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SUFFICIENCY CLASSES OF LCA GROUPS

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Abstract. By the sufficiency class $S(H)$ of a locally compact Abelian (LCA) group H we shall mean the class of LCA groups G having sufficiently many continuous homomorphisms into H to separate the points of G. In this paper we determine the sufficiency classes of a number of LCA groups and indicate how these determinations may help to describe the structure of certain classes of LCA groups. In particular, we give a new proof of a theorem of Robertson which states that an LCA group is torsion-free if and only if its dual contains a dense divisible subgroup. We shall also derive some facts about the compact connected Abelian groups and a result about topological p-groups containing dense divisible subgroups. We conclude by giving a necessary condition for two LCA groups to have the same sufficiency class.

By a *generalized character* of a topological group G we mean a member of Hom (G, H), the set of all continuous homomorphisms from G into H, where H is an Abelian topological group. The set Hom (G, H) can be turned into an Abelian topological group in a natural way (see [1, 23.34]), but we shall not be directly concerned with this. Throughout this paper, all groups will be assumed to be locally compact Abelian (LCA) Hausdorff topological groups. If H is the circle group, then Hom (G, H) becomes $\hat{G}$, the ordinary character group of G. The usefulness of generalized characters has been illustrated (especially in the case where H is the group of real numbers with its usual topology) for many years in a variety of situations. It is our present intention to illustrate the use of generalized characters as a vehicle for describing certain natural classes of LCA groups.

The LCA groups of which we shall make constant mention are the circle group T, the additive real numbers R (both with their usual topologies), the integers Z, the cyclic groups $Z(n)$, the quasicyclic groups $Z(p^\infty)$, the discrete additive rationals Q, the p-adic integers J_p and the p-adic numbers F_p. Precise definitions of, and information concerning, all these groups may be found in [1]. Finally, let $\{G_\iota\}_{\iota\in I}$ be a collection of groups indexed by a set I. We denote by $P_{\iota\in I}\, G_\iota$ the full direct product, and by $P^*_{\iota\in I}\, G_\iota$ the weak direct product, of the groups G_ι (see [1, 2.3]).

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DEFINITION 1. Let H be LCA.

(a) We denote by $S(H)$ the class of all LCA groups G having sufficiently many continuous homorphisms into H to separate the points of G.

(b) A subgroup K of H is called a G-subgroup of H if $K=f(G)$ for some f in Hom (G, H), where G is LCA.

(c) We denote by $S^*(H)$ the class of all LCA groups G whose H-subgroups generate a dense subgroup of G.

We call $S(H)$ the *sufficiency class* of H, while $S^*(H)$ is called the *dual sufficiency class* of H. In this new terminology, several well-known theorems take on a more compact form. For example, we can state that a group G is connected if and only if G is in $S^*(R)$ [1, 25.20], and that a group G is in $S(R)$ if and only if $\hat{G}$ is connected [1, 24.35]. From these two statements we deduce immediately that G is in $S(R)$ if and only if $\hat{G}$ is in $S^*(R)$. This is a special case of a completely general dual phenomenon, which we state as a lemma. It will be fundamental throughout the sequel.

LEMMA 1 (MOSKOWITZ). *Let G and H be LCA. Then G is in $S(H)$ if and only if $\hat{G}$ is in $S^*(\hat{H})$.*

Proof. This is proved, with different notation, in the paper of Moskowitz [3, Theorem 2.8].

It is not in general a simple matter to determine $S(H)$ for a given LCA group H. The process is immensely simplified if H is divisible, since then one may extend homomorphisms, often continuously. This is the case if $H=R$. It is found that $S(R)$ consists of all LCA groups G having no compact elements [1, 24.34]. This pattern of proof can be used for various divisible groups H. To make our proofs run more smoothly, we isolate two lemmas for future reference.

LEMMA 2. *Let H be a divisible LCA group. If U is in $S(H)$ for every compactly generated open subgroup U of an LCA group G, then G is in $S(H)$.*

Proof. We must show that if $x\neq 0$ is an element of G, there exists $f\in$ Hom (G, H) such that $f(x)\neq 0$. By [1, 5.14] there is an open compactly generated subgroup U of G which contains x. By hypothesis, U is in $S(H)$, so there is a homomorphism $f_0\in$ Hom (U, H) such that $f_0(x)\neq 0$. Since H is divisible, f_0 may be extended to a member of Hom (G, H), by [1, A.7] and the fact that U is open. Thus we have found an $f\in$ Hom (G, H) such that $f(x)\neq 0$.

