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QUASICONFORMAL MAPPINGS AND ROYDEN ALGEBRAS IN SPACE(1)

BY
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Abstract. On every open connected set G in Euclidean n-space R^n and for every index $p>1$, we define the Royden p-algebra $M_p(G)$. We use results by F. W. Gehring and W. P. Ziemer to prove that two such sets G and G' are quasiconformally equivalent if and only if their Royden n-algebras are isomorphic as Banach algebras. Moreover, every such algebra isomorphism is given by composition with a quasiconformal homeomorphism between G and G'. This generalizes a theorem by M. Nakai concerning Riemann surfaces. In case $p \neq n$, the only homeomorphisms which induce an isomorphism of the p-algebras are the locally bi-Lipschitz mappings, and for $1<p<n$, every such isomorphism arises this way. Under certain restrictions on the domains, these results extend to the Sobolev space $H_p^1(G)$ and characterize those homeomorphisms which preserve the H_p^1 classes.

Introduction. In 1960 Nakai proved [12] that two Riemann surfaces are quasiconformally equivalent if and only if their Royden algebras are isomorphic. In this paper we characterize a class of homeomorphisms and a family of Banach algebras which extend this result to higher dimensions.

On each finite subset G of Euclidean n-space R^n we define the Royden p-algebra $M_p(G)$ for arbitrary $p>1$. $M_p(G)$ is a commutative semisimple Banach algebra with identity. We then form the Gelfand compactification of G with respect to the Royden p-algebra and call it the Royden p-compactification of G [15].

We also define Q_p-mappings of G onto G' and characterize them to be exactly those homeomorphisms which induce by composition an isomorphism between the respective Royden p-algebras. Using the Gelfand theory we prove that for $1<p\leqq n$ every such isomorphism is obtained by composition with a Q_p-mapping.

If G has finite measure we may consider $M_p(G)$ to be a subset of the Sobolev space $H_p^1(G)$. With certain restrictions on the domains, we also prove that the Q_p-mappings are exactly those homeomorphisms which leave the H_p^1 classes invariant.

Recently F. W. Gehring has shown that all Q_p-mappings are quasiconformal mappings. In fact for every $p\neq n$ they are precisely the class of all bi-Lipschitz

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mappings. Thus the case $p=n$ is distinguished when studying either the Royden algebras or the Sobolev spaces under composition. This special case was introduced by C. Loewner [8] as "conformal capacity" and studied by F. W. Gehring [4].

H. M. Reimann [14] has recently considered a class of homeomorphisms similar to the Q_p-mappings.

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1. **Preliminaries.** The following notation is used: m_n is n-dimensional Lebesgue measure, m_{n-1} is $(n-1)$-dimensional Hausdorff measure, $L_p(G)$ is the usual space of equivalence classes of functions f for which $|f|^p$ is m_n-integrable, with norm $\|f\|_p$, and $C_0^\infty(G)$ is the space of infinitely differentiable functions with compact support contained in G. Unless otherwise indicated, all functions are complex-valued and all integrals are taken over G (or G'). $x=(x^1,\dots,x^n)$ denotes an arbitrary point in R^n, $B(x,r)$ and $S(x,r)$ are the usual ball and sphere, abbreviated B^n and S^{n-1} in case $x=0$ and $r=1$, and $\omega_n=m_{n-1}(S^{n-1})$.

Denote by ACL (G) the collection of all functions which are absolutely continuous along the intersection of every compact n-interval with m_{n-1}-almost every line parallel to the coordinate axes. Note that every $f\in$ ACL (G) has a gradient.

More generally, a function f has a (weak) gradient

$$\nabla f = (r^1,\dots,r^n)$$

if there exist distributional derivatives r^i satisfying

$$\int wr^i\,dm_n = -\int f\frac{\partial w}{\partial x^i}\,dm_n, \qquad w\in C_0^\infty(G), \tag{1.1}$$

for $i=1,\dots,n$. Then the *Dirichlet p-integral* of f is

$$D_p[f] = \int|\nabla f|^p\,dm_n = \int\left[\sum_{i=1}^n |r^i|^2\right]^{p/2} dm_n.$$

In case f has locally integrable partial derivatives, they are equal m_n-a.e. to the r^i, and hence satisfy (1.1). Thus, for $f\in$ ACL (G) with $D_p[f]<\infty$, ∇f as defined above is the usual gradient.

