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SOME ANALYTIC VARIETIES IN THE POLYDISC AND THE MÜNTZ-SZASZ PROBLEM IN SEVERAL VARIABLES

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Abstract. For $1 \leqq p_1 < p_2 < \infty$ and $n \geqq 2$ it is shown that there exists a sequence of monomials $\{\prod_{j=1}^n s_j^{\lambda_{mj}}\}$ with $\lambda_{mj} \sim m$ for each $j=1, \ldots, n$ whose linear span is dense in $L^{p_1}(I^n)$ but not in $L^{p_2}(I^n)$ (I^n is the Cartesian product of n copies of the closed unit interval [0, 1]). Construction of the examples is via duality, making use of suitable analytic varieties in the polydisc.

The object of this note is to exhibit some contrasts between the Müntz-Szasz theorem in one variable and the analogous problem in several variables.

The following notation will be used. The closed unit interval [0, 1] will be denoted by I and I^n will denote the unit cube $I \times I \times \cdots \times I$ in $\mathbf{R}^n$ ($n = 2, 3, \ldots$). The Banach spaces $L^p(I^n)$, $1 \leqq p < \infty$, will be the usual spaces of complex valued functions on I^n taken with respect to Lebesgue measure. I shall use $\tilde{C}(I^n)$ to denote the Banach space with supremum norm of continuous functions f on I^n which satisfy $f(s_1, \ldots, s_n) = 0$ if $s_j = 0$ for any $j = 1, 2, \ldots, n$. In addition if $\{f_m\}$ is a sequence of functions in $L^p(I^n)$ then $S_p(\{f_m\})$ will denote the closed linear span of $\{f_m\}$ in $L^p(I^n)$.

One version of the theorem of Müntz and Szasz is the following [3, p. 23]:

THEOREM A. *Suppose $\{s^{\lambda_m}\}$ is a sequence of monomials with $0 < \lambda_1 < \lambda_2 < \cdots$, then the condition*

$$\sum \frac{1}{\lambda_m} = +\infty$$

is necessary and sufficient in order that $S_p(\{s^{\lambda_m}\}) = L^p(I)$ for all $p = 1, 2, \ldots$ and also in order that $\{s^{\lambda_m}\}$ have a dense linear span in $\tilde{C}(I)$.

In particular Theorem A asserts that, for $1 \leqq p < \infty$, $\{s^{\lambda_m}\}$ is a spanning set in $L^p(I)$ for one value of p if and only if it is a spanning set in every $L^p(I)$.

I shall show that the analogous statement in $L^p(I^n)$ is false if $n \geqq 2$. In fact,

THEOREM 1. *If $1 \leqq p_1 < p_2 < \infty$ and $n \geqq 2$, then there exists a sequence $\{s_m\}$ of monomials $\{s_1^{\lambda_{m1}} s_2^{\lambda_{m2}} \cdots s_n^{\lambda_{mn}}\}$ with positive real powers satisfying*

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(i) $\lambda_{mj} \nearrow \infty$ *as* $m \to \infty$ $(j=1, 2, \ldots, n)$,

(ii) $S_{p_1}(\{s_m\}) = L^{p_1}(I^n)$,

(iii) $S_{p_2}(\{s_m\}) \neq L^{p_2}(I^n)$.

One consequence of Theorem 1 is the corollary:

COROLLARY 1.1. *Suppose that* $\{s_1^{\lambda_{m1}} s_2^{\lambda_{m2}} \cdots s_n^{\lambda_{mn}}\}$ *is a sequence of monomials satisfying properties* (i)–(iii) *of Theorem* 1. *Then the sequence* $\{s_m^1\}$ *of the monomials* $\{s_1^{\lambda_{m1}-1} s_2^{\lambda_{m2}-1} \cdots s_n^{\lambda_{mn}-1}\}$ *satisfies* $S_{p_1}(\{s_m^1\}) \neq L^{p_1}(I^n)$.

Moreover, for any p, $1<p<\infty$, there exists a monomial spanning sequence $\{s_1^{\lambda_{m1}} \cdots s_n^{\lambda_{mn}}\}$ in $\tilde{C}(I^n)$ such that $\{s_1^{\lambda_{m1}-1} \cdots s_n^{\lambda_{mn}-1}\}$ is not a spanning sequence in $L^p(I^n)$.

In other words a shift to the left in the powers of a spanning sequence may destroy the spanning property. Clearly, if $n=1$ no such sequences exist. It is not difficult to see that shifts to the right do preserve the spanning property.

