{A}$ which is the algebra generated by the universal representation $\pi_u(\mathfrak{A})$ [5]. This identification is such that every state of $\mathfrak{A}$ lifts to a vector state of $\mathscr{A}$. Moreover each α_g lifts to a weakly continuous automorphism of $\mathscr{A}$ [11].

Since $\mathscr{G}$ is amenable there exists an invariant mean on $\mathscr{G}$, i.e. an invariant state on $C(\mathscr{G})$, the bounded continuous functions on $\mathscr{G}$, and let $x, y \in \mathscr{H}_u$, the Hilbert space of the universal representation. By Riesz' lemma, define for each $A \in \mathscr{A}$ an operator $\Phi(A)$ such that

$$(\Phi(A)x, y) = \eta\{(\alpha_g(A)x, y)\}.$$

$\Phi(A)$ lies in the weak (strong) closure of co $\{\alpha_g(A) \mid g \in \mathscr{G}\}$ (the convex hull of the $\alpha_g(A)$) and hence in $\mathscr{A}$. Furthermore Φ is an expectation of $\mathscr{A}$ onto $\mathscr{A}^{\mathscr{G}}$, the fixed point von Neumann subalgebra of $\mathscr{A}$. By this we mean Φ is a positive linear map of $\mathscr{A}$ onto $\mathscr{A}^{\mathscr{G}}$ such that $\Phi(I) = I$ and $\Phi(AB) = A\Phi(B)$ for $A \in \mathscr{A}^{\mathscr{G}}$ and $B \in \mathscr{A}$. This map is actually an extension of the map M_η defined in [6].

Definition. Let ρ be an invariant state of $\mathfrak{A}$, i.e., for $g \in \mathscr{G}$, $\rho(\alpha_g(A)) = \rho(A)$. We say that ρ is $\mathscr{G}$-clustering if $\rho(A\Phi(B)) = \rho(A)\rho(B)$ for $A, B \in \mathscr{A}$. (The extension of ρ to $\mathscr{A}$ is denoted by the same symbol.)

This is a stronger notion of clustering than that presented in [6] in that there the clustering need only hold for $A, B \in \mathfrak{A}$. We have identified $\mathfrak{A}$ with its canonical image $\pi_u(\mathfrak{A})$. In §I.3 we shall compare the two ideas.

Remark 1. The expectation Φ has the property that $\rho \circ \Phi = \rho$ for any invariant state of $\mathfrak{A}$. By the invariance of ρ it follows that for $T \in$ co $\{\alpha_g(A) \mid g \in \mathscr{G}\}$, $\rho(T) = \rho(A)$. Since ρ is weakly continuous on bounded sets $\rho(\Phi(A)) = \rho(A)$ by the remarks following the definition of Φ.

Let ρ be an invariant state $\mathfrak{A}$. Since ρ acts normally on $\mathscr{A}$ we can define its support projection E_ρ [4]. We now show that all parts are one point in

Theorem 1. *Suppose ρ and ψ are invariant $\mathscr{G}$ clustering states on $\mathfrak{A}$ then $\|\rho - \psi\| = 2$ if and only if $\rho \neq \psi$.*

Proof. Let E_ρ and E_ψ denote the supports of ρ and ψ acting on $\mathscr{A}$. Since ρ and ψ are invariant one can verify that E_ρ and E_ψ belong to $\mathscr{A}^{\mathscr{G}}$. Now consider $\rho|\mathscr{A}^{\mathscr{G}}$ and $\psi|\mathscr{A}^{\mathscr{G}}$. Since $\rho(A\Phi(B)) = \rho(A)\rho(B)$ and Φ is an expectation onto $\mathscr{A}^{\mathscr{G}}$ both ρ and ψ are multiplicative there. Also since E_ρ and $E_\psi \in \mathscr{A}^{\mathscr{G}}$ they are respectively the supports of $\rho|\mathscr{A}^{\mathscr{G}}$ and $\psi|\mathscr{A}^{\mathscr{G}}$. $I-E_\rho$ is the projection onto the ideal of $T\in\mathscr{A}^{\mathscr{G}}$ such that $\rho(T^*T)=0$, which by the multiplicativity is two sided and coincides with the kernel of $\rho|\mathscr{A}^{\mathscr{G}}$. Similar remarks hold for ψ. Hence E_ρ and E_ψ are minimal in $\mathscr{A}^{\mathscr{G}}$ and belong to the center of $\mathscr{A}^{\mathscr{G}}$. Thus $E_\rho E_\psi=0$ or $E_\rho=E_\psi$. If $E_\rho=E_\psi$ then $\ker \rho|\mathscr{A}^{\mathscr{G}}=\mathscr{A}^{\mathscr{G}}_{I-E_\rho}=\mathscr{A}^{\mathscr{G}}_{I-E_\psi}=\ker \psi|\mathscr{A}^{\mathscr{G}}$. So $\rho=\psi$ on $\mathscr{A}^{\mathscr{G}}$ since $I\in\mathscr{A}^{\mathscr{G}}$. By Remark 1 $\rho=\psi$ on $\mathscr{A}$. If $\rho\neq\psi$ then their restrictions are not equal so E_ρ is orthogonal to E_ψ in which case