A real-valued function u is of class H_p^1 on G if $u\in L_p(G)$ and if u has a gradient and satisfies $D_p[u]<\infty$. A homeomorphism is of class H_p^1 on G if each of its coordinate functions is. $H_p^1(G)$ is the space of all such functions with the norm

$$|||u||| = \left\{\int[|u|^2+|\nabla u|^2]^{p/2}\,dm_n\right\}^{1/p}.$$

As is customary with $L_p(G)$ we write $u\in H_p^1(G)$ and mean that u is of class H_p^1 on G. $H_p^1(G)$ is complete in this norm, and each of its equivalence classes contains a representative in ACL (G) [10, p. 66]. This fact together with Fubini's theorem implies the following.

LEMMA 1.1. *A continuous real-valued function u which is locally of class H_p^1 on G is in* ACL (G).

A *ring* R is a connected open set in R^n whose complement consists of a bounded component C_0 and an unbounded one C_1. For $1<p<\infty$, the *p-capacity* of R is

$$\operatorname{cap}_p R = \inf\left\{\int_R |\nabla u|^p \, dm_n\right\},$$

where the infimum is taken over all continuous functions $u \in$ ACL (R) with constant boundary values 0 and 1. As in [3, p. 358] each such function may be extended to R^n and the integral taken over R^n without affecting $\operatorname{cap}_p R$. Such an extended function is *admissible for R.*

In case R is the spherical ring $\{x : a<|x-x_0|<b\}$, then

$$\text{(1.2)} \qquad \begin{aligned} \operatorname{cap}_p R = \omega_n\left\{\int_a^b t^{q-1}\, dt\right\}^{1-p} &= \omega_n(\log b/a)^{1-n}, \qquad p = n, \\ &= \omega_n((b^q-a^q)/q)^{1-p}, \qquad p \neq n, \end{aligned}$$

where $q=(p-n)/(p-1)$. Note that this gives easily the existence of a unique extremal function for $\operatorname{cap}_p R$ whenever R is a spherical ring; for $p=n$ this is well known and for $p\neq n$ it is the function $|x|^q$ with proper normalization. The uniqueness follows from Clarkson's inequality as in [11, pp. 74, 77].

2. **The Royden p-algebra.** Denote by $M_p(G)$ the collection of all continuous functions f such that

(i) $f \in$ ACL (G),

(ii) $\|f\|_\infty = \sup_G |f| < \infty$,

(iii) $D_p[f] < \infty$.

Since Minkowski's inequality implies

$$\text{(2.1)} \qquad D_p[fg]^{1/p} \leqq \|f\|_\infty D_p[g]^{1/p} + \|g\|_\infty D_p[f]^{1/p},$$

it follows that $M_p(G)$ is an algebra under pointwise multiplication with the constant function 1 as identity.

THEOREM 2.1. *With the norm*

$$\text{(2.2)} \qquad \|f\| = \|f\|_\infty + D_p[f]^{1/p},$$

$M_p(G)$ is a complex commutative Banach algebra with identity.

Proof. We conclude from (2.1) that $\|fg\| \leqq \|f\| \, \|g\|$, and since $\|1\|=1$ we only need to show that $M_p(G)$ is complete in the norm (2.2) to finish the proof.

Let $\{f_j\}$ be a Cauchy sequence in $M_p(G)$. Then $\{f_j\}$ converges uniformly on G to a bounded continuous function f, and the derivatives $\{r_j^i\}$ form a Cauchy sequence in

$L_p(G)$, $i=1,\ldots,n$. There exist functions $r^1,\ldots,r^n \in L_p(G)$ such that $\|r^i_j - r^i\|_p \to 0$, and an easy calculation shows that f and r^i satisfy (1.1). Thus f has a gradient satisfying $D_p[f]<\infty$, and f is locally of class H^1_p on G. Finally Lemma 1.1 implies that $f\in M_p(G)$, proving the theorem.

A sequence $\{f_j\}$ of functions in $M_p(G)$ converges in the BD_p-*topology* to f if

(i) $\|f_j-f\|_\infty \leqq M<\infty$,

(ii) $f_j\to f$ uniformly on compact subsets of G,

(iii) $D_p[f_j-f]\to 0$.

COROLLARY 2.2. *$M_p(G)$ is complete in the BD_p-topology.*

Proof. Any Cauchy sequence in the BD_p-topology converges uniformly on compact subsets to a bounded continuous function, and the proof is similar to the above proof.

