QUASICONFORMAL MAPPINGS
AND ROYDEN ALGEBRAS IN SPACE(1)

BY
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Abstract. On every open connected set $G$ in Euclidean $n$-space $\mathbb{R}^n$ and for every index $p > 1$, we define the Royden $p$-algebra $M_p(G)$. We use results by F. W. Gehring and W. P. Ziemer to prove that two such sets $G$ and $G'$ are quasiconformally equivalent if and only if their Royden $n$-algebras are isomorphic as Banach algebras. Moreover, every such algebra isomorphism is given by composition with a quasiconformal homeomorphism between $G$ and $G'$. This generalizes a theorem by M. Nakai concerning Riemann surfaces. In case $p \neq n$, the only homeomorphisms which induce an isomorphism of the $p$-algebras are the locally bi-Lipschitz mappings, and for $1 < p < n$, every such isomorphism arises this way. Under certain restrictions on the domains, these results extend to the Sobolev space $H_2^p(G)$ and characterize those homeomorphisms which preserve the $H_2^p$ classes.

Introduction. In 1960 Nakai proved [12] that two Riemann surfaces are quasiconformally equivalent if and only if their Royden algebras are isomorphic. In this paper we characterize a class of homeomorphisms and a family of Banach algebras which extend this result to higher dimensions.

On each finite subset $G$ of Euclidean $n$-space $\mathbb{R}^n$ we define the Royden $p$-algebra $M_p(G)$ for arbitrary $p > 1$. $M_p(G)$ is a commutative semisimple Banach algebra with identity. We then form the Gelfand compactification of $G$ with respect to the Royden $p$-algebra and call it the Royden $p$-compactification of $G$ [15].

We also define $Q_p$-mappings of $G$ onto $G'$ and characterize them to be exactly those homeomorphisms which induce by composition an isomorphism between the respective Royden $p$-algebras. Using the Gelfand theory we prove that for $1 < p \leq n$ every such isomorphism is obtained by composition with a $Q_p$-mapping.

If $G$ has finite measure we may consider $M_p(G)$ to be a subset of the Sobolev space $H_2^p(G)$. With certain restrictions on the domains, we also prove that the $Q_p$-mappings are exactly those homeomorphisms which leave the $H_2^p$ classes invariant.

Recently F. W. Gehring has shown that all $Q_p$-mappings are quasiconformal mappings. In fact for every $p \neq n$ they are precisely the class of all bi-Lipschitz mappings.

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mappings. Thus the case \( p = n \) is distinguished when studying either the Royden algebras or the Sobolev spaces under composition. This special case was introduced by C. Loewner [8] as "conformal capacity" and studied by F. W. Gehring [4].

H. M. Reimann [14] has recently considered a class of homeomorphisms similar to the \( Q_p \)-mappings.

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1. Preliminaries. The following notation is used: \( m_n \) is \( n \)-dimensional Lebesgue measure, \( m_{n-1} \) is \((n-1)\)-dimensional Hausdorff measure, \( L_p(G) \) is the usual space of equivalence classes of functions \( f \) for which \( |f|^p \) is \( m_n \)-integrable, with norm \( \|f\|_p \), and \( C_0^\infty(G) \) is the space of infinitely differentiable functions with compact support contained in \( G \). Unless otherwise indicated, all functions are complex-valued and all integrals are taken over \( G \) (or \( G' \)). \( x = (x^1, \ldots, x^n) \) denotes an arbitrary point in \( \mathbb{R}^n \), \( B(x, r) \) and \( S(x, r) \) are the usual ball and sphere, abbreviated \( B_n \) and \( S^{n-1} \) in case \( x = 0 \) and \( r = 1 \), and \( \omega_n = m_{n-1}(S^{n-1}) \).

Denote by \( ACL(G) \) the collection of all functions which are absolutely continuous along the intersection of every compact \( n \)-interval with \( m_{n-1} \)-almost every line parallel to the coordinate axes. Note that every \( f \in ACL(G) \) has a gradient.