$$|(\rho-\psi)(E_\rho-E_\psi)| = 1-(-1) = 2 \quad\text{and}\quad \|\rho-\psi\| = 2. \qquad \text{Q.E.D.}$$

Remark 2. We can conclude from the proof of Theorem 1 that the fixed point algebra $\mathscr{A}^{\mathscr{G}}$ contains an abelian subalgebra $\mathscr{C}$ as a direct summand and that a state ρ which is invariant and $\mathscr{G}$-clustering on $\mathfrak{A}$ has its support in $\mathscr{C}$. One merely takes $E=\sup E_\rho$ where the ρ are distinct invariant $\mathscr{G}$-clustering states on $\mathfrak{A}$ and defines $\mathscr{C}=\mathscr{A}^{\mathscr{G}}_E$.

If π_ρ is the representation of $\mathfrak{A}$ due to ρ then one has $\rho(A)=(\pi_\rho(A)x_\rho, x_\rho)$ and the invariance of ρ implies the existence of unitary operators $U_\rho(g)$ such that

$$\pi_\rho(\alpha_g(A)) = U_\rho(g)\pi_\rho(A)U_\rho(g)^{-1}, \qquad U_\rho(g)x_\rho = x_\rho.$$

The representation π_ρ extends to a normal representation [5], $\tilde{\pi}_\rho$ of $\mathscr{A}$ such that $\tilde{\pi}_\rho(\mathscr{A})=\pi_\rho(\mathfrak{A})''$.

We adapt a proof of Størmer [16] for the following

Proposition 1. *Let ρ and ψ be invariant $\mathscr{G}$-clustering states and suppose that ω_{x_ρ} is faithful on the fixed points in the von Neumann algebra $\tilde{\pi}_\rho(\mathscr{A})$. Then π_ρ is quasi-equivalent* [5] *to π_ψ if and only if $\rho=\psi$.*

Proof. By the quasi-equivalence there exists a normal isomorphism R of $\pi_\rho(\mathfrak{A})''$ onto $\pi_\psi(A)''$ such that $R(\pi_\rho(A))=\pi_\psi(A)$, $A\in\mathfrak{A}$. By the normality of $\tilde{\pi}_\rho$, $\tilde{\pi}_\psi$ and R we have $R(\tilde{\pi}_\rho(A))=\tilde{\pi}_\psi(A)$, $A\in\mathscr{A}$.

$$\psi = \omega_{x_\psi}\circ\tilde{\pi}_\psi = \omega_{x_\psi}\circ R\circ\tilde{\pi}_\rho = \varphi\circ\tilde{\pi}_\rho,$$

where φ is a normal, invariant state. We now restrict $\psi=\varphi\circ\pi_\rho$ to $\mathscr{A}^{\mathscr{G}}$. Suppose now that $T\in\ker \rho|\mathscr{A}^{\mathscr{G}}$. Since ρ is multiplicative, $T^*T\in\ker \rho|\mathscr{A}^{\mathscr{G}}$ and $\tilde{\pi}_\rho(T^*T)$ is a fixed point in $\tilde{\pi}_\rho(\mathscr{A})$. We have

$$(\tilde{\pi}_\rho(T^*T)x_\rho, x_\rho) = \rho(T^*T) = 0.$$

By assumption $\tilde{\pi}_\rho(T^*T)=0$ hence $\tilde{\pi}_\rho(T)=0$. Thus $\psi(T)=\varphi\circ\tilde{\pi}_\rho(T)=0$ or $\ker \psi|\mathscr{A}^{\mathscr{G}} \supseteq \ker \rho|\mathscr{A}^{\mathscr{G}}$. By the above arguments $\rho=\psi$.