3. **Induced isomorphisms.** Let T be a homeomorphism of G onto G'. Denote $T\in Q_p(K)=Q_p(K;G)$ if for some $K<\infty$

$$\operatorname{cap}_p T(R) \leqq K \operatorname{cap}_p R$$

for every spherical ring R satisfying $\bar{R}\subset G$. T is called a *Q_p-mapping* if $T\in Q_p(K;G)$ and $T^{-1}\in Q_p(K;G')$ for some K. T is a *quasiconformal mapping* if for some $1\leqq K<\infty$

$$K^{-1}\operatorname{cap}_n R \leqq \operatorname{cap}_n T(R) \leqq K \operatorname{cap}_n R$$

for every ring R satisfying $\bar{R}\subset G$.

The restriction to spherical rings in the definition of $Q_p(K)$ is necessary for the additivity of the p-capacity in (4.3).

REMARK 3.1. It is well known that a Q_n-mapping is a quasiconformal mapping (see [3], [11] or [19]). Thus every Q_p-mapping is a quasiconformal mapping [5, Theorem 1] and is in fact a bi-Lipschitz mapping for $p\neq n$ [5, Theorem 2]. It follows easily (see Corollary 3.3) that the class of Q_p-mappings for all $p\neq n$ is precisely the class of all homeomorphisms T which together with their inverses satisfy the following Lipschitz condition: There exists an $M<\infty$ such that

$$\limsup_{|x-y|\to 0} \frac{|T(x)-T(y)|}{|x-y|} \leqq M, \qquad x\in G. \tag{3.1}$$

THEOREM 3.2. *Every Q_p-mapping T of G onto G' induces an algebra isomorphism φ_T of $M_p(G')$ onto $M_p(G)$ defined by*

$$\varphi_T(f) = f\circ T, \qquad f\in M_p(G'). \tag{3.2}$$

Proof. Since the composition φ_T clearly preserves the algebraic operations, we prove only $\varphi_T(M_p(G'))\subset M_p(G)$. This will complete the proof since the opposite inclusion follows similarly.

Let $f \in M_p(G')$ and set $u = \operatorname{Re} f$ and $v = u \circ T$. In case $p \neq n$ it follows from (3.1) and the Rademacher-Stepanoff theorem [17, p. 310] that T and T^{-1} are differentiable m_n-a.e. and that

$$M^{-n} \leqq |JT| \leqq M^n \qquad m_n\text{-a.e. in } G, \tag{3.3}$$

for some $M < \infty$. Again using (3.1), we get

$$|\nabla v| \leqq M |\nabla u| \circ T \qquad m_n\text{-a.e. in } G. \tag{3.4}$$

Then (3.3) and (3.4) yield

$$\begin{aligned} \int_G |\nabla v|^p \, dm_n &\leqq M^p \int_G |\nabla u|^p \circ T \, dm_n \\ &\leqq M^{p+n} \int_G |\nabla u|^p \circ T |JT| \, dm_n = M^{p+n} \int_{G'} |\nabla u|^p \, dm_n. \end{aligned}$$

For $p = n$ let Q be any n-cube satisfying $\bar{Q} \subset G$ and denote $Q' = T(Q)$, $u_0 = u | Q'$ and $v_0 = u_0 \circ T | Q$. Then $u_0 \in H_n^1(Q')$ and it follows from [20, Remark 4.2] that $v_0 \in H_1^1(Q)$ and

$$\int_Q |\nabla v_0|^n \, dm_n \leqq K \int_{Q'} |\nabla u_0|^n \, dm_n.$$

Exhausting G by a sequence of disjoint cubes gives

$$\int_G |\nabla v|^n \, dm_n \leqq K \int_{G'} |\nabla u|^n \, dm_n.$$

We apply the above argument to $\operatorname{Im} f$ and see that for any $1 < p < \infty$ there exists a constant K_2 depending only on K, M, p and n such that

$$D_p[g] \leqq K_2 D_p[f], \tag{3.5}$$

where $g = f \circ T$, $f \in M_p(G')$. Lemma 1.1 then implies that $g \in M_p(G)$, concluding the proof.