LEMMA 3. *Let G be a compactly generated LCA group. Then G is topologically isomorphic with $R^n\times Z^m\times F$ for some nonnegative integers m and n and some compact group F.*

Proof. See [1, 9.8].

PROPOSITION 1. *Let G be LCA. The following are equivalent*:

(1) *G is torsion-free.*

(2) *G is in* $S(\hat{Q})$.

Proof. Since $\hat{Q}$ is torsion-free, it is obvious that (2) ⇒ (1). Conversely, assume (1) and let U be a compactly generated open subgroup of G. By Lemma 3, U has the form $R^n \times Z^m \times F$, where F is compact. Since F is torsion-free, it can be written as a full direct product of groups $\hat{Q}$ and J_p, for various primes p (see [1, 25.4 and 25.8]). Now it is easy to see that if $\{G_i\}_{i\in I}$ is a family of LCA groups indexed by a set I, and if G_i is in $S(H)$ for each $i \in I$, where H is some fixed LCA group, then the direct product of the groups G_i is an LCA group in $S(H)$, provided that all but a finite number of the G_i are compact. Thus if we can show that each of the groups R, Z, $\hat{Q}$ and J_p is in $S(\hat{Q})$, then we can conclude immediately that U is in $S(\hat{Q})$. The proof will then be completed by an appeal to Lemma 2, since $\hat{Q}$ is divisible.

It is obvious that $\hat{Q}$ and Z are in $S(\hat{Q})$. As for R, the natural injection $i: Q \to R$ has dense image, so that its transpose $i^*: R \to \hat{Q}$ is one-one [1, 24.41]; hence R is in $S(\hat{Q})$. Finally, since $Z(p^\infty)$ is a quotient of Q, say by π, then $\pi^*: J_p \to \hat{Q}$ is one-one (see [1, 24.41 and 25.2]), so that J_p is in $S(\hat{Q})$. This completes the proof.

Our object in determining $S(Q)$ is to provide a different proof of Robertsons' theorem [4, 5.2], stated below. Robertson based his proof upon the theory of homogeneous groups.

THEOREM 1 (ROBERTSON). *Let G be LCA. The following are equivalent*:

(1) *G is torsion-free.*

(2) $\hat{G}$ *contains a divisible dense subgroup.*

Proof. If (2) holds, then every nontrivial continuous character of $\hat{G}$ must have infinite range, since homomorphic images of divisible groups are divisible and therefore either infinite or trivial. Therefore every nontrivial continuous character of $\hat{G}$ must be of infinite order, since a subgroup H of T such that nH is trivial for some n is finite. (*Referee's note*: This is really a special case of Burnside's theorem that, if H is a subgroup of $G(n, C)$ of bounded order, then H is finite.) Thus (2) ⇒ (1). Conversely, if G is torsion-free, then G is in $S(\hat{Q})$ by Proposition 1. Hence, by Lemma 1, $\hat{G}$ is in $S^*(Q)$. Let D be the largest divisible subgroup of $\hat{G}$ [1, A.6]. Since Q is divisible, it follows that $f(Q) \subseteq D$ for every f in Hom $(Q, \hat{G})$, so that D is dense in $\hat{G}$, proving that (1) ⇒ (2).

We next determine $S(Q)$. This is a very simple task; when we combine the result with Lemma 1, we obtain a simple characterization of the compact connected Abelian groups, analogous to the characterization of the connected Abelian groups in terms of one-parameter subgroups [1, 25.20].

PROPOSITION 2. *Let G be LCA. The following are equivalent*:

(1) *G is discrete and torsion-free.*

(2) *G is in* $S(Q)$.

Proof. If G is in $S(Q)$, then G is in $S(R)$, and so $G \cong R^n \times D$, where n is a nonnegative integer and D is discrete and torsion-free [1, 24.35(iii)]. But certainly $n=0$ here, so that (2) $\Rightarrow$ (1). Conversely, if (1) holds, and if U is a compactly generated subgroup of G, it is obvious from Lemma 3 that $U \cong Z^m$, and since Z is in $S(Q)$, Lemma 2 shows that G is in $S(Q)$, so (1) $\Rightarrow$ (2).

THEOREM 2. *The following are equivalent for an LCA group G:*

(1) *G is compact and connected.*

(2) *G is in $S^*(\hat{Q})$.*

Proof. Since a group is compact and connected if and only if its dual is discrete and torsion-free, this follows immediately from Proposition 2 and Lemma 1.

We next determine $S(Z(p^\infty))$, where p is a prime. To describe this class of groups, we need a definition:

DEFINITION 2. Let G be an LCA group, written additively. We say that G is a *topological p-group* if and only if $\lim_{n\to\infty} p^n x = 0$ for every x in G (see [4, 3.1] for more information).