In [16] Størmer shows Proposition 1 to be true for extremal invariant states provided the group of automorphisms is "large."

Remark 3. It is not unreasonable to suppose that ω_{x_ρ} is faithful. This is the case, for example, if ρ is a state satisfying the KMS boundary condition [10], [18] with respect to a representation, $t\to\sigma_t$, of the real line as automorphisms of $\mathfrak{A}$.

2. For the $\mathscr{G}$-clustering of an invariant state ρ we have required $\rho(A\Phi(B)) = \rho(A)\rho(B)$ for $A, B \in \mathscr{A}$. It is clearly more desirable to require this relation only for elements $A, B \in \mathfrak{A}$. We shall call any invariant state with the latter property *weakly $\mathscr{G}$-clustering*. The purpose of dealing with weakly $\mathscr{G}$-clustering states is that they coincide with the extremal invariant states of $\mathfrak{A}$ provided that $\mathfrak{A}$ is η-asymptotically abelian [6, Theorem 4a].

If we have sufficiently many normal invariant states on the fixed point algebra of $\tilde{\pi}_\rho(\mathscr{A})$ then weak $\mathscr{G}$-clustering implies $\mathscr{G}$-clustering.

Definition [12]. $\tilde{\pi}_\rho(\mathscr{A})$ is said to be $\mathscr{G}$-finite if, for T, positive, belonging to the fixed point subalgebra of $\tilde{\pi}_\rho(\mathscr{A})$ $(=[\tilde{\pi}_\rho(\mathscr{A})]^{\mathscr{G}})$, there exists a normal, invariant state σ on $\tilde{\pi}_\rho(\mathscr{A})$ with $\sigma(T)\neq 0$.

Proposition 2. *Let ρ be an invariant weakly $\mathscr{G}$-clustering state. If $\tilde{\pi}_\rho(\mathscr{A})$ is $\mathscr{G}$-finite then ρ is $\mathscr{G}$-clustering.*

Proof. Theorem 1 of [12] states that under these conditions there exists, for $S \in \tilde{\pi}_\rho(\mathscr{A})$, a unique element in

$$\overline{\mathrm{co}}\,\{U_\rho(g)SU_\rho(g)^{-1} \mid g \in \mathscr{G}\} \cap [\tilde{\pi}_\rho(\mathscr{A})^{\mathscr{G}}],$$

which we denote $S^{\mathscr{G}}$. The closure is taken in the strong topology. Further Theorem 2 of that paper shows that the map $S \to S^{\mathscr{G}}$ is ultra weakly and ultra strongly continuous. If $S=\tilde{\pi}_\rho(A)$ then $S^{\mathscr{G}}=\tilde{\pi}_\rho(\Phi(A))$. Clearly $\tilde{\pi}_\rho(\Phi(A)) \in [\tilde{\pi}_\rho(\mathscr{A})]^{\mathscr{G}}$. Let $T_\gamma \in \mathrm{co}\,\{\alpha_g(A) \mid g \in \mathscr{G}\}$ with $T_\gamma \to \Phi(A)$ strongly. Then since $\tilde{\pi}_\rho$ is a normal representation of $\mathscr{A}$, it is strongly continuous [5, p. 56] on bounded sets. Hence $\tilde{\pi}_\rho(T_\gamma) \to \tilde{\pi}_\rho(\Phi(A))$.

We now show the $\mathscr{G}$-clustering of ρ.

If $B \in \mathscr{A}$ the Kaplansky density theorem provides a bounded net $B_\gamma \in \mathfrak{A}$ such that B_γ converges strongly to B. Now by hypothesis

$$\rho(A\Phi(B_\gamma)) = \rho(A)\rho(B_\gamma), \qquad A \in \mathfrak{A}.$$

We then have

$$\begin{aligned}\rho(A)\rho(B_\gamma) &= (\pi_\rho(\Phi(B_\gamma))x_\rho, \pi_\rho(A)x_\rho) = ([\pi_\rho(B_\gamma)]^{\mathscr{G}}x_\rho, \pi_\rho(A^*)x_\rho)\\ &\to ([\tilde{\pi}_\rho(B)]^{\mathscr{G}}x_\rho, \pi_\rho(A^*)x_\rho) = (\tilde{\pi}_\rho(\Phi(B))x_\rho, \pi_\rho(A^*)x_\rho) = \rho(A\Phi(B))\end{aligned}$$

by the above remarks. However $\rho(A)\rho(B_\gamma) \to \rho(A)\rho(B)$. The extension to $A \in \mathscr{A}$ is clear so ρ is indeed $\mathscr{G}$-clustering.