COROLLARY 3.3. *Every homeomorphism T of G onto G' such that T and T^{-1} satisfy* (3.1) *is a Q_p-mapping for all p.*

Proof. Such a T satisfies (3.3) and (3.4). Let R' be a spherical ring in G', $\bar{R} \subset G'$, and let u be the extremal function for $\operatorname{cap}_p R'$. Setting $v = u \circ T$, the above proof gives

$$\operatorname{cap}_p R \leqq \int_G |\nabla v|^p \, dm_n \leqq M^{p+n} \int_{G'} |\nabla u|^p \, dm_n = M^{p+n} \operatorname{cap}_p R',$$

where $R = T^{-1}(R')$. Hence $T^{-1} \in Q_p(M^{p+n})$ and the corollary follows by applying the same argument to T.

4. **Characterization of Q_p-mappings.**

In this section we give a necessary and sufficient condition for a given homeomorphism T of G onto G' to be a Q_p-mapping.

Suppose that T has the property that $u=v\circ T^{-1}$ is admissible for $T(R)$ whenever v is the extremal function for $\operatorname{cap}_p R$ for any spherical ring satisfying $\bar{R}\subset G$; i.e., suppose T^{-1} preserves ACL functions. Suppose further that there exists a constant $K_1<\infty$ independent of R such that

$$D_p[u]^{1/p} \leqq K_1\|v\| \tag{4.1}$$

whenever u and v are as above.

LEMMA 4.1. *$T\in Q_p(K)$ for some $K\leqq K_1^p$.*

Proof. If the lemma were false there would exist a spherical ring $R=\{x : a<|x-x_0|<b\}$ with $\bar{R}\subset G$ such that

$$\operatorname{cap}_p T(R) \geqq d^p \operatorname{cap}_p R \tag{4.2}$$

for some $d>K_1$. We may assume that $\operatorname{cap}_p R$ is arbitrarily large because for any positive integer m we may define

$$R_j = \{x : r_{j-1} < |x-x_0| < r_j\}, \qquad j = 1,\dots,m,$$

where

$$\begin{aligned} r_j &= [a^{m-j}b^j]^{1/m}, && p = n,\\ &= [((m-j)a^q+jb^q)/m]^{1/q}, && p \neq n, \end{aligned}$$

and $q=(p-n)/(p-1)$. Then $\operatorname{cap}_p R_j=m^{p-1}\operatorname{cap}_p R$ and

$$\sum_{j=1}^{m} (\operatorname{cap}_p R_j)^{1/(1-p)} = (\operatorname{cap}_p R)^{1/(1-p)}, \tag{4.3}$$

which together with the p-capacity version of [2, Lemma 2] gives

$$d^p \operatorname{cap}_p R \leqq \left\{\sum_{j=1}^{m} (\operatorname{cap}_p T(R_j))^{1/(1-p)}\right\}^{1-p}$$

An easy calculation shows that for some $1\leqq k\leqq m$ we must have

$$\operatorname{cap}_p T(R_k) \geqq m^{p-1}\, d^p \operatorname{cap}_p R = d^p \operatorname{cap}_p R_k,$$

and R_k satisfies both (4.2) and $\operatorname{cap}_p R_k=m^{p-1}\operatorname{cap}_p R$. Thus we could use R_k instead of R.

Now choose some c, with $K_1<c<d$, and assume

$$\operatorname{cap}_p R > c^p(d-c)^{-p}.$$

Let v be the extremal function for $\operatorname{cap}_p R$ and set $u=v\circ T^{-1}$. Then by hypothesis $D_p[u]\geqq\operatorname{cap}_p T(R)$, and it follows from (4.1) and (4.2) that

$$\begin{aligned} K_1\|v\| &\geqq D_p[u]^{1/p} \geqq (\operatorname{cap}_p T(R))^{1/p} \geqq d(\operatorname{cap}_p R)^{1/p}\\ &= (d-c)(\operatorname{cap}_p R)^{1/p}+cD_p[v]^{1/p} > c(1+D_p[v]^{1/p}) = c\|v\|. \end{aligned}$$

But this gives $c<K_1$, contradicting the choice of $c>K_1$ and proving the lemma.

THEOREM 4.2. *Every homeomorphism T of G onto G' for which φ_T is an algebra isomorphism of $M_p(G')$ onto $M_p(G)$ is a Q_p-mapping.*

Proof. Let R be a spherical ring, $\bar{R}\subset G$, and let v be the extremal function for $\text{cap}_p\, R$. Then $\varphi_T^{-1}(v)=v\circ T^{-1}=u\in M_p(G')$, and u is clearly admissible for $T(R)$. In §6 we apply some Banach algebra results (Lemma 6.1) to prove that φ_T^{-1} is a bounded operator. Setting $K_1=\|\varphi_T^{-1}\|$ satisfies (4.1) and by Lemma 4.1 we have $T\in Q_p(K)$ for some $K\leqq K_1^p$. Applying the same argument to T completes the proof.