More generally, a function \( f \) has a (weak) gradient

\[
\nabla f = (r^1, \ldots, r^n)
\]

if there exist distributional derivatives \( r^i \) satisfying

\[
\int w r^i \, dm_n = -\int f \frac{\partial w}{\partial x^i} \, dm_n, \quad w \in C_0^\infty(G),
\]

for \( i = 1, \ldots, n \). Then the Dirichlet \( p \)-integral of \( f \) is

\[
D_p[f] = \int |\nabla f|^p \, dm_n = \int \left[ \sum_{i=1}^n |r^i|^2 \right]^{p/2} \, dm_n.
\]

In case \( f \) has locally integrable partial derivatives, they are equal \( m_n \)-a.e. to the \( r^i \), and hence satisfy (1.1). Thus, for \( f \in ACL(G) \) with \( D_p[f] < \infty \), \( \nabla f \) as defined above is the usual gradient.

A real-valued function \( u \) is of class \( H^1_p \) on \( G \) if \( u \in L_p(G) \) and if \( u \) has a gradient and satisfies \( D_p[u] < \infty \). A homeomorphism is of class \( H^1_p \) on \( G \) if each of its coordinate functions is. \( H^1_p(G) \) is the space of all such functions with the norm

\[
\|u\| = \left\{ \int |u|^2 + |\nabla u|^2 \right\}^{1/2} \, dm_n.
\]

As is customary with \( L_p(G) \) we write \( u \in H^1_p(G) \) and mean that \( u \) is of class \( H^1_p \) on \( G \). \( H^1_p(G) \) is complete in this norm, and each of its equivalence classes contains a
representative in ACL (G) [10, p. 66]. This fact together with Fubini’s theorem implies the following.

**Lemma 1.1.** A continuous real-valued function \( u \) which is locally of class \( H^1_p \) on \( G \) is in ACL (\( G \)).

A ring \( R \) is a connected open set in \( \mathbb{R}^n \) whose complement consists of a bounded component \( C_0 \) and an unbounded one \( C_1 \). For \( 1 < p < \infty \), the \( p \)-capacity of \( R \) is

\[
\text{cap}_p R = \inf \left\{ \int_R |\nabla u|^p \, dm_n \right\},
\]

where the infimum is taken over all continuous functions \( u \in \text{ACL} (R) \) with constant boundary values 0 and 1. As in [3, p. 358] each such function may be extended to \( \mathbb{R}^n \) and the integral taken over \( \mathbb{R}^n \) without affecting \( \text{cap}_p R \). Such an extended function is admissible for \( R \).

In case \( R \) is the spherical ring \( \{x : a < |x - x_0| < b\} \), then

\[
\text{cap}_p R = \omega_n \left( \int_a^b t^{n-1} \, dt \right)^{1-p} = \omega_n (\log b/a)^{1-p}, \quad p = n,
\]

\[
= \omega_n ((b^q - a^q)/q)^{1-p}, \quad p \neq n,
\]

where \( q = (p-n)/(p-1) \). Note that this gives easily the existence of a unique extremal function for \( \text{cap}_p R \) whenever \( R \) is a spherical ring; for \( p=n \) this is well known and for \( p \neq n \) it is the function \( |x|^q \) with proper normalization. The uniqueness follows from Clarkson’s inequality as in [11, pp. 74, 77].

2. **The Royden \( p \)-algebra.** Denote by \( M_p(G) \) the collection of all continuous functions \( f \) such that

(i) \( f \in \text{ACL} (G) \),

(ii) \( \|f\|_\infty = \sup_{G} |f| < \infty \),

(iii) \( D_p[f] < \infty \).

Since Minkowski’s inequality implies

\[
D_p[f g]^{1/p} \leq \|f\|_\infty D_p[g]^{1/p} + \|g\|_\infty D_p[f]^{1/p}, \tag{2.1}
\]

it follows that \( M_p(G) \) is an algebra under pointwise multiplication with the constant function 1 as identity.

**Theorem 2.1.** With the norm

\[
\|f\| = \|f\|_\infty + D_p[f]^{1/p}, \tag{2.2}
\]

\( M_p(G) \) is a complex commutative Banach algebra with identity.

**Proof.** We conclude from (2.1) that \( \|fg\| \leq \|f\| \|g\| \), and since \( \|1\| = 1 \) we only need to show that \( M_p(G) \) is complete in the norm (2.2) to finish the proof.