Alternatively, the normality of the map Φ is sufficient to insure that weak $\mathscr{G}$-clustering implies $\mathscr{G}$-clustering. This, however, requires some restrictions. We show that normality occurs under a continuity condition on the map $g \to \alpha_g$, with $\mathscr{G}$ compact.

We follow the notation of [1] and impose the following mild condition:

(+) Let Γ be the map

$$\Gamma\colon (g, A) \to \alpha_g(A) \quad \text{from } \mathscr{G} \times \mathfrak{A} \to \mathfrak{A}$$

and suppose that Γ is jointly continuous at $(e, 0)$ where $\mathfrak{A}$ is given the weak topology, $\sigma(\mathfrak{A}, \mathfrak{A}^*)$.

REMARK 4. If $\mathcal{G}$ is a Baire space and $\mathfrak{A}$ is separable, then this joint continuity condition is automatically a consequence of separate continuity [17]. Of course one then has that the map $g \to \alpha_g(\cdot)$ is strongly continuous on $\mathfrak{A}$.

PROPOSITION 3. *Let $\mathcal{G}$ be a compact group and suppose Γ satisfies $(+)$, then the map Φ is normal.*

Proof. η is given by Haar measure and

$$(\Phi(A)x, y) = \int_{\mathcal{G}} (\alpha_g(A)x, y)\, d\eta.$$

Let $A_\gamma \uparrow A$; $A_\gamma, A \in \mathcal{A}^+$. $(\alpha_g(A)x, x) = \nu_g(\omega_{x,x})(A)$ where ν_g is the affine isomorphism of the state space of $\mathfrak{A}$ defined by $\nu_g(\rho)(A) = \rho(\alpha_g(A))$ [11]. Now $\omega_{x,x}$ is normal on $\mathcal{A}$ hence the orbit piece, $\{\nu_g(\omega_{x,x}) \mid g \in \mathcal{G}\}$, is compact, [1], in the weak topology, $\sigma(\mathfrak{A}^*, \mathcal{A})$. It then follows from a result of Akemann [2] that $\nu_g(\omega_{x,x})(A_\gamma)$ converges uniformly in γ on $\mathcal{G}$ to $\nu_g(\omega_{x,x})(A)$. Thus $(\Phi(A_\gamma)x, x) \to (\Phi(A)x, x)$ and Φ is normal.

In general, however, one should not expect the normality of Φ for noncompact $\mathcal{G}$, as the following example shows:

EXAMPLE. Let η be an invariant mean on $\mathcal{C}(\boldsymbol{R})$, the bounded continuous functions on the real line. Suppose that $t \to U_t$ is a strongly continuous representation of the real line as unitary operators in a Hilbert space $\mathcal{H}$. With $U_t = e^{iHt}$, let us suppose that H has no point spectrum. Suppose $\mathfrak{A}$ is the $\mathcal{C}^*$ algebra $\mathcal{LC}(\mathcal{H})$ of compact operators on $\mathcal{H}$. Then the second dual of $\mathfrak{A}$ is $\mathcal{L}(\mathcal{H})$, all bounded operators on $\mathcal{H}$ [5]. Clearly $\alpha_t(A) = U_t A U_{-t}$ defines an automorphism of $\mathcal{LC}(\mathcal{H})$. If $M = \{H\}'$ then the map Φ constructed above is a projection of $\mathcal{L}(\mathcal{H})$ onto M. If Φ is normal then there exists a nonzero, positive compact operator such that $\Phi(A) \neq 0$. If P is any projection in M such that $P\Phi(A) = \Phi(A)P \geqq \delta P$, $\delta > 0$, then P is finite dimensional [3]. (This is done by showing M_P is separable.) By assumption H has no point spectrum and thus we have a contradiction. Hence Φ is not normal.