Theorems 3.2 and 4.2 give the following characterization of a Q_p-mapping.

COROLLARY 4.3. *Let T be a homeomorphism of G onto G'. Then T is a Q_p-mapping if and only if φ_T is an algebra isomorphism of $M_p(G')$ onto $M_p(G)$.*

Another corollary follows from Theorem 4.2 and Remark 3.1.

COROLLARY 4.4. *Let T be a Q_p-mapping of G onto G', $p\neq n$. Then $M_{p'}(G)$ and $M_{p'}(G')$ are algebra isomorphic for every p'.*

COROLLARY 4.5. *φ_T is an isometry if and only if $T, T^{-1}\in Q_p(1)$.*

Proof. If $T, T^{-1}\in Q_p(1)$ then it follows that the constant $K_2=1$ in (3.5). For $p=n$ this is well known and for $p\neq n$ it follows from Remark 3.1 and the fact that $K_0=K^{2n/p}$ in the proof of [5, Theorem 1]. Then (3.5) applied to T and T^{-1} implies that $D_p[f]=D_p[\varphi_T(f)]$, $f\in M_p(G')$. But $\|f\|_\infty=\|\varphi_T(f)\|_\infty$ trivially, proving the isometry.

Conversely, if φ_T is an isometry then (4.1) holds with $K_1=1$ and Lemma 4.1 implies $T\in Q_p(1)$. The same is true for T^{-1}, concluding the proof.

5. **Sobolev spaces.** With certain geometrical restrictions on the domains G and G' the techniques of the previous section may be used to characterize those homeomorphisms whose composition maps the Sobolev spaces $H_p^1(G)$ and $H_p^1(G')$ onto each other.

Let G_1 be a convex subdomain of G. G is *star-shaped with respect to* G_1 if G contains every cone whose vertex is in G and whose generators terminate on G_1, i.e., if G is star-shaped with respect to every point of G_1. We first give a Sobolev imbedding lemma.

LEMMA 5.1. *Let G be a domain which is star-shaped with respect to some convex subdomain and which satisfies $m_n(G)<\infty$. Then there exists a constant M, depending only on p and G, such that every $v\in H_p^1(G)$ satisfies*

$$\|v\|_p \leqq M(D_p[v]^{1/p}+m_n(G)^{-1}\|v\|_1). \tag{5.1}$$

Proof. The cases $1<p\leqq n$ and $p>n$ are just special cases of Theorems 2 and 1, respectively, of [18, pp. 56, 57]. Cf. also [7, pp. 369–380].

REMARK 5.2. Actually (5.1) holds for more general domains than those considered in Lemma 5.1. For example [7, Remark 4 and Theorem 2, p. 376], if

$G=G_1 \cup G_2$, where (5.1) holds for G_1 and G_2, and if $m_n(G_1 \cap G_2)>0$, then (5.1) holds for G. See [10, Theorem 3.2.1 and pp. 72–74] for a still more general class of domains for which (5.1) holds.

THEOREM 5.3. *Let G and G' be domains of finite measure for which* (5.1) *holds, and let T be a homeomorphism of G onto G'. Then T is a Q_p-mapping if and only if φ_T maps $H_p^1(G')$ onto $H_p^1(G)$.*

Proof. If T is a Q_p-mapping we use Ziemer's theorem [20] to prove that composition with T preserves the H_p^1 classes. In case $p \neq n$ we conclude from [5, Lemma 7] and Remark 3.1 that T is bi-measurable. (3.1) implies that $\|dT^{-1}\| \leqq M$, hence that T^{-1} is of class $H_{p'}^1$ on G, where $p'=p(n-1)/(p-1)$. Then for any $1<p<\infty$ it follows from [20, Theorem 1.1], as in the proof of (3.5) for $p=n$, that $D_p[v] \leqq K_2 D_p[u]<\infty$ and $v \in H_1^1(G)$ whenever $v=u \circ T$, $u \in H_p^1(G')$. Thus $\|v\|_1<\infty$ and $D_p[v]<\infty$, and (5.1) implies $\|v\|_p<\infty$. Finally this means $v \in H_p^1(G)$ and $\varphi_T(H_p^1(G')) \subset H_p^1(G)$. The opposite inclusion follows similarly, proving half of the theorem.