Let \( \{f_j\} \) be a Cauchy sequence in \( M_p(G) \). Then \( \{f_j\} \) converges uniformly on \( G \) to a bounded continuous function \( f \), and the derivatives \( \{r_j\} \) form a Cauchy sequence in
There exist functions $r_1, \ldots, r^n \in L_p(G)$ such that $\|r_i - r^i\|_p \to 0$, and an easy calculation shows that $f$ and $r^i$ satisfy (1.1). Thus $f$ has a gradient satisfying $D_p[f] < \infty$, and $f$ is locally of class $H^p_2$ on $G$. Finally Lemma 1.1 implies that $f \in M_p(G)$, proving the theorem.

A sequence $\{f_j\}$ of functions in $M_p(G)$ converges in the $BD_p$-topology to $f$ if

(i) $\|f_j - f\|_\infty \leq M < \infty$,

(ii) $f_j \to f$ uniformly on compact subsets of $G$,

(iii) $D_p[f_j - f] \to 0$.

**Corollary 2.2.** $M_p(G)$ is complete in the $BD_p$-topology.

**Proof.** Any Cauchy sequence in the $BD_p$-topology converges uniformly on compact subsets to a bounded continuous function, and the proof is similar to the above proof.

3. **Induced isomorphisms.** Let $T$ be a homeomorphism of $G$ onto $G'$. Denote $T \in Q_p(K) = Q_p(K; G)$ if for some $K < \infty$

$$\text{cap}_p T(R) \leq K \text{ cap}_p R$$

for every spherical ring $R$ satisfying $\bar{R} \subset G$. $T$ is called a $Q_p$-mapping if $T \in Q_p(K; G)$ and $T^{-1} \in Q_p(K; G')$ for some $K$. $T$ is a quasiconformal mapping if for some $1 \leq K < \infty$

$$K^{-1} \text{ cap}_n R \leq \text{ cap}_n T(R) \leq K \text{ cap}_n R$$

for every ring $R$ satisfying $\bar{R} \subset G$.

The restriction to spherical rings in the definition of $Q_p(K)$ is necessary for the additivity of the $p$-capacity in (4.3).

**Remark 3.1.** It is well known that a $Q_\alpha$-mapping is a quasiconformal mapping (see [3], [11] or [19]). Thus every $Q_p$-mapping is a quasiconformal mapping [5, Theorem 1] and is in fact a bi-Lipschitz mapping for $p \neq n$ [5, Theorem 2]. It follows easily (see Corollary 3.3) that the class of $Q_p$-mappings for all $p \neq n$ is precisely the class of all homeomorphisms $T$ which together with their inverses satisfy the following Lipschitz condition: There exists an $M < \infty$ such that

$$\limsup_{|x - y| \to 0} \frac{|T(x) - T(y)|}{|x - y|} \leq M, \quad x \in G.$$  

**Theorem 3.2.** Every $Q_p$-mapping $T$ of $G$ onto $G'$ induces an algebra isomorphism $\varphi_T$ of $M_p(G')$ onto $M_p(G)$ defined by

$$\varphi_T(f) = f \circ T, \quad f \in M_p(G').$$

**Proof.** Since the composition $\varphi_T$ clearly preserves the algebraic operations, we prove only $\varphi_T(M_p(G')) \subset M_p(G)$. This will complete the proof since the opposite inclusion follows similarly.
Let $f \in M_p(G')$ and set $u = \Re f$ and $v = u \circ T$. In case $p \neq n$ it follows from (3.1) and the Rademacher-Stepanoff theorem [17, p. 310] that $T$ and $T^{-1}$ are differentiable $m_n$-a.e. and that

\[ M^{-n} \leq |JT| \leq M^n \quad m_n\text{-a.e. in } G, \]

for some $M < \infty$. Again using (3.1), we get

\[ |\nabla v| \leq M |\nabla u| \circ T \quad m_n\text{-a.e. in } G. \]

Then (3.3) and (3.4) yield

\[
\int_G |\nabla v|^p \, dm_n \leq M^p \int_G |\nabla u|^p \circ T \, dm_n \\
\leq M^{p+n} \int_G |\nabla u|^p \circ T |JT| \, dm_n = M^{p+n} \int_G |\nabla u|^p \, dm_n.
\]

For $p = n$ let $Q$ be any $n$-cube satisfying $\bar{Q} \subseteq G$ and denote $Q' = T(Q)$, $u_0 = u|Q'$ and $v_0 = u_0 \circ T|Q$. Then $u_0 \in H^1_n(Q')$ and it follows from [20, Remark 4.2] that $v_0 \in H^1_n(Q)$ and

\[ \int_Q |\nabla v_0|^n \, dm_n \leq K \int_{Q'} |\nabla u_0|^n \, dm_n. \]

Exhausting $G$ by a sequence of disjoint cubes gives

\[ \int_G |\nabla v|^n \, dm_n \leq K \int_{Q'} |\nabla u|^n \, dm_n. \]

We apply the above argument to $\text{Im} f$ and see that for any $1 < p < \infty$ there exists a constant $K_2$ depending only on $K, M, p$ and $n$ such that

\[ D_p[g] \leq K_2 D_p[f], \]

where $g = f \circ T, f \in M_p(G')$. Lemma 1.1 then implies that $g \in M_p(G)$, concluding the proof.