II. 1. **Central decomposition of invariant states.** Suppose $\mathfrak{A}$ is a separable $\mathcal{C}^*$-algebra and $\mathfrak{S}$ is its state space. For $\psi \in \mathfrak{S}$ we have

DEFINITION. A Radon measure μ_ψ on $\mathfrak{S}$ is a central Radon measure if there exists a σ-continuous homomorphism $\Lambda\psi$ taking $\mathcal{L}$, the center of $\mathcal{A}$ onto $L^\infty(\mathfrak{S}, \mu_\psi)$ such that

(i) $$\psi(zA) = \int_{\mathfrak{S}} \Lambda_\psi(z)(\phi)\phi(A)\, d\mu_\psi(\phi) \quad \text{for } z \in \mathcal{L} \text{ and } A \in \mathcal{A}.$$

Sakai shows [14] that for every $\psi \in \mathfrak{S}$ there exists a unique pair, μ_ψ, Λ_ψ, such that (i) holds. Moreover $\mu_\psi(K) = \mu_\psi(\mathfrak{S})$ where K is the Borel subset of factor states of $\mathfrak{S}$. That is (i) gives the central decomposition of π_ψ.

This decomposition has been used in [9] to deal with invariant states. In particular a result obtained there is the following:

THEOREM. *Let $\mathfrak{A}$ be a separable C^*-algebra and $g \to \alpha_g$ a homomorphism of the group $\mathcal{G}$ into the automorphism group of $\mathfrak{A}$. Let ρ be a $\mathcal{G}$-invariant state of $\mathfrak{A}$ with central measure μ_ρ. If ρ is extremal invariant then $\{\alpha_g\}$ acts ergodically on $L^\infty(\mathfrak{S}, \mu_\rho)$. If the system $\{\mathfrak{A}, \alpha\}$ is "M-abelian" then the ergodic action of $\{\alpha_g\}$ on $L^\infty(\mathfrak{S}, \mu_\rho)$ implies that ρ is extremal invariant.*

The notion of weak mixing from classical ergodic theory and that of weak clustering of states [6] are analogous. The latter under any of the "asymptotic abelian" conditions (see the definition preceding Theorem 1) is equivalent to extremal invariance. In this vein we show that, in the above setting, the clustering of the state ρ is equivalent to the "mixing" of $\{\alpha_g\}$ on $L^\infty(\mathfrak{S}, \mu_\rho)$.

DEFINITION. If $\mathcal{G}$ is a locally compact, noncompact group then an invariant state ρ on $\mathfrak{A}$ is said to be *clustering* if for $A, B \in \mathfrak{A}$

$$\lim_{g \to \infty} \rho(\alpha_g(A)B) = \rho(A)\rho(B).$$

We shall say that the measure μ_ρ is *mixing* if for measurable sets $E, F \subseteq \mathfrak{S}$

$$\lim_{g \to \infty} \mu_\rho(\alpha_g(E)F) = \mu_\rho(E)\mu_\rho(F).$$

LEMMA 1. *Let $\mathfrak{A}$ be a C^*-algebra and ρ a state on $\mathfrak{A}$. Then there exists a projection ε, from $M = \pi_\rho(\mathfrak{A})''$ onto the center, $\mathfrak{Z}$, of M, such that $\rho \circ \varepsilon = \rho$. If $\{\alpha_g : g \in \mathcal{G}\}$ acts as a group of automorphisms of $\mathfrak{A}$ and ρ is invariant under this action, then, for $A \in M$, $\alpha_g(\varepsilon(A)) = \varepsilon(\alpha_g(A))$.*

Proof. Suppose that $x_\rho \in \mathcal{H}_\rho$ is such that $\rho(A) = (\pi_\rho(A)x_\rho, x_\rho)$ for $A \in \mathfrak{A}$. The projection $P = [\mathfrak{Z}x_p]$ belongs to $\mathfrak{Z}$. Thus $M_P \subset \mathfrak{Z}_P = \mathfrak{Z}_P$ since P contains x_P and $\mathfrak{Z}_P$ is abelian. Further one has the existence of an isomorphism $\tau: \mathfrak{Z}_P \to \mathfrak{Z}$ [4]. For $A \in M$ write $\varepsilon(A) = \tau(PAP)$. If $z \in \mathfrak{Z}$ $\varepsilon(zA)$ is the composition of the maps $zA \to PzAP = zPAP = \tau(zP)\tau(PAP) = z\tau(PAP)$, and thus equals $z\varepsilon(A)$. For $A \in M$ we write

$$(Ax_\rho, x_\rho) = (PAPx_\rho, x_\rho) = (\tau(PAP)x_\rho, x_\rho).$$

Thus $\omega_{x_\rho} \circ \varepsilon = \omega_{x_\rho}$ on M. The above method was shown to the author by Professor M. Takesaki.

