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GENERALLY p^α-TORSION COMPLETE ABELIAN GROUPS

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Abstract. A generalized p-primary cotorsion abelian group G is a p^α-injective, that is satisfies $p^\alpha \operatorname{Ext}(-, G)=0$, iff G_t is p^α-injective in the category of torsion abelian groups. Such a torsion group is generally p^α-torsion complete, but an example shows that all its Ulm factors need not be complete. The injective properties of generally p^α-torsion complete groups are investigated. They are an injectively closed class, and the corresponding class of sequences is the class of p^α-pure sequences with split completion when α is "accessible". Also, these groups are the p^α-high injectives.

This paper is motivated by the problem of determining the structure, say in the sense of [2], of the p^α-injective abelian groups, i.e. those groups G such that $p^\alpha \operatorname{Ext}(-, G)=0$. It is shown that a generalized p-primary cotorsion group G is p^α-injective iff its torsion subgroup is p^α-injective in the category of torsion abelian groups. Properties similar to those in [19] are shown for this latter concept, which we call "p^α-torsion injective". An example is then given to show that unlike the general case a $p^{\omega 2}$-torsion injective need not have complete groups as Ulm factors. The injective properties of the generally p^α-torsion complete groups are investigated. It is shown that they form an injectively closed class and the corresponding sequences are characterized for accessible ordinals. Finally, the p^α-high injectives are characterized.

It is assumed that the reader is familiar with elementary homological algebra such as in [5] and [1, p. 116] and with §§1 and 2 of Chapter III in [14]. We follow Mines [15] in calling an ordinal *accessible* if it is the limit of a countable increasing sequence of lesser ordinals. The notation is that of [3] and/or [12].

1. **Background.** Let G be an additively written abelian group. G is called a generalized p-primary group or g.p. group if $qG=G$ for all primes q different from the given prime p. If α is a limit ordinal, we can make G a topological group by letting a fundamental neighborhood system of zero be $\{p^\beta G\}_{\beta<\alpha}$ (see [15]). This induces a Hausdorff topology on $G/p^\alpha G$. Completing this we get a group $L_\alpha(G)$. We denote the torsion subgroup of $L_\alpha(G)$ by $T_\alpha(G)$. $L_\alpha(G)$ can also be viewed as

$$L_\alpha(G) = \operatorname*{proj\,lim}_{\beta<\alpha} G/p^\beta G$$

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where the inverse limit is taken over the family of natural homomorphisms $f_{\beta\gamma}: G/p^\beta G \to G/p^\gamma G$ for each pair γ, β with $\alpha > \beta > \gamma$ [21]. There is a natural map $\delta: G \to L_\alpha(G)$ given by $\delta(g) = (g + p^\beta G)_{\beta < \alpha}$, viewing $L_\alpha(G)$ as a subgroup of $\prod_{\beta < \alpha} G/p^\beta G$. This induces an injection $\delta_*: G/p^\alpha G \to L_\alpha(G)$.

If α is an accessible ordinal then $L_\alpha(G)/\delta(G)$ is divisible. An example showing this to be not the case for $\alpha = \Omega$, the first uncountable ordinal, is given in [15].

For ease of reference and to refresh the reader's memory we list three propositions. Recall that A is p^α-pure in B if $0 \to A \to B \to B/A \to 0$ represents an element of $p^\alpha \operatorname{Ext}(B/A, A)$.

PROPOSITION 1 (NUNKE [18]). *A p^α-injective group has the form $D \oplus C$ where D is divisible and C is a cotorsion group such that $p^\alpha C = 0$.*

PROPOSITION 2 (MINES [15]). *Let G be a g.p. group. Then $\delta(G)$ is p^α-pure in $L_\alpha(G)$ for limit ordinals α.*

PROPOSITION 3 (NUNKE [17]). *A p^α-pure extension $0 \to A \to B \to C \to 0$ has the following properties*:

(1) *$p^\beta B \cap A = p^\beta A$ for all $\beta < \alpha$.*

(2) *$(p^\beta C)[p] = (A + (p^\beta B)[p])/A$ for all $\beta < \alpha$.*

Let $0 \to A \to B \to C \to 0$ be exact with C a p-group. Suppose

(1) *$C/p^\beta C$ is p^β-projective for all $\beta < \alpha$,*

(2) *$(p^\beta C)[p] = (A + (p^\beta B)[p])/A$ for all $\beta < \alpha$, and*

(3) *$p^\beta C = (A + p^\beta B)/A$ for all $\beta < \alpha$.*

Then the extension is p^α-pure. If α is a limit ordinal or the reduced part of A satisfies $p^\alpha(\text{red } A) = 0$ then (3) *is redundant. If A is divisible,* (1) *is redundant and* (3) *may be omitted.*

2. It is known that every p^α-injective is cotorsion but that the converse is false if α is greater than or equal to $\omega 2$ [19]. The problem of describing the structure of these groups has been outstanding for some time. The following theorem transfers the problem of distinguishing the injectives among the g.p. cotorsions to the category of torsion groups, where more is known.