Conversely, if φ_T maps $H_p^1(G')$ onto $H_p^1(G)$, then by Lemma 1.1 its restriction is actually an isomorphism between the Royden p-algebras, and the other half follows from Theorem 4.2.

REMARK 5.4. The boundedness of the partial derivatives of both T and T^{-1} is known to be sufficient for T to preserve the H_p^1 classes [10, Theorems 3.1.5 and 3.1.6]. Theorem 5.3 shows that this condition is also necessary for $p \neq n$. It is not necessary for $p=n$, since the partial derivatives of a quasiconformal mapping need not be bounded.

6. **The Royden p-compactification.** We recall first some well-known facts about Banach algebras. Let A be a normed algebra of continuous functions defined on G which contains the constant functions. Suppose that A is *regular*, i.e., that for every closed set $W \subset G$ and every $x \in G-W$ there exists some $f \in A$ such that $f=0$ on W and $f(x) \neq 0$.

Denote by $G^*=G_A^*$ the collection of all nonzero bounded complex linear functionals χ on A which satisfy

$$\chi(fg) = \chi(f)\chi(g) \quad \text{and} \quad \chi(\bar{f}) = \overline{\chi(f)}, \qquad f, g \in A.$$

Then $\|\chi\|=1$ for every $\chi \in G^*$, and G^* is contained in the unit sphere of the dual space of A, inheriting the relative weak* topology generated by A. That is, $\chi_\alpha \to \chi$ in G^* if and only if

$$\lim_{\alpha} |\chi_\alpha(f)-\chi(f)| = 0, \qquad f \in A.$$

In this topology G^* is closed, and hence is a compact Hausdorff space by Alaoglu's theorem [16, p. 202]. For each $x \in G$ define

$$\hat{x}(f) = f(x), \qquad f \in A.$$

Then since A is regular, $x \to \hat{x}$ is a homeomorphism of G onto a subset $\hat{G}=\hat{G}_A$ of G^*.

Suppose in addition that A is *selfadjoint* and *inverse-closed*, i.e., that $f \in A$ implies $\bar{f} \in A$ and that $f \in A$ and $\inf_G |f| > 0$ imply $1/f = f^{-1} \in A$, respectively. For each $f \in A$ define

$$\hat{f}(\chi) = \chi(f), \qquad \chi \in G^*.$$

Then $f \to \hat{f}$ is a homomorphism of A onto a subset $\hat{A}$ of $C(G^*)$, which by the Stone-Weierstrass theorem is dense. It follows [12, p. 163] that $\hat{G}$ is dense in G^* and that

$$G^* - \hat{G} = \{\chi \in G^* : \chi(f) = 0, f \in A_0\}, \tag{6.1}$$

where A_0 denotes those functions in A with compact support in G. Since $G^* - \hat{G}$ is thus the intersection of a family of zero sets of continuous functions, $\hat{G}$ is open in G^* and $\Delta = \Delta_A = G^* - \hat{G}$ is called the *A-ideal boundary of* G. For every $x \in G$, $\hat{f}(\hat{x}) = \hat{x}(f) = f(x)$, and by identifying G with its homeomorphic image $\hat{G}$ we may consider $\hat{f}$ to be a continuous extension of f to G^*. Then $G^* = G^*_A$ is the *A-compactification of* G [1, Chapter 9] (which is unique up to a homeomorphism which leaves G fixed).

Note that $M_p(G)$ is regular since it contains $C_0^\infty(G)$. That $M_p(G)$ is selfadjoint is trivial, and it is easy to verify that it is inverse-closed. Thus we may apply the above theory to the Royden p-algebra, in which case Δ and G^* are called the *Royden p-ideal boundary* and the *Royden p-compactification*, respectively, of G.

Since $M_p(G)$ separates points, it is semisimple, i.e., $f \to \hat{f}$ is one-to-one. Thus we have the following result [9, p. 76] which has already been used in the proof of Theorem 4.2.