The second part of the lemma is seen as follows: For $A \in M$, $\alpha_g(A) = U_\rho(g)AU_\rho(g^{-1})$. Since $U_\rho(g)x_\rho = x_\rho$, for all $g \in \mathcal{G}$, we have $U_\rho(g)P = PU_\rho(g)$. If $A \in M$, then

$$PU_\rho(g)AU_\rho(g^{-1})P = U_\rho(g)(PAP)U_\rho(g^{-1}) \in \mathfrak{Z}P.$$

Since τ is an isomorphism, $\varepsilon(\alpha_g(A)) = \alpha_g(\varepsilon(A))$.

LEMMA 2. *Let $\{\mathfrak{A}, \alpha_g, \rho\}$ be as in the second part of Lemma 1. Then ρ is clustering on $\mathfrak{A}$ iff ρ is clustering on $\mathscr{A}$.*

Proof. The lemma is an immediate consequence of the fact that clustering on $\mathfrak{A}$ implies that in $\mathscr{H}_\rho$, $U_\rho(g) \to P_{x_\rho}$ weakly as $g \to \infty$.

DEFINITION. Let $\mathfrak{A}$ be a $\mathscr{C}^*$-algebra acted upon by a group of automorphisms $\{\alpha_g : g \in \mathscr{G}\}$. Let $\mathscr{G}$ be locally compact, noncompact. The system $(\mathfrak{A}, \mathscr{G}, \alpha)$ is said to be weakly asymptotically abelian if for $\psi \in \mathfrak{A}^*$, $A, B \in \mathfrak{A}$ it follows that

$$\lim_{g \to \infty} \psi([\alpha_g(A), B]) = 0.$$

We are now ready to prove

THEOREM 2. *Let $\mathfrak{A}$ be a separable $\mathscr{C}^*$-algebra acted upon by a group of automorphisms $\{\mathscr{A}_g : g \in \mathscr{G}\}$. Suppose that $\mathscr{G}$ is locally compact and noncompact and that for $\psi \in \mathfrak{A}^*$ the function $g \to \psi(\alpha_g(A))$ is continuous. Suppose that ρ is an invariant state with central Radon measure μ_ρ. If ρ is clustering on $\mathfrak{A}$ then $\{\alpha_g\}$ is mixing on $L^\infty(\mathfrak{S}, \mu_\rho)$. If the system $(\mathfrak{A}, \mathscr{G}, \alpha)$ is weakly asymptotically abelian then the converse is true.*

Proof. By Lemma 2, ρ is clustering on $\mathscr{A}$. Taking elements $z_1, z_2 \in \mathscr{L}$ corresponding to characteristic functions χ_{E_1} and $\chi_{E_2} \in L^\infty(\mathfrak{S}, \mu_\rho)$ one easily obtains the desired result.

If $z(\tilde{\pi}_\rho)$ is the central projection in $\mathscr{L}$ such that ker $\tilde{\pi}_\rho = \mathscr{A}I - z(\tilde{\pi}_\rho)$ we may restrict our considerations of the converse to $\mathscr{A}z(\tilde{\pi}_\rho)$ and thus to $M = \pi_\rho(\mathfrak{A})''$. Thus let $A, B \in \mathfrak{Z}$ (the center of M) correspond to simple functions in $L^\infty(\mathfrak{S}, \mu_\rho)$. The mixing of $\{\alpha_g\}$ on $L^\infty(\mathfrak{S}, \mu_\rho)$ clearly give $\rho(\alpha_g(A)B) \to \rho(A)\rho(B)$ as $g \to \infty$.

(We are abusing the notation and writing $\rho(B)$ for $\omega_{x_\rho}(B)$ if $B \in M$.) One then has clustering on all of $\mathfrak{Z}$ by the density of simple functions in $L^\infty(\mathfrak{S}, \mu_\rho)$.

Suppose now that A is arbitrary in M and $B \in \mathfrak{Z}$. Let ε be the projection in Lemma 1. We have

$$\begin{aligned} \rho(\alpha_g(A)B) = \rho \circ \varepsilon(\alpha_g(A)B) &= \rho(\varepsilon(\alpha_g(A)B)) \\ &= \rho(\alpha_g(\varepsilon(A)B)). \end{aligned}$$

However, by the preceding remarks as $g \to \infty$ the right-hand side goes to $\rho(\varepsilon(A))\rho(B) = \rho(A)\rho(B)$.