PROPOSITION 3. *The following are equivalent for an LCA group G:*

(1) *G is totally disconnected, and every compact open subgroup of G is a topological p-group.*

(2) *G is in $S(Z(p^\infty))$.*

Proof. First assume that G is compact. If (2) holds, then by Lemma 1 the discrete group $\hat{G}$ is in $S^*(J_p)$. It is clear that no f in Hom $(J_p, \hat{G})$ could be one-one, so that $f(J_p)$ has the form $Z(p^n)$ for every f in Hom $(J_p, \hat{G})$. Hence every element of $\hat{G}$ has order a power of p, so that G is a topological p-group by [4, 3.14]. Certainly then G is totally disconnected, so that (2) $\Rightarrow$ (1) in case G is compact. Conversely, if G is compact and if (1) holds, then G is itself a topological p-group; thus every element of the discrete group $\hat{G}$ has order a power of p [4, 3.14]. Since every group $Z(p^n)$ can be realized as a quotient of J_p by a closed subgroup, it follows that $\hat{G}$ is in $S^*(J_p)$, or G is in $S(Z(p^\infty))$ by Lemma 1. Thus for compact G, (1) $\Leftrightarrow$ (2).

Next we observe that, since the subgroups of T of the form $Z(p^n)$ generate the dense subgroup $Z(p^\infty)$ of T, T is a member of $S^*(J_p)$. Hence Z is in $S(Z(p^\infty))$ by Lemma 1. We could also see this directly by observing that if $\pi_n: Z \to Z(p^n)$ is the canonical epimorphism and if i_n is the injection of $Z(p^n)$ into $Z(p^\infty)$, then the homomorphisms $i_n \circ \pi_n$ separate the points of Z.

Now let G be arbitrary. If (2) holds, then certainly G is totally disconnected. Moreover, (2) will hold for every compact subgroup of G, so that every compact subgroup of G is a topological p-group, by the first paragraph. Hence (2) $\Rightarrow$ (1). Conversely, assume that (1) holds, and let U be a compactly generated open subgroup of G. By Lemma 3, U has the form $Z^m \times F$, where m is a nonnegative integer and F is compact. Since F must be open in G, (1) implies that F is a topological p-group, so that F is in $S(Z(p^\infty))$ by the first paragraph. Also, Z is in

$S(Z(p^\infty))$ by the second paragraph. Hence U is a member of $S(Z(p^\infty))$, so Lemma 2 implies that G is in $S(Z(p^\infty))$, since $Z(p^\infty)$ is divisible, i.e. (1) $\Rightarrow$ (2), completing the proof.

The dual of this proposition states that a group G is in $S^*(J_p)$ if and only if every element of G is compact and every quotient of G by a compact open subgroup is a p-group (a discrete topological p-group). In particular, if G is compact and connected, the latter condition is trivially satisfied. Hence a compact connected group is a member of $S^*(J_p)$ for every prime p. This can be shown much more directly (see [5, 1.9]). Below we give a strengthened converse, which is our second characterization of compact connected Abelian groups.

THEOREM 3. *Let G be LCA. The following are equivalent:*

(1) *G is compact and connected.*

(2) *There exist distinct primes p and q such that G is in $S^*(J_p) \cap S^*(J_q)$.*

Proof. We have already shown that if (1) holds, then G is in $S^*(J_p)$ for every prime p, so that (1) $\Rightarrow$ (2). Conversely, if (2) holds, then $\hat{G}$ is in $S(Z(p^\infty)) \cap S(Z(q^\infty))$ by Lemma 1. Thus by Proposition 3, $\hat{G}$ is totally disconnected and every compact open subgroup of $\hat{G}$ is at once a topological p-group and a topological q-group. But a nontrivial group can be a topological p-group for at most one prime p [4, 3.17], so that every compact open subgroup of $\hat{G}$ is trivial. Since $\hat{G}$ is totally disconnected, it follows that $\hat{G}$ is discrete and torsion-free; that is, G is compact and connected. Hence (2) $\Rightarrow$ (1).

In [5, 3.13 and 1.3] it is shown, in different terminology, that a topological p-group G contains a dense divisible subgroup if and only if G is in $S^*(F_p)$. We shall prove this in a new way by determining $S(F_p)$; at the same time we shall provide a slightly more detailed description of the topological p-groups containing dense divisible subgroups.