Corollary 1.1 answers the question raised in [1] at least for monomial sequences with positive powers increasing to ∞.

In [1], J. Korevaar and I considered the Müntz-Szasz problem in 2 variables for monomials with positive integral powers and proved the following:

THEOREM B. *Suppose that* Ω *is a sequence* $\{(m_k, n_k)\}$ *of positive lattice points in* $\mathbf{R}^2$. *Denote by* $N(r, \Omega)$ *the number of lattice points in* Ω *satisfying* $m_k^2+n_k^2 \leq r^2$. *Assume that* $\limsup_{r\to\infty} N(r, \Omega)/r^2 > 0$, *then* $\{s_1^{m_k} s_2^{m_k}\}$ *is a spanning sequence in* $\tilde{C}(I^2)$.

Examples given in [1] show that the condition of positive upper density is not necessary. One such example is the set of all lattice points bounded by the curves $y = x^{1/2}$ and $y = (2x)^{1/2}$.

The present methods will show that if the lattice condition on the powers is dropped, then "very sparse" sequences in $\mathbf{R}^n$ may serve as powers of monomial spanning sequences. For example it will be shown that there exists h holomorphic in $\operatorname{Re} z > \alpha$, for some $\alpha > 0$, such that $h(x)$ is real for $x = \operatorname{Re} z > \alpha$, $h(x) \nearrow \infty$ as $x \nearrow \infty$ with $(h(x)-x) \to 0$ as $x \nearrow \infty$ and such that $\{s_1^m s_2^{h(m)}\}_{m=1}^{\infty}$ is a spanning sequence in $\tilde{C}(I^2)$.

Some analytic varieties and uniqueness sets in the polydisc. The unit disc centered at the origin in the complex plane $\mathbf{C}$ will be denoted by U and its boundary by T. Let $U^n = U \times \cdots \times U$ be the unit polydisc in $\mathbf{C}^n$ centered at the origin with $T^n = T \times \cdots \times T$ its distinguished boundary. The space of bounded holomorphic functions in U^n with supremum norm will be given as usual by $H^\infty(U^n)$.

The following lemma is central to this work.

LEMMA 1. *Suppose B is a Blaschke product in U having T as a natural boundary. Let V be the analytic variety defined by*

$$V = \{(z, B(z), B_2(z), \ldots, B_{n-1}(z)) \mid z \in U\}$$

where $B_j = B \circ B_{j-1}$ $(2 \leq j \leq n-1)$, *and* $B_1 = B$. *Then the closure of V contains T^n.*

Proof. Take $(w_1, w_2, \ldots, w_n) \in T^n$. For each $m=1, 2, \ldots$ and $j=1, 2, \ldots, n$ denote by V_{mj} the intersection of U with the disc of radius $1/m$ centered at w_j. Since T is a natural boundary for B, the cluster set of B at any $w \in T$ is $\bar{U}$ [4]. Put $V_{m2}^* = B(V_{m1}) \cap V_{m2}$ and recursively define $V_{mj}^* = B(V_{m(j-1)}^*) \cap V_{mj}$ $(j=3, \ldots, n)$. Since $B(V_{mj})$ is dense in U for each j $(1 \leqq j \leqq n-1)$, it follows that V_{m2}^* is dense in V_{m2}, V_{m3}^* is dense in $V_{m3}, \ldots, V_{mn}^*$ in V_{mn}. Now select any $\zeta_{mn} \in V_{mn}^*$, then pick $\zeta_{m(n-1)} \in V_{m(n-1)}^*$ so that $B(\zeta_{m(n-1)}) = \zeta_{mn}$ and continue step by step choosing $\zeta_{m(n-k)} \in V_{m(n-k)}^*$ so that $B(\zeta_{m(n-k)}) = \zeta_{m(n-k+1)}$ for $k=2, 3, \ldots, n-1$. Put $z_m = \zeta_{m1}$. Then $\{(z_m, B(z_m), B_2(z_m), \ldots, B_{n-1}(z_m))\}$ converges to $(w_1, w_2, \ldots, w_n)$ as $m \to \infty$.

REMARKS. The same proof shows that the closure of $\{(B(z), B_2(z), \ldots, B_n(z))\}$ as z ranges through U is $\bar{U}^n$.

I first employed the variety $\{(z, B(z))\}$ in C^2. I am grateful to J. Zinn for conjecturing that the variety in C^n obtained by using successive iteration of the coordinate functions has the desired property.