Now suppose B is arbitrary. Then

$$\rho(\alpha_g(A)B) = \rho(\alpha_g(A)(I-\varepsilon)(B)) + \rho(\alpha_g(A) \in (B)).$$

By the preceding arguments the proof will be complete if we show

$$\lim_{g \to \infty} \rho(\alpha_g(A)(I-\varepsilon)(B)) = 0.$$

Suppose $A = \pi_\rho(S)$, $\|A\| \leqq 1$, $S \in \mathfrak{A}$ and g_γ is an arbitrary net of elements in $\mathscr{G}$, $g_\gamma \to \infty$. The unit ball of M contains $\{\alpha_{g_\gamma}(A)\}$ and thus there is a subset $\{\alpha_{g_\nu}(A)\}$

converging weakly to an element $T \in M$. Since $(\mathfrak{A}, \mathcal{G}, \alpha)$ is weakly asymptotically abelian, $T \in \mathfrak{Z}$.

Returning to our proof, let $\{g_\gamma\}$ be an arbitrary net of elements in $\mathcal{G}$ converging to ∞. If $A \in \pi_\rho(\mathfrak{A})$ we have shown there exists a subset $\{g_\nu\}$ of $\{g\}$ such that

$$\lim_{g_\nu \to \infty} \rho(\alpha_{g_\nu}(A)(I-\varepsilon)(B)) = \rho(T(I-\varepsilon)(B)).$$

But $T = \varepsilon(T)$ and $\rho \circ \varepsilon = \rho$ so that this last term is zero.

If A is arbitrary in M, $\|A\| \leqq 1$. Let A_β be a net in $\pi_\rho(\mathfrak{A})$ converging strongly to A. Then

$$\begin{aligned} |\rho(\alpha_g(A)(I-\varepsilon)(B))| &= |\rho(\alpha_g(A-A_\beta)(I-\varepsilon)(B)) + \rho(\alpha_g(A_\beta)(I-\varepsilon)(B))| \\ &\leqq \rho((A-A_\beta)^*(A-A_\beta))\rho(C^*C) + |\rho(\alpha_g(A_\beta)(I-\varepsilon)(B))| \end{aligned}$$

where $C = (I-\varepsilon)(B)$. Choosing β so that the first term is sufficiently small we have our result since we know the second term goes to zero.

2. *Ergodic decomposition.* As in §1 let us consider an invariant state ρ and its central decomposition

$$\rho = \int \psi \, d\mu_\rho(\psi).$$

Assume further that the measure μ_ρ is transitive, i.e. there exists a state $\varphi \in \mathfrak{S}$ such that if $\mathcal{O}_\varphi = \{\alpha_g^t(\varphi) \mid g \in \mathcal{G}\}$, then $\mu_\rho(\mathcal{O}_\rho) = \mu_\rho(\mathfrak{S})$. Further let $S_\varphi = \{g \in \mathcal{G} \mid \alpha_g^t(\varphi) = \varphi\}$. If we assume that S_φ is a normal subgroup of $\mathcal{G}$ then it is natural to ask when the components ψ are extremal S_φ invariant. We assure the existence of an ergodic decomposition by requiring S_φ to be "large" and then show that the central decomposition is it.

We first need some preliminary remarks and notation. $\mathfrak{S} \cap \mathcal{L}_{S_\varphi}^\perp$ will denote the compact convex subset of S_φ-invariant state of $\mathfrak{A}$. $\mathcal{E}(\mathfrak{S} \cap \mathcal{L}_{S_\varphi}^\perp)$ will be the extreme points of that set. Since ρ is $\mathcal{G}$-invariant we know that there is a unitary group representation $U_\rho(g)$ of $\mathcal{G}$ on $\mathcal{H}_\rho$. Let $U_\rho(S_\varphi) = \{U_\rho(g) \mid g \in S_\varphi\}$ and write $\mathcal{R}_\rho(S_\varphi) = \{\pi_\rho(\mathfrak{A}), U_\rho(S_\varphi)\}''$.

If K (see [13]) is a compact, convex subset of a locally convex topological vector space E, let $\mathcal{C}$ be the (real) convex continuous function on K. If λ and μ are non-negative regular Borel measures on K write $\lambda \succ \mu$ if $\lambda(f) \geqq \mu(f)$ for each $f \in \mathcal{C}$. Such a measure is said to be maximal if it is maximal for the ordering $\succ$. A measure ν is said to have resultant $\rho \in K$ if $f(\rho) = \int f(\sigma)\, d\nu(\sigma)$ for all $f \in E'$ (the dual space of E). We write $\nu \sim \varepsilon_\rho$.