**Corollary 3.3.** Every homeomorphism $T$ of $G$ onto $G'$ such that $T$ and $T^{-1}$ satisfy (3.1) is a $Q_p$-mapping for all $p$.

**Proof.** Such a $T$ satisfies (3.3) and (3.4). Let $R'$ be a spherical ring in $G'$, $\bar{R} \subseteq G'$, and let $u$ be the extremal function for $\text{cap}_p R'$. Setting $v = u \circ T$, the above proof gives

\[ \text{cap}_p R \leq \int_G |\nabla v|^p \, dm_n \leq M^{p+n} \int_{Q'} |\nabla u|^p \, dm_n = M^{p+n} \text{cap}_p R', \]

where $R = T^{-1}(R')$. Hence $T^{-1} \in Q_p(M^{p+n})$ and the corollary follows by applying the same argument to $T$.

4. **Characterization of $Q_p$-mappings.** In this section we give a necessary and sufficient condition for a given homeomorphism $T$ of $G$ onto $G'$ to be a $Q_p$-mapping.
Suppose that \( T \) has the property that \( u = v \circ T^{-1} \) is admissible for \( T(R) \) whenever \( v \) is the extremal function for \( \text{cap}_p R \) for any spherical ring satisfying \( \bar{R} \subset G \); i.e., suppose \( T^{-1} \) preserves ACL functions. Suppose further that there exists a constant \( K_1 < \infty \) independent of \( R \) such that

\[
D_p[u]^{1/p} \leq K_1\|v\|
\]

whenever \( u \) and \( v \) are as above.

**Lemma 4.1.** \( T \in Q_p(K) \) for some \( K \leq K_1 \).

**Proof.** If the lemma were false there would exist a spherical ring \( R = \{ x : a < |x - x_0| < b \} \) with \( \bar{R} \subset G \) such that

\[
\text{cap}_p T(R) \geq d^p \text{cap}_p R
\]

for some \( d > K_1 \). We may assume that \( \text{cap}_p R \) is arbitrarily large because for any positive integer \( m \) we may define

\[
R_j = \{ x : r_{j-1} < |x - x_0| < r_j \}, \quad j = 1, \ldots, m,
\]

where

\[
r_j = [a^{m-j}b^{j}]^{1/m}, \quad p = n,
\]

\[
= \left[ \frac{(m-j)a^q + jb^p}{m} \right]^{1/q}, \quad p \neq n,
\]

and \( q = (p-n)/(p-1) \). Then \( \text{cap}_p R_j = m^{p-1} \text{cap}_p R \) and

\[
\sum_{j=1}^{m} (\text{cap}_p R_j)^{1/(1-p)} = (\text{cap}_p R)^{1/(1-p)},
\]

which together with the \( p \)-capacity version of [2, Lemma 2] gives

\[
d^p \text{cap}_p R \leq \left\{ \sum_{j=1}^{m} (\text{cap}_p T(R_j))^{1/(1-p)} \right\}^{1-p}
\]

An easy calculation shows that for some \( 1 \leq k \leq m \) we must have

\[
\text{cap}_p T(R_k) \geq m^{p-1} d^p \text{cap}_p R = d^p \text{cap}_p R_k,
\]

and \( R_k \) satisfies both (4.2) and \( \text{cap}_p R_k = m^{p-1} \text{cap}_p R \). Thus we could use \( R_k \) instead of \( R \).