COROLLARY 1.2. *Let J be a set of positive measure on T^n and $f \in H^\infty(U^n)$ which is continuous on $U^n \cup J$. Suppose that $\{z_m\}$ is a sequence in U satisfying*

(i) $\sum (1-|z_m|) = +\infty$,

(ii) $f(z_m, B(z_m), \ldots, B_{n-1}(z_m)) = 0$, $m=1, 2, \ldots$ (*B as in the preceding lemma*).

Then $f \equiv 0$.

In other words, with B as in the lemma, the sequence $\{z_m, B(z_m), \ldots, B_{n-1}(z_m)\}$ is a set of uniqueness (or determining set [3]) for the space of functions in $H^\infty(U^n)$ which have a continuous extension to J, and in particular then for $A(U^n)$, the polydisc algebra.

Proof. Consider $\tilde{f}(z) = f(z, B(z), \ldots, B_{n-1}(z))$. Then $\tilde{f} \in H^\infty(U)$ and $\tilde{f}(z_m) = 0$. From (i) and the Blaschke condition it follows that $\tilde{f} \equiv 0$ or equivalently $f = 0$ on the variety V of the lemma. But $\bar{V} \supset T^n$ and f is continuous on J, so $f=0$ on J. Since J has positive measure and $f \in H^\infty(U^n)$, $f \equiv 0$.

I now introduce a particular Blaschke product B which in addition to having T as a natural boundary has the property that both B and B' have radial limits equal to one as z approaches 1 along the positive ray, with $B''(z) \to 0$ along this ray.

LEMMA 2. *Suppose that B is the Blaschke product whose zeros are simple and situated at the kth roots of $-r_k$ for each $k=1, 2, \ldots$ where r_k is defined by*

$$r_k = \left(1 - \frac{1}{k2^k}\right) \Big/ \left(1 + \frac{1}{k2^k}\right).$$

Then

(i) *T is a natural boundary for B,*

(ii) $B(r) \nearrow 1$ *as* $r \nearrow 1$,

(iii) $\lim_{r \nearrow 1} B'(r) = 1$,

(iv) $\lim_{r \nearrow 1} B''(r) = 0$.

Proof. Easy estimates show that $\sum k(1-r_k^{1/k})<\infty$, so that B is a convergent Blaschke product. To prove (i) observe that every point of T is a limit point of zeros of B.

For the proof of (ii) notice that

$$B(z) = \prod (r_k+z^k)/(1+r_k z^k)$$

and that for r (>0) fixed, for θ arbitrary, and for each k

$$|(r_k+r^k e^{ik\theta})/(1+r_k r^k e^{ik\theta})| \leqq (r_k+r^k)/(1+r_k r^k).$$

Hence, $\max_{|z|=r} |B(z)|=B(r)$ and $B(r)$ is an increasing function of r for $0<r<1$. Since $\|B\|_\infty=1$, (ii) follows.

To see that (iii) holds, write

$$B'(r)/B(r) = \sum [(1-r_k^2)kr^{k-1}]/[(r_k+r^k)(1+r_k r^k)].$$

Letting $r \nearrow 1$ and using property (ii) gives

$$\lim_{r\nearrow 1} B'(r) = \sum k(1-r_k)/(1+r_k) = \sum 1/2^k = 1.$$

In order to prove (iv), differentiate $B'(r)/B(r)$ to obtain

$$\frac{B''(r)}{B(r)}-\left[\frac{B'(r)}{B(r)}\right]^2 = \sum\left\{\frac{k(k-1)r^{k-2}}{(r_k+r^k)(1+r_k r^k)}-\frac{(1+2r_k r^k+r_k^2)k^2 r^{2(k-1)}}{(r_k+r^k)^2(1+r_k r^k)^2}\right\}(1-r_k^2).$$

Since $B(r)$ and $B'(r)$ tend to 1 as $r \nearrow 1$,

$$\lim_{r\nearrow 1} B''(r)-1 = \sum\left\{\frac{k(k-1)}{(1+r_k)^2}-\frac{k^2}{(1+r_k)^2}\right\}(1-r_k^2) = -\sum k\frac{1-r_k}{1+r_k} = -1,$$

and thus the assertion.

The interchanges of limit and sum operations in the proofs of (iii) and (iv) as well as the term by term differentiation in (iv) may be justified by observing that the series in question are uniformly convergent on [0, 1], as is readily verified.