PROPOSITION 4. *Let G be LCA, and let p be a prime. The following are equivalent:*

(1) *G is totally disconnected, and every compact open subgroup of G has the form $(J_p)^M$, where M is a cardinal number.*

(2) *G is in $S(F_p)$.*

Proof. If (2) holds for G, then certainly G is totally disconnected and torsion-free. Let K be a compact subgroup of G. Then by [1, 25.8], K can be written as a full direct product of groups of type J_q for various primes q; clearly, only such groups with $q = p$ can appear, so (2) $\Rightarrow$ (1).

Conversely, suppose (1) holds, and let U be a compactly generated open subgroup of G. By Lemma 3, U has the form $Z^m \times F$, where F is compact. Since F is also open in G, F has the form $(J_p)^M$, where M is a cardinal number. Since both Z and J_p are in $S(F_p)$, it follows that U is in $S(F_p)$. Since F_p is divisible, we have that (1) $\Rightarrow$ (2) by Lemma 2, completing the proof.

THEOREM 4. *Let G be a topological p-group, where p is a fixed prime. The following are equivalent*:

(1) *G contains a dense divisible subgroup.*

(2) *Every quotient of G by a compact open subgroup is a weak direct product of groups $Z(p^\infty)$.*

(3) *G is in $S^*(F_p)$.*

Proof. Assume (1). Then $\hat{G}$ is a torsion-free topological p-group by Theorem 1 and [4, 3.18]. Hence $\hat{G}$ is totally disconnected, and it follows from [1, 25.8] that every compact subgroup of $\hat{G}$ is a product of groups J_p. Thus $\hat{G}$ is in $S(F_p)$ by Proposition 4. Lemma 1 together with the fact that F_p is self-dual now implies that (3) holds. But (3) is equivalent to (2) by Proposition 4, Lemma 1 and a straightforward duality argument. Finally, the fact that F_p is divisible shows that (3) $\Rightarrow$ (1). This completes the proof.

We have now described the sufficiency classes of the divisible groups R, $\hat{Q}$, Q, $Z(p^\infty)$ and F_p. It should perhaps be mentioned that $S(T)$ consists of all LCA groups; this fundamental fact cannot, however, be regarded as a consequence of Lemma 1; such reasoning would be circular, since the Duality Theorem depends upon the fact that the continuous characters of an LCA group separate its points.

It would be interesting to ascertain necessary and sufficient conditions for a class of LCA groups to be a sufficiency class. Two conditions are obviously necessary:

(1) if G is a member of the class, then $S(G)$ is a subset of the class, and

(2) if G_1 and G_2 are members of the class, then so is $G_1 \times G_2$.

These conditions are not sufficient, however, since the class of LCA torsion groups of bounded order satisfies these two conditions, and yet it is not hard to see that this class cannot be the sufficiency class of any LCA group. It would seem natural to strengthen condition (2) to require that all local direct products of groups in the class also belong to the class whenever they are locally compact. The author has not succeeded in showing that these strengthened conditions (obviously necessary) are sufficient to guarantee that the class in question is a sufficiency class. For example, is the class of all reduced (i.e. containing no divisible subgroups) LCA groups a sufficiency class?

There are several other, hitherto unmentioned, natural classes of LCA groups which turn out to be sufficiency classes (again, of divisible groups). We mention two of these in the next two propositions.

PROPOSITION 5. *$S(Q/Z)$ is the class of totally disconnected LCA groups.*

Proof. If G is in $S(Q/Z)$, then G is certainly totally disconnected. Conversely, if G is totally disconnected, and if U is a compactly generated subgroup of G, then U has the form $Z^m \times F$, where F is compact. Since Z is in $S(Z(p^\infty))$ for each prime p, we know that Z is in $S(Q/Z)$. Moreover, since the continuous characters of a compact totally disconnected group have range contained in the rational circle

[1, 24.26], it follows that F is in $S(Q/Z)$. Hence U is in $S(Q/Z)$, and the result follows from Lemma 2.

Using this proposition and Lemma 1, together with the fact that the dual of Q/Z is $P_{p\in I} J_p$, where I is the set of primes, we see that an LCA group G consists entirely of compact elements if and only if the subgroups of G of the form $Z(p^n)$ and J_p generate a dense subgroup of G.

For the next proposition, we let G_0 be the local direct product of the groups F_p with respect to the compact subgroups J_p over all primes p. The group G_0 may be alternatively described as the minimal divisible extension of the group $(Q/Z)^\wedge$; see [1, 25.32].

PROPOSITION 6. *Let G_0 be as above. Then $S(G_0)$ is the class of totally disconnected torsion-free LCA groups.*

Proof. If G is in $S(G_0)$, then certainly G is totally disconnected and torsion-free. The converse follows from [1, 25.8] and Lemmas 2 and 3.