LEMMA 6.1. *Let ψ be an algebra homomorphism from a commutative Banach algebra onto $M_p(G)$. Then ψ is a bounded linear operator.*

LEMMA 6.2. *For $1 < p \leqq n$, no point of the Royden p-ideal boundary Δ has a countable neighborhood basis.*

Proof. Assuming the contrary as in [13, p. 558], let $\{U_j\}$ be a countable neighborhood basis for the topology at $\chi \in \Delta$ and let $\hat{V}_j = U_j \cap \hat{G}$. Then $\{V_j\}$ is a sequence of nonempty open subsets of G, and we may assume that $\overline{V}_{j+1} \subset V_j$, $j = 1, 2, \ldots$. For each j, $V_j - \overline{V}_{j+1}$ contains some ball $B(x_j, b_j)$. Define $R_j = B(x_j, b_j) - \mathrm{Cl}\,(B(x_j, a_j))$, where

$$\begin{aligned} a_j &= b_j \exp -(2^j \omega_n)^{1/(n-1)}, & p &= n, \\ &= b_j[1 - q b_j^{-q}(2^j \omega_n)^{1/(p-1)}]^{1/q}, & p &< n; \end{aligned}$$

then $\mathrm{cap}_p\, R_j = 2^{-j}$, $j = 1, 2, \ldots$.

Denote the extremal function for $\mathrm{cap}_p\, R_j$ by v_j and set

$$w_k(x) = \sum_{j=1}^{k} v_j(x) \quad \text{and} \quad w(x) = \sum_{j=1}^{\infty} v_j(x).$$

Since we may choose $v_j=0$ on $G-B(x_j, b_j)$ and since $\{B(x_j, b_j)\}$ are all disjoint, $\{w_k\}$ converges uniformly on compact subsets of G to w, and $0 \leqq w \leqq 1$. Also $D_p[w_k-w]=2^{-k} \to 0$, and $w_k \to w$ in the BD_p-topology. Then $w \in M_p(G)$ by Corollary 3.2, and w has a continuous extension to G^*. Choosing $y_j \in S(x_j, b_j)$ gives $\hat{y}_j \to \chi$ with $\hat{w}(\hat{y}_j)=0$. But $\hat{x}_j \to \chi$ also and $\hat{w}(\hat{x}_j)=1$, contradicting the continuity of $\hat{w}$ and proving the lemma.

7. **Induced homeomorphisms.** Let ψ be an algebra isomorphism of $M_p(G')$ onto $M_p(G)$. Then the adjoint mapping T^*, defined by

$$T^*(\chi) = \chi \circ \psi, \qquad \chi \in G^*,$$

is a homeomorphism of G^* onto a closed subset of G'^* [9, p. 76]. Since ψ^{-1} satisfies the same conditions it is clear that $T^*(G^*)=G'^*$.

Suppose that $T^*(\hat{G})=\hat{G}'$. Then composition with $x \to \hat{x}$ and its inverse would induce a homeomorphism T of G onto G' satisfying

$$T(x) = y, \quad \text{where } \hat{y} = T^*(\hat{x}). \tag{7.1}$$

Note that the isomorphism ψ would be given by composition with T, for let $f \in M_p(G')$. Then for any $x \in G$

$$\psi(f)(x) = \hat{x} \circ \psi(f) = T^*(\hat{x})(f) = \hat{y}(f) = f \circ T(x), \tag{7.2}$$

where $y=T(x)$.

THEOREM 7.1. *For $1<p \leqq n$, every isomorphism ψ of $M_p(G')$ onto $M_p(G)$ induces a Q_p-mapping T of G onto G' such that $\psi=\varphi_T$.*

Proof. For $1<p\leqq n$, Lemma 6.2 implies that $T^*(\hat{G})\subset \hat{G}'$ and that $T^{*-1}(\hat{G}')\subset \hat{G}$. Thus the above argument holds, and (7.2) gives $\psi=\varphi_T$. Theorem 4.2 concludes the proof.

COROLLARY 7.2. *For $1<p\leqq n$ there is a one-to-one correspondence between Q_p-mappings of G onto G' and algebra isomorphisms of $M_p(G)$ onto $M_p(G')$ given by $T \to \varphi_{T^{-1}}$.*

COROLLARY 7.3. *G and G' are quasiconformally equivalent if and only if $M_p(G)$ and $M_p(G')$ are algebra isomorphic for some p satisfying $1<p\leqq n$.*

REMARK 7.4. Using [6, §10] we may now observe that $M_3(G)$ and $M_3(B^3)$ are not isomorphic if the boundary of G has an inward directed spire or an outward directed ridge.