Now choose some \( c \), with \( K_1 < c < d \), and assume

\[
\text{cap}_p R > c^p(d-c)^{-p}.
\]

Let \( v \) be the extremal function for \( \text{cap}_p R \) and set \( u = v \circ T^{-1} \). Then by hypothesis \( D_p[u] \geq \text{cap}_p T(R) \), and it follows from (4.1) and (4.2) that

\[
K_1\|v\| \geq D_p[u]^{1/p} \geq (\text{cap}_p T(R))^{1/p} \geq d(\text{cap}_p R)^{1/p}
\]

\[
= (d-c)(\text{cap}_p R)^{1/p} + cD_p[v]^{1/p} > c(1 + D_p[v]^{1/p}) = c\|v\|.
\]

But this gives \( c < K_1 \), contradicting the choice of \( c > K_1 \) and proving the lemma.
Theorem 4.2. Every homeomorphism $T$ of $G$ onto $G'$ for which $\varphi_T$ is an algebra isomorphism of $M_p(G')$ onto $M_p(G)$ is a $Q_p$-mapping.

Proof. Let $R$ be a spherical ring, $\bar{R}\subseteq G$, and let $v$ be the extremal function for cap$_p R$. Then $\varphi_T^{-1}(v) = v \circ T^{-1} = u \in M_p(G')$, and $u$ is clearly admissible for $T(R)$. In §6 we apply some Banach algebra results (Lemma 6.1) to prove that $\varphi_T^{-1}$ is a bounded operator. Setting $K_1 = \|\varphi_T^{-1}\|$ satisfies (4.1) and by Lemma 4.1 we have $T \in Q_p(K)$ for some $K \leq K_1$. Applying the same argument to $T$ completes the proof.

Theorems 3.2 and 4.2 give the following characterization of a $Q_p$-mapping.

Corollary 4.3. Let $T$ be a homeomorphism of $G$ onto $G'$. Then $T$ is a $Q_p$-mapping if and only if $\varphi_T$ is an algebra isomorphism of $M_p(G')$ onto $M_p(G)$.

Another corollary follows from Theorem 4.2 and Remark 3.1.

Corollary 4.4. Let $T$ be a $Q_p$-mapping of $G$ onto $G'$, $p \neq n$. Then $M_p(G)$ and $M_p(G')$ are algebra isomorphic for every $p'$.

Corollary 4.5. $\varphi_T$ is an isometry if and only if $T, T^{-1} \in Q_p(1)$.

Proof. If $T, T^{-1} \in Q_p(1)$ then it follows that the constant $K_1 = 1$ in (3.5). For $p = n$ this is well known and for $p \neq n$ it follows from Remark 3.1 and the fact that $K_0 = K_2n/p$ in the proof of [5, Theorem 1]. Then (3.5) applied to $T$ and $T^{-1}$ implies that $D_p[f] = D_p[\varphi_T(f)]$, $f \in M_p(G')$. But $\|f\|_\infty = \|\varphi_T(f)\|_\infty$ trivially, proving the isometry.

Conversely, if $\varphi_T$ is an isometry then (4.1) holds with $K_1 = 1$ and Lemma 4.1 implies $T \in Q_p(1)$. The same is true for $T^{-1}$, concluding the proof.

5. Sobolev spaces. With certain geometrical restrictions on the domains $G$ and $G'$ the techniques of the previous section may be used to characterize those homeomorphisms whose composition maps the Sobolev spaces $H^1_p(G)$ and $H^1_p(G')$ onto each other.

Let $G_1$ be a convex subdomain of $G$. $G$ is star-shaped with respect to $G_1$ if $G$ contains every cone whose vertex is in $G$ and whose generators terminate on $G_1$, i.e., if $G$ is star-shaped with respect to every point of $G_1$. We first give a Sobolev imbedding lemma.

Lemma 5.1. Let $G$ be a domain which is star-shaped with respect to some convex subdomain and which satisfies $m_n(G) < \infty$. Then there exists a constant $M$, depending only on $p$ and $G$, such that every $v \in H^1_p(G)$ satisfies

$$\|v\|_p \leq M(D_p[v]^{1/p} + m_n(G)^{-1}\|v\|_1).$$

Proof. The cases $1 < p \leq n$ and $p > n$ are just special cases of Theorems 2 and 1, respectively, of [18, pp. 56, 57]. Cf. also [7, pp. 369–380].

Remark 5.2. Actually (5.1) holds for more general domains than those considered in Lemma 5.1. For example [7, Remark 4 and Theorem 2, p. 376], if
$G = G_1 \cup G_2$, where (5.1) holds for $G_1$ and $G_2$, and if $m_n(G_1 \cap G_2) > 0$, then (5.1) holds for $G$. See [10, Theorem 3.2.1 and pp. 72–74] for a still more general class of domains for which (5.1) holds.