Before proceeding to our final results, we inquire into the problem of describing the class of LCA groups having dense torsion subgroup. This class is a dual sufficiency class. To see this, let $B=P_{n\in N} Z(n)$, where N is the set of positive integers and B is given the (compact) product topology. Then $\hat{B}=P^*_{n\in N} Z(n)$, taken discrete, and it is clear that the class of LCA groups with dense torsion subgroup is the dual sufficiency class $S^*(\hat{B})$. Thus a determination of the sufficiency class $S(B)$ would yield a description of the class of all LCA groups whose duals have dense torsion subgroup. While this determination appears difficult, owing to the fact that B is not divisible, we shall nevertheless use the observations just made to give a proof of the following result.

PROPOSITION 7. *Let G be a compact and connected Abelian group. Then G has dense torsion subgroup if and only if $\hat{G}$ is reduced.*

Proof. If G has dense torsion subgroup, then $\hat{G}$ is in $S(B)$. Since B is reduced, it is obvious that $\hat{G}$ must be reduced as well. Conversely, suppose $\hat{G}$ is reduced, and let $\hat{G}(n)=\{\gamma^n : \gamma \in \hat{G}\}$. Since $\hat{G}$ is torsion-free, $\bigcap_{n=1}^{\infty} \hat{G}(n)=\{1\}$, since otherwise $\hat{G}$ would contain a copy of the rationals. Thus if $\gamma\neq 1$ is in $\hat{G}$, there is an n such that $\gamma \notin \hat{G}(n)$. Since the quotient group $\hat{G}/\hat{G}(n)$ is a torsion group, it is a weak direct product of cyclic groups and is hence a member of $S(B)$. Thus by using the projection from $\hat{G}$ onto $\hat{G}/\hat{G}(n)$ we can find $f\in \operatorname{Hom}(\hat{G}, B)$ such that $f(\gamma)\neq 0$. Since $\gamma\neq 1$ was arbitrary, it follows that $\hat{G}$ is in $S(B)$, so that G has dense torsion subgroup. This completes the proof.

REMARK. In light of the fact that every (discrete) Abelian group is the product of a divisible group and a reduced group (see [1, A.8]), the proposition above is equivalent to a theorem of Itzkowitz [2, Theorem 2], which states that the closure of the torsion subgroup of a compact connected Abelian group is a topological direct summand of the group. Itzkowitz proved his theorem by quite different methods.

We conclude our findings by giving necessary conditions so that two LCA groups G_1 and G_2 have the same sufficiency class. The problem of specifying intrinsic sufficient conditions seems to be much more difficult.

PROPOSITION 8. *Suppose that* $S(G_1)=S(G_2)$, *where* G_1 *and* G_2 *are both LCA. Then* G_1 *and* G_2 *contain closed subgroups* F_1 *and* F_2, *respectively, such that* $F_1\cong F_2\cong F$, *where* F *is either* Z, J_p *or* $Z(p)$, *where* p *is a prime.*

Proof. First assume that G_1 is torsion-free. Then so is G_2. Let C_i be the identity component of G_i, for $i=1, 2$. Then we can write $C_i=R^{n_i}\times(\hat{Q})^{M_i}$ where n_i is a nonnegative integer and M_i is a cardinal number, for $i=1$, 2, by [1, 9.14 and 25.8]. If C_1 is not trivial, then it is clear that C_1 and C_2 must both contain a copy of either R or $\hat{Q}$. Since Z is a closed subgroup of R and J_p is a closed subgroup of $\hat{Q}$ for any prime p, we conclude that G_1 and G_2 must both contain a copy of Z or of J_p. If C_1 is trivial, then G_1 either contains compact elements or it does not. If G_1 contains no compact elements, then neither does G_2, and so both groups contain copies of Z. Otherwise, G_1 contains a copy of J_p for some p, by [1, 25.8], and, since G_2 is torsion-free and G_1 is in $S(G_2)$, G_2 contains a copy of the same J_p. Finally, if G_1 is not torsion-free, it contains a copy of $Z(p)$ for some prime p, and therefore G_2 contains a copy of the same $Z(p)$. This completes the proof.

REMARK. It is not difficult to show that if G is indecomposable (that is, it cannot be written as the direct product of two nontrivial LCA groups) and if $S(G)=S(T)$ (respectively, $S(R)$, $S(\hat{Q})$, $S(Q)$, $S(Z(p^\infty))$, $S(F_p)$) then $G=T$ (respectively, R, $\hat{Q}$, Q, $Z(p^\infty)$, F_p).

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