8. **The Royden p-ideal boundary.** We conclude with some remarks about the behavior of Δ under Q_p-mappings defined on G. The *Royden p-harmonic boundary* Γ is defined by

$$\Gamma = \{\chi \in G^* : \chi(f) = 0, f \in M_\Delta(G)\},$$

where $M_\Delta(G)$ denotes the closure in the BD_p-topology of the functions in $M_p(G)$ with compact support in G. It follows from Corollary 2.2 and (6.1) that Γ is a closed subset of Δ.

THEOREM 8.1. *Every Q_p-mapping T of G onto G' induces a homeomorphism of Δ onto Δ' whose restriction to Γ is a homeomorphism onto Γ'.*

Proof. By Theorem 3.2, T induces an isomorphism φ_T of $M_p(G')$ onto $M_p(G)$ which in turn induces a homeomorphism T^* of G^* onto G'^* defined by

$$T^*(\chi) = \chi \circ \varphi_T, \qquad \chi \in G^*.$$

But for $x \in G$

$$\begin{aligned} T^*(\hat{x})(f) &= \hat{x} \circ \varphi_T(f) \\ &= \hat{x}(f \circ T) = f(y) = \hat{y}(f) \end{aligned}$$

for all $f \in M_p(G')$, where $y = T(x)$. Thus $T^*(\hat{G}) \subset \hat{G}'$, and the opposite inclusion implies that the restriction of T^* to Δ is the desired homeomorphism.

Now let $\chi \in \Gamma$ and let $\{f_n\}$ be a sequence of functions in $M_p(G)$ with compact support which converges in the BD_p-topology to f. We show that $T^*(\chi)(f) = 0$. Denote $g_n = f_n \circ T$ and $g = f \circ T$. Then (3.5) implies that $D_p[g_n - g] \to 0$, and $\{g_n\}$ converges to g in the BD_p-topology. But each g_n has compact support and

$$T^*(\chi)(f) = \chi \circ \varphi_T(f) = \chi(g) = 0.$$

Thus $T^*(\Gamma) \subset \Gamma'$, and the opposite inclusion completes the proof.

Added in proof. H. M. Reimann has recently made me aware of a simple example which shows that Theorem 7.1 is false for $p > n$.

BIBLIOGRAPHY

1. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, Berlin, 1963. MR **28** #3151.

2. F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519. MR **24** #A2677.

3. ———, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393. MR **25** #3166.

4. ———, *Extremal length definitions for the conformal capacity of rings in space*, Michigan Math. J. **9** (1962), 137–150. MR **25** #4098.

5. ———, *Lipschitz mappings and the p-capacity of rings in n-space*, Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., Providence, R. I., 1971.

6. F. W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70. MR **31** #4905.

7. L. V. Kantorovič and G. P. Akilov, *Functional analysis in normed spaces*, Fizmatgiz, Moscow, 1959; English transl., Internat. Series of Monographs in Pure and Appl. Math., vol. 46, Macmillan, New York, 1964. MR **22** #9837; MR **35** #4699.

8. C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414. MR **21** #3538.

9. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Princeton, N. J., 1953. MR **14,** 883.

10. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966. MR **34** #2380.

11. G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 53–104. MR **38** #4679.

12. M. Nakai, *Algebraic criterion on quasiconformal equivalence of Riemann surfaces*, Nagoya Math. J. **16** (1960), 157–184. MR **22** #1669.

13. ———, *Some topological properties on Royden's compactification of a Riemann surface*, Proc. Japan. Acad. **36** (1960), 555–559. MR **23** #A102.

14. H. M. Reimann, *Über harmonische Kapazität und quasikonforme Abbildungen im Raum*, Comment. Math. Helv. **44** (1969), 284–307. MR **40** #5856.

15. H. Royden, *On the ideal boundary of a Riemann surface*, Contributions to the Theory of Riemann Surfaces, Ann. of Math. Studies, no. 30, Princeton Univ. Press, Princeton, N. J. 1953. MR **15,** 25.

16. ———, *Real analysis*, 2nd ed., Macmillan, New York, 1968.

17. S. Saks, *Theory of the integral*, 2nd rev. ed., Dover, New York, 1964. MR **29** #4850.

18. S. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; English transl., Transl. Math. Monographs, vol. 7, Amer. Math. Soc., Providence, R. I., 1963. MR **14,** 565; MR **29** #2624.

19. J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A. I. No. 298 (1961), 1–36. MR **25** #4100a.

20. W. Ziemer, *Change of variables for absolutely continuous functions*, Duke Math. J. **36** (1969), 171–178. MR **38** #6006.

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