**Theorem 5.3.** Let $G$ and $G'$ be domains of finite measure for which (5.1) holds, and let $T$ be a homeomorphism of $G$ onto $G'$. Then $T$ is a $Q_p$-mapping if and only if $\varphi_T$ maps $H^1_p(G')$ onto $H^1_p(G)$.

**Proof.** If $T$ is a $Q_p$-mapping we use Ziemer's theorem [20] to prove that composition with $T$ preserves the $H^1_p$ classes. In case $p \neq n$ we conclude from [5, Lemma 7] and Remark 3.1 that $T$ is bi-measurable. (3.1) implies that $\|dT^{-1}\| \leq M$, hence that $T^{-1}$ is of class $H^1_p$, on $G$, where $p' = p(n-1)/(p-1)$. Then for any $1 < p < \infty$ it follows from [20, Theorem 1.1] for $p = n$, that $D_p[v] \leq K_p D_p[u] < \infty$ and $v \in H^1_p(G)$ whenever $v = u \circ T$, $u \in H^1_p(G')$. Thus $\|v\|_1 < \infty$ and $\|v\|_p < \infty$, and (5.1) implies $\|v\|_p < \infty$. Finally this means $v \in H^1_p(G)$ and $\varphi_T(H^1_p(G')) \subset H^1_p(G)$. The opposite inclusion follows similarly, proving half of the theorem.

Conversely, if $\varphi_T$ maps $H^1_p(G')$ onto $H^1_p(G)$, then by Lemma 1.1 its restriction is actually an isomorphism between the Royden $p$-algebras, and the other half follows from Theorem 4.2.

**Remark 5.4.** The boundedness of the partial derivatives of both $T$ and $T^{-1}$ is known to be sufficient for $T$ to preserve the $H^1_p$ classes [10, Theorems 3.1.5 and 3.1.6]. Theorem 5.3 shows that this condition is also necessary for $p \neq n$. It is not necessary for $p = n$, since the partial derivatives of a quasiconformal mapping need not be bounded.

6. **The Royden $p$-compactification.** We recall first some well-known facts about Banach algebras. Let $A$ be a normed algebra of continuous functions defined on $G$ which contains the constant functions. Suppose that $A$ is regular, i.e., that for every closed set $W \subset G$ and every $x \in G - W$ there exists some $f \in A$ such that $f = 0$ on $W$ and $f(x) \neq 0$.

Denote by $G^* =$ the collection of all nonzero bounded complex linear functionals $\chi$ on $A$ which satisfy

$$\chi(fg) = \chi(f)\chi(g) \quad \text{and} \quad \chi(\bar{f}) = \overline{\chi(f)}, \quad f, g \in A.$$  

Then $\|\chi\| = 1$ for every $\chi \in G^*$, and $G^*$ is contained in the unit sphere of the dual space of $A$, inheriting the relative weak* topology generated by $A$. That is, $\chi_a \to \chi$ in $G^*$ if and only if

$$\lim_{a} |\chi_a(f) - \chi(f)| = 0, \quad f \in A.$$  

In this topology $G^*$ is closed, and hence is a compact Hausdorff space by Alaoglu's theorem [16, p. 202]. For each $x \in G$ define

$$\hat{x}(f) = f(x), \quad f \in A.$$
Then since $A$ is regular, $x \rightarrow \dot{x}$ is a homeomorphism of $G$ onto a subset $\dot{G} = \dot{G}_A$ of $G^*$. 

Suppose in addition that $A$ is selfadjoint and inverse-closed, i.e., that $f \in A$ implies $\dot{f} \in A$ and that $f \in A$ and $\inf_G |f| > 0$ imply $1/f = f^{-1} \in A$, respectively. For each $f \in A$ define

$$\dot{f}(x) = \chi(f), \quad x \in G^*.$$ 

Then $f \rightarrow \dot{f}$ is a homomorphism of $A$ onto a subset $\dot{A}$ of $C(G^*)$, which by the Stone-Weierstrass theorem is dense. It follows [12, p. 163] that $\dot{G}$ is dense in $G^*$ and that