Proof of Theorem 1. Choose p so that $1\leqq p_1<p<p_2<\infty$ and set

$$\mathscr{H}_p^n = \{(z_1,\dots,z_n) \mid \operatorname{Re} z_j > -1/p,\ 1 \leqq j \leqq n\}.$$

Map U^n onto $\mathscr{H}_p^n$ by

$$z_j = \varphi(\zeta_j) = (1+\zeta_j)/(1-\zeta_j)-1/p \qquad (1 \leqq j \leqq n).$$

Let B be the Blaschke product of Lemma 2 and consider the sequence $\{(\lambda_{m1},\dots,\lambda_{mn})\}$ defined by $\lambda_{m1}=m$ and $\lambda_{mj}=\varphi\circ B_{j-1}\circ\varphi^{-1}(m)$ for $j=2,3,\dots,n$ and $m=1,2,\dots$. Denote this sequence by $\{\lambda_m\}$.

Also, for notational simplicity let the variable $(z_1,\dots,z_n)\in \boldsymbol{C}^n$ be given by z with the n-tuples $(s_1,\dots,s_n)$ and $(y_1,\dots,y_n)$ in $\boldsymbol{R}^n$ denoted by s and y respectively. The monomial $s_1^{z_1}\cdots s_n^{z_n}$ will be given by s^z. If $\alpha\in\boldsymbol{C}$, then $z+\alpha=(z_1+\alpha,\dots,z_n+\alpha)$ and $s^{z+\alpha}=s_1^{z_1+\alpha}\cdots s_n^{z_n+\alpha}$.

Claim. With $n \geqq 2$ fixed, $\{s^{\lambda_m}\}$ satisfies assertions (i)–(iii) of the theorem.

(i) Since B increases to 1 along the positive ray and $\{\varphi^{-1}(m)\}$ is a positive sequence increasing to 1, $B_{j-1} \circ \varphi^{-1}(m)$ increases to 1 as $m \to \infty$ for $j=2, 3, \ldots, n$.

(ii) Let $g \in L^{q_1}(I^n)$ where $1/p_1 + 1/q_1 = 1$. I shall show that if

$$\int_{I^n} g(s) s^{\lambda_m}\, ds = 0$$

for $m=1, 2, \ldots$, with ds denoting Lebesgue measure on I^n, then $g=0$ a.e. on I^n. Introduce

$$F(z) = \int_{I^n} g(s) s^z\, ds.$$

Then F is holomorphic in $\mathscr{H}^n_{p_1}$, bounded in $\mathscr{H}^n_p$, and $F(\lambda_m)=0$ for $m=1, 2, \ldots$. It follows that $\tilde{F}(\zeta)=F(\Phi(\zeta))$ $(\zeta=(\zeta_1, \ldots, \zeta_n)$, $\Phi(\zeta)=(\varphi(\zeta_1), \ldots, \varphi(\zeta_n)))$ is in $H^\infty(U^n)$ and is continuous on $J=T^n-\{(w_1, \ldots, w_n) \mid w_j=1 \text{ for some } j=1, 2, \ldots, n\}$ which is a set of full measure on T^n. Also $\tilde{F}$ vanishes on the sequence $\{\zeta_m\}$ given by $\zeta_{m1}=\varphi^{-1}(m)$ and $\zeta_{mj}=B_{j-1} \circ \varphi^{-1}(m)$ for each $m=1, 2, \ldots$ and $j=2, 3, \ldots, n$. Since $\sum (1-\varphi^{-1}(m)) = +\infty$, Corollary 1.2 applies. It follows that $\tilde{F}(\zeta) \equiv 0$ on U^n and $F \equiv 0$ on $\mathscr{H}^n_p$. In particular $F=0$ on the set of all nonnegative lattice points in $\boldsymbol{R}^n$ so that g annihilates all polynomials and $g=0$ a.e. on I^n.

(iii) I shall show that there exists $g \in L^{q_2}(I^n)$, $1/p_2 + 1/q_2 = 1$, such that $g=0$ a.e. on I^n does not hold while

$$\int_{I^n} g(s) s^{\lambda_m}\, ds = 0$$

for each $m=1, 2, \ldots$ where $\{\lambda_m\}$ is the sequence of n-tuples defined above.