In the theorem below we shall assume $\mathcal{R}_\rho(S_\varphi)' \subseteq \mathfrak{Z}$, the center of $\pi_\rho(\mathfrak{A})''$. This is satisfied if S_φ acts as a large group of automorphisms in the sense of Størmer [15]. This condition is less strict than weak asymptotic abelianness. If the entire center $\mathfrak{Z}$ were fixed elementwise, a trivial application of decomposition theory would give the desired result. It is not necessary to stipulate this as we have.

THEOREM 3. *Let ρ, $\{\alpha_g\}$ and $\mathscr{G}$ be as in Theorem 2 with the further assumption that $\mathscr{G}$ is separable. Suppose μ_ρ is transitive with $\mathcal{O}_\varphi$ and S_φ as above. If S_φ is normal and $\mathscr{R}_\rho(S_\varphi)' \subseteq \mathfrak{Z}$, then each $\psi \in \mathcal{O}_\varphi$ belongs to $\mathscr{E}(\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi})$.*

Proof. Each $\psi \in \mathcal{O}_\varphi$ belongs to $\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}$. Since $\mathfrak{A}$ is separable $\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}$ is a compact metric space. Thus its extreme points are a G_δ set [13, p. 7]. Suppose that μ_ρ is maximal in the ordering above, then μ_ρ is concentrated in the *Borel* sense on $\mathscr{E}(\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi})$ by the separability of $\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}$ and [13, p. 27]. Further $\mathcal{O}_\varphi$ is a Borel set (in fact it is closed [17]). One concludes that $\mathcal{O}_\varphi \cap \mathscr{E}(\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}) \neq \varnothing$, since $\mu_\rho(\mathcal{O}_\varphi) = \mu_\rho(\mathscr{E}(E \cap \mathscr{L}^{\perp}_{S_\varphi}))$ by assumption of transitivity. Then $\mathcal{O}_\varphi \subseteq \mathscr{E}(\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi})$. Thus the theorem is proven once we show the maximality of μ_ρ.

We adapt, in a straightforward manner, a proof in [7]. Let $\mathscr{C}$ be the space of all real-valued, convex, continuous functions on $\mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}$ and suppose $\nu \sim \varepsilon_\rho$ there. Let $f \in \mathscr{C}$ and $\varepsilon > 0$. Then [13, p. 67] there is a discrete measure $\nu_{f,\varepsilon} = \sum_1^n \alpha_i \rho_i$ with $\sum \alpha_i = 1$, $\alpha_i > 0$ and $\rho_i \in \mathfrak{S} \cap \mathscr{L}^{\perp}_{S_\varphi}$ such that $\nu_{f,\varepsilon} \sim \varepsilon_\rho$ and $\int f\, d\nu - \int f\, d\nu_{f,\varepsilon} < \varepsilon$. One concludes that $\rho = \sum_1^n \alpha_i \rho_i$. Hence $\rho_i(\cdot) = (\pi_\rho(\cdot) z_i x_\rho, x_\rho)$ with $0 \leqq z_i \leqq 1$, a unique element in $\mathscr{R}_\rho(S_\varphi)'$. By hypothesis, however, $z_i \in \mathfrak{Z}$. The proof then proceeds as in [7, Theorem 1.1].

It has been shown in [7, Theorem 2.1] that if $\mathscr{G}$ is the Euclidean group, then the transitivity of μ_ρ implies that the coset space $\mathscr{G}/S_\varphi$ is compact. With the assumption of Theorem 3, one then has that S_φ is noncompact, locally compact. Thus if the system $\{\mathfrak{A}, \alpha, \mathscr{G}\}$ is for example weakly asymptotically abelian, then so is the system $\{\mathfrak{A}, \alpha, S_\varphi\}$. This together with the remarks preceding Theorem 3 show that the assumption $\mathscr{R}_\rho(S_\varphi)' \subseteq \mathfrak{Z}$ is not unreasonable.

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Added in proof. We note that Theorem 3 follows, essentially, from the theorem of [9], quoted above, and Theorem 5.2 of [17]; without the assumption of normality. However, our proof yields information in the *nontransitive* case, viz.; if $\mathscr{G}_0$ is a normal subgroup of $\mathscr{G}$ satisfying the hypothesis of Theorem 3, then the restriction of the central measure to the set of $\mathscr{G}_0$-invariant states is maximal, and is thus concentrated, in the Borel sense, on the extreme points of that set.

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University of California,
Los Angeles, California 90024