$$G^* - \dot{G} = \{x \in G^* : \chi(f) = 0, f \in A_0\},$$

where $A_0$ denotes those functions in $A$ with compact support in $G$. Since $G^* - \dot{G}$ is thus the intersection of a family of zero sets of continuous functions, $\dot{G}$ is open in $G^*$ and $\Delta = \Delta_A = G^* - \dot{G}$ is called the A-ideal boundary of $G$. For every $x \in G$, $\dot{f}(x) = \dot{x}(f) = f(x)$, and by identifying $G$ with its homeomorphic image $\dot{G}$ we may consider $\dot{f}$ to be a continuous extension of $f$ to $G^*$. Then $G^* = G_A^*$ is the A-compactification of $G$ [1, Chapter 9] (which is unique up to a homeomorphism which leaves $G$ fixed).

Note that $M_p(G)$ is regular since it contains $C^\infty_0(G)$. That $M_p(G)$ is selfadjoint is trivial, and it is easy to verify that it is inverse-closed. Thus we may apply the above theory to the Royden $p$-algebra, in which case $\Delta$ and $G^*$ are called the Royden $p$-ideal boundary and the Royden $p$-compactification, respectively, of $G$.

Since $M_p(G)$ separates points, it is semisimple, i.e., $f \rightarrow \dot{f}$ is one-to-one. Thus we have the following result [9, p. 76] which has already been used in the proof of Theorem 4.2.

**Lemma 6.1.** Let $\psi$ be an algebra homomorphism from a commutative Banach algebra onto $M_p(G)$. Then $\psi$ is a bounded linear operator.

**Lemma 6.2.** For $1 < p \leq n$, no point of the Royden $p$-ideal boundary $\Delta$ has a countable neighborhood basis.

**Proof.** Assuming the contrary as in [13, p. 558], let $\{U_i\}$ be a countable neighborhood basis for the topology at $x \in \Delta$ and let $\dot{V}_j = U_j \cap \dot{G}$. Then $\{V_j\}$ is a sequence of nonempty open subsets of $G$, and we may assume that $\overline{V}_{j+1} \subset V_j$, $j = 1, 2, \ldots$. For each $j$, $V_j - \overline{V}_{j+1}$ contains some ball $B(x_j, b_j)$. Define $R_j = B(x_j, b_j) - \text{Cl}(B(x_j, a_j))$, where

$$a_j = b_j \exp - (2^{j+1} \omega_n)^{1/(n-1)}, \quad p = n,$$

$$= b_j [1 - q b_j^{-q} (2^{j+1} \omega_n)^{1/(p-1)}]^{1/q}, \quad p < n;$$

then $\text{cap}_p R_j = 2^{-j}, j = 1, 2, \ldots$.

Denote the extremal function for $\text{cap}_p R_j$ by $v_j$ and set

$$w_k(x) = \sum_{j=1}^k v_j(x) \quad \text{and} \quad w(x) = \sum_{j=1}^\infty v_j(x).$$
Since we may choose \( v_j = 0 \) on \( G - B(x_j, b_j) \) and since \( \{B(x_j, b_j)\} \) are all disjoint, \( \{w_k\} \) converges uniformly on compact subsets of \( G \) to \( w \), and \( 0 \leq w \leq 1 \). Also \( D_p[w_k - w] = 2^{-k} \to 0 \), and \( w_k \to w \) in the \( BD_p \)-topology. Then \( w \in M_p(G) \) by Corollary 3.2, and \( w \) has a continuous extension to \( G^* \). Choosing \( y_j \in S(x_j, b_j) \) gives \( \hat{y}_j \to \chi \) with \( \hat{w}(\hat{y}_j) = 0 \). But \( \hat{x}_j \to \chi \) also and \( \hat{w}(\hat{x}_j) = 1 \), contradicting the continuity of \( \hat{w} \) and proving the lemma.

7. Induced homeomorphisms. Let \( \psi \) be an algebra isomorphism of \( M_p(G') \) onto \( M_p(G) \). Then the adjoint mapping \( T^* \), defined by

\[
T^*(\chi) = \chi \circ \psi, \quad \chi \in G^*,
\]

is a homeomorphism of \( G^* \) onto a closed subset of \( G'^* \) [9, p. 76]. Since \( \psi^{-1} \) satisfies the same conditions it is clear that \( T^*(G^*) = G'^* \).