To this end, set

$$f(z) = f(z_1, \ldots, z_n) = \varphi^{-1}(z_2) - (B \circ \varphi^{-1})(z_1)$$

with B and φ as before. Then f is holomorphic and bounded in $\mathscr{H}^n_p$ and $f(\lambda_m)=0$ for $m=1, 2, \ldots$. Define F by

$$F(z) = f(z) \Big/ \prod_{j=1}^{n} (2+z_j)^2.$$

Then F is holomorphic and bounded in $\mathscr{H}^n_p$ and the zero sets of F and f coincide. Also if F^* is given by

$$F^*(y) = F^*(y_1, \ldots, y_n) = \lim_{x \swarrow -1/p} F(x+iy_1, \ldots, x+iy_n),$$

then F^* exists a.e. on $\boldsymbol{R}^n$ and is in $L^1(\boldsymbol{R}^n)$, [5, Chapter XVII, §4].

It is now sufficient to find $g \in L^{q_2}(I^n)$ such that, for all $z \in \mathscr{H}^n_{p_2}$,

$$F(z) = \int_{I^n} g(s) s^z\, ds.$$

By the Cauchy formula and the relations

$$F(z_1, \ldots, z_n) = O(1/|z_j|^2) \qquad \text{as } |z_j| \to \infty \text{ in } \mathscr{H}_p^n \qquad (1 \leqq j \leqq n),$$

it follows that

$$F(z) = \left(\frac{-1}{2\pi}\right)^n \int_{R^n} F^*(y) \Big/ \left(\prod_{j=1}^n \left(-\frac{1}{p}+iy_j-z_j\right)\right) dy;$$

where dy is Lebesgue measure in $\boldsymbol{R}^n$. But

$$\left(\prod_{j=1}^n \left(\frac{1}{p}+z_j-iy_j\right)\right)^{-1} = \int_{I^n} s^{z-iy+1/p-1}\, ds,$$

so that

$$F(z) = (1/2\pi)^n \int_{R^n} F^*(y) \left\{\int_{I^n} s^{z-iy+1/p-1}\, ds\right\} dy.$$

Since $F^* \in L^1(\boldsymbol{R}^n)$, by Fubini's theorem,

$$F(z) = \int_{I^n} s^z \left\{(1/2\pi)^n s^{1/p-1} \int_{R^n} F^*(y) e^{-i\langle y, \log s\rangle}\, dy\right\} ds,$$

where $\langle y, \log s\rangle = \sum_{j=1}^n y_j \log s_j$. The integral inside the brackets is a bounded continuous function of s on $\bar{I}^n$ ($\bar{I} = (0, 1]$). In addition, since $p < p_2$ and $1/p_2 + 1/q_2 = 1$, $q_2(1/p-1) > -1$, so that

$$\int_{I^n} s^{q_2(1/p-1)}\, ds < +\infty.$$

Denoting the bracketed expression in the last integral but one by $g(s)$, it follows that, for all $z \in \mathscr{H}_p^n$,

$$F(z) = \int_{I^n} s^z g(s)\, ds,$$

with $g \in L^{q_2}(I^n)$.

Proof of Corollary 1.1. Let $\{\lambda_m\}$ be any sequence in $\boldsymbol{R}^n$ satisfying properties (i)–(iii) of Theorem 1. Since $S_{p_2}\{s^{\lambda_m}\} \neq L^{p_2}(I^n)$ there exists $g \in L^{q_2}(I^n)$, $1/p_2 + 1/q_2 = 1$, such that g is not 0 a.e. on I^n and

$$\int_{I^n} g(s) s^{\lambda_m}\, ds = 0 \qquad (m = 1, 2, \ldots).$$

Put $f(z) = \int_{I^n} g(s) s^z\, ds$, and fix $p > p_2$. Then f is holomorphic in $\mathscr{H}_{p_2}^n$ and is bounded in $\mathscr{H}_p^n$. Consider

$$F(z) = f(z) \Big/ \prod_{j=1}^n (2+z_j)^2.$$

Proceeding as in the proof of assertion (iii) of Theorem 1 gives the representation

$$F(z) = \left(\frac{1}{2\pi}\right)^n \int_{R^n} s^{z-1} g^*(s)\, ds$$

for all $z \in \mathscr{H}_p^n$, where $g^*(s) = s^{1/p}h(s)$ and h is bounded and continuous on I_0^n. Since g^* is continuous on I^n and $F(\lambda_m)=0$ for $m=1, 2, \ldots$, $S_p\{s^{\lambda_m - 1}\} \neq L^p(I^n)$ for *any p*, $1 \leqq p < \infty$.