Suppose that \( T^*(\hat{G}) = \hat{G}' \). Then composition with \( x \to \hat{x} \) and its inverse would induce a homeomorphism \( T \) of \( G \) onto \( G' \) satisfying

\[
T(x) = y, \quad \text{where } \hat{y} = T^*(\hat{x}).
\]

Note that the isomorphism \( \psi \) would be given by composition with \( T \), for let \( f \in M_p(G') \). Then for any \( x \in G \)

\[
\psi(f)(x) = \hat{x} \circ \psi(f) = T^*(\hat{x})(f) = \hat{y}(f) = f \circ T(x),
\]

where \( y = T(x) \).

**Theorem 7.1.** For \( 1 < p \leq n \), every isomorphism \( \psi \) of \( M_p(G') \) onto \( M_p(G) \) induces a \( Q_p \)-mapping \( T \) of \( G \) onto \( G' \) such that \( \psi = \varphi_T \).

**Proof.** For \( 1 < p \leq n \), Lemma 6.2 implies that \( T^*(\hat{G}) \subseteq \hat{G}' \) and that \( T^{*-1}(\hat{G}') \subseteq \hat{G} \). Thus the above argument holds, and (7.2) gives \( \psi = \varphi_T \). Theorem 4.2 concludes the proof.

**Corollary 7.2.** For \( 1 < p \leq n \) there is a one-to-one correspondence between \( Q_p \)-mappings of \( G \) onto \( G' \) and algebra isomorphisms of \( M_p(G) \) onto \( M_p(G') \) given by \( T \to \varphi_T^{-1} \).

**Corollary 7.3.** \( G \) and \( G' \) are quasiconformally equivalent if and only if \( M_p(G) \) and \( M_p(G') \) are algebra isomorphic for some \( p \) satisfying \( 1 < p \leq n \).

**Remark 7.4.** Using [6, §10] we may now observe that \( M_3(G) \) and \( M_3(B^3) \) are not isomorphic if the boundary of \( G \) has an inward directed spire or an outward directed ridge.

8. The Royden \( p \)-ideal boundary. We conclude with some remarks about the behavior of \( \Delta \) under \( Q_p \)-mappings defined on \( G \). The **Royden \( p \)-harmonic boundary** \( \Gamma \) is defined by

\[
\Gamma = \{ \chi \in G^*: \chi(f) = 0, f \in M_\Delta(G) \},
\]
where $M_p(G)$ denotes the closure in the $BD_p$-topology of the functions in $M_p(G)$ with compact support in $G$. It follows from Corollary 2.2 and (6.1) that $\Gamma$ is a closed subset of $\Delta$.

**Theorem 8.1.** Every $Q_p$-mapping $T$ of $G$ onto $G'$ induces a homeomorphism of $\Delta$ onto $\Delta'$ whose restriction to $\Gamma$ is a homeomorphism onto $\Gamma'$.

**Proof.** By Theorem 3.2, $T$ induces an isomorphism $\varphi_T$ of $M_p(G')$ onto $M_p(G)$ which in turn induces a homeomorphism $T^*$ of $G^*$ onto $G'^*$ defined by

$$T^*(\chi) = \chi \circ \varphi_T, \quad \chi \in G^*.$$ 

But for $x \in G$

$$T^*(\hat{x})(f) = \hat{x} \circ \varphi_T(f)$$

$$= \hat{x}(f \circ T) = f(y) = \hat{y}(f)$$

for all $f \in M_p(G')$, where $y = T(x)$. Thus $T^*(\hat{G}) \subset \hat{G}'$, and the opposite inclusion implies that the restriction of $T^*$ to $\Delta$ is the desired homeomorphism.

Now let $\chi \in \Gamma$ and let $\{f_n\}$ be a sequence of functions in $M_p(G)$ with compact support which converges in the $BD_p$-topology to $f$. We show that $T^*(\chi)(f) = 0$. Denote $g_n = f_n \circ T$ and $g = f \circ T$. Then (3.5) implies that $D_p[g_n - g] \to 0$, and $\{g_n\}$ converges to $g$ in the $BD_p$-topology. But each $g_n$ has compact support and

$$T^*(\chi)(f) = \chi \circ \varphi_T(f) = \chi(g) = 0.$$ 

Thus $T^*(\Gamma) \subset \Gamma'$, and the opposite inclusion completes the proof.

**Added in proof.** H. M. Reimann has recently made me aware of a simple example which shows that Theorem 7.1 is false for $p > n$.

**Bibliography**


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