REMARKS. To obtain an example of a sequence $\{s^{\lambda_m}\}$ which spans in $\breve{C}(I^n)$ and has the property that $\{s^{\lambda_m - 1}\}$ does not span a given $L^p(I^n)$, $1 < p < \infty$, it is sufficient to define $\{\lambda_m\}$ as in the proof of Theorem 1 by

$$\lambda_{m1} = m \quad \text{and} \quad \lambda_{mj} = \varphi \circ B_{j-1} \circ \varphi^{-1}(m) \quad (m = 1, 2, \ldots; j = 2, 3, \ldots, n)$$

with B as in Lemma 2 and φ given this time by

$$\varphi(\zeta) = (1+\zeta)/(1-\zeta)+\alpha,$$

with α fixed and $0 < \alpha < 1 - 1/p$.

The same methods also show that the spanning property in $L^p(I^n)$ or $\breve{C}(I^n)$ may be undone by the addition to all the powers of any vector $(\alpha_1, \ldots, \alpha_n)$ if $\alpha_j < 0$ for $j = 1, 2, \ldots, n$.

Moreover, in the following section we shall also show that as $m \nearrow \infty$, the sequence $\{\lambda_m\}$ defined in the preceding proof satisfies $\lambda_{mj} \sim m$ for $j = 1, 2, \ldots, n$.

Sparse spanning sets. The sequences of powers appearing in the monomials in the proof of Theorem 1 are of the form

$$\{(m, \varphi \circ B \circ \varphi^{-1}(m), \ldots, \varphi \circ B_{n-1} \circ \varphi^{-1}(m))\}$$

where B is the Blaschke product of Lemma 2 and φ is the conformal map of U onto $\mathscr{H}_p$ defined at the outset of the proof.

Recall that by Lemma 2, $B(r) \nearrow 1$, $B'(r) \to 1$ and $B''(r) \to 0$ as $r \nearrow 1$. Then

$$(\varphi \circ B)(r) - \varphi(r) = \frac{1+B(r)}{1-B(r)} - \frac{1+r}{1-r} = \frac{2(r-B(r))}{(1-B(r))(1-r)},$$

and therefore

$$\lim_{r \nearrow 1} [(\varphi \circ B)(r) - \varphi(r)] = 2 \lim_{r \nearrow 1} \frac{r - B(r)}{(1-r)^2} = \lim_{r \nearrow 1} B''(r) = 0.$$

It follows that each of the power sequences above lies on an analytic curve of the form

$$\{(t, \varphi \circ B \circ \varphi^{-1}(t), \ldots, \varphi \circ B_{n-1} \circ \varphi^{-1}(t)) \mid (0 < t < \infty)\}$$

which has monotone increasing coordinate functions and which as $t \to \infty$ is asymptotic to the diagonal $\{(t, t, \ldots, t)\}$, although the monomials $\{(s_1^m, s_2^m \cdots s_n^m)\}$ do not span in any $L^p(I^n)$ if $n \geqq 2$.

If φ is defined by $\varphi(\zeta) = (1+\zeta)/(1-\zeta) + \alpha$ $(\alpha > 0)$, then the corresponding power sequences, although asymptotic to the diagonal sequence, serve as powers for monomials whose linear span is dense in $\breve{C}(I^n)$.

It may be worth remarking that in fact the full power sequences given above are not needed. Any subsequence $\{(m_k, \varphi \circ B \circ \varphi^{-1}(m_k), \ldots, \varphi \circ B_{n-1} \circ \varphi^{-1}(m_k))\}$ where $\sum 1/m_k = +\infty$ would suffice.

I am grateful to W. Rudin for some stimulating conversations and helpful suggestions.

Note. J. Korevaar has informed me that he has also proved Theorem 1 and Corollary 1.1 using monomials with lattice powers.

References

1. J. Korevaar and S. Hellerstein, *Discrete sets of uniqueness for bounded holomorphic functions $f(z, w)$*, Proc. Sympos. Pure Math., vol. 11, Amer. Math. Soc., Providence, R. I., 1968, pp. 273–284. MR **38** #3462.

2. W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

3. L. Schwartz, *Étude des sommes exponentielles*, 2ième éd., Actualités Sci. Indust., no. 959, Hermann, Paris, 1959. MR **21** #5116.

4. W. Seidel, *On the distribution of values of bounded analytic functions*, Trans. Amer. Math. Soc. **36** (1934), 201–226.

5. A. Zygmund, *Trigonometrical series*, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

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