DEFINITION. A torsion group G is called p^α-torsion injective if $p^\alpha \operatorname{Ext}(A, G) = 0$ for every torsion group A.

A torsion group G is called p^α-divisible injective if $p^\alpha \operatorname{Ext}(A, G) = 0$ for every torsion divisible group A.

THEOREM 1. *Let G be a cotorsion, g.p. group, and α a limit ordinal. Then G is p^α-injective iff its torsion subgroup G_t is p^α-torsion injective. Given a reduced torsion group G_t which is p^α-torsion injective,* $\operatorname{Ext}(Q/Z, G_t)$ *is a p^α-injective whose torsion subgroup is exactly G_t.*

Proof. Let G be p^α-injective and A a torsion group. Then $0 \to G_t \xrightarrow{i} G \to G/G_t$ induces the exact sequence

$$\operatorname{Hom}(A, G/G_t) \longrightarrow \operatorname{Ext}(A, G_t) \xrightarrow{i_*} \operatorname{Ext}(A, G).$$

Since A is torsion, Hom $(A, G/G_t)=0$ and so i_* is an inclusion. Thus p^α Ext $(A, G_t) \subseteq p^\alpha$ Ext $(A, G)=0$. Thus G_t is p^α-torsion injective.

Suppose G_t is p^α-torsion injective and A is any group. Then $0 \to A_t \to A \to A/A_t \to 0$ induces an exact sequence

$$\text{Ext}\,(A/A_t, G) \to \text{Ext}\,(A, G) \to \text{Ext}\,(A_t, G) \to 0.$$

Since G is cotorsion, Ext $(A/A_t, G)=0$. Thus we need only show p^α Ext $(A, G)=0$ for torsion groups A. Since G is g.p., Ext $(A_q, G)=0$ for all $q \neq p$ so it suffices to show p^α Ext $(A, G)=0$ for p-primary groups A.

By results in [5], each cotorsion group G satisfies

$$G \simeq \text{Ext}\,(Q/Z, G_t) \oplus \text{Ext}\,(Q/Z, G/G_t). \tag{1}$$

Thus we have

$$\begin{aligned} \text{Ext}\,(A, G) &\simeq \text{Ext}\,(A, \text{Ext}\,(Q/Z, G_t)) \oplus \text{Ext}\,(A, \text{Ext}\,(Q/Z, G/G_t)) \\ &\simeq \text{Ext}\,(A, G_t) \oplus \text{Ext}\,(A, G/G_t). \end{aligned} \tag{2}$$

Since G_t is p^α-torsion injective, (2) yields

$$p^\alpha\,\text{Ext}\,(A, G) \simeq p^\alpha\,\text{Ext}\,(A, G/G_t). \tag{3}$$

Letting $H=G/G_t$ we have an exact sequence

$$\begin{aligned} 0 = \text{Hom}\,(A, Q \otimes H) \to \text{Hom}\,(A, H \otimes (Q/Z)) &\to \text{Ext}\,(A, H) \\ &\to \text{Ext}\,(A, Q \otimes H) = 0. \end{aligned}$$

Thus p^α Ext $(A, G)=p^\alpha$ Ext $(A, H) \simeq p^\alpha$ Hom $(A, H \otimes (Q/Z))=0$ by [3, p. 206], since $\alpha \geqq \omega$. Hence G is p^α-injective.

If G_t is a reduced, torsion, p^α-torsion injective group, and A is any group then

$$p^\alpha\,\text{Ext}\,(A, \text{Ext}\,(Q/Z, G_t)) = p^\alpha\,\text{Ext}\,(\text{Tor}\,(A, Q/Z), G_t) = p^\alpha\,\text{Ext}\,(A_t, G_t) = 0.$$

Further,

$$0 = \text{Hom}\,(Q, G_t) \to \text{Hom}\,(Z, G_t) = G_t \to \text{Ext}\,(Q/Z, G_t) \to \text{Ext}\,(Q, G_t)$$

is exact. Since Ext (Q, G_t) is torsion free [5, p. 370], we conclude that

$$G_t = (\text{Ext}\,(Q/Z, G_t))_t,$$

concluding the proof.

Mines, in [15], shows that any g.p. cotorsion group must be generally complete, that is, that $L_\beta(G)/\delta(G)$ is reduced for each limit ordinal β. (For accessible ordinals β, this means $G/p^\beta G$ is complete in the p^β-topology.) This result means that the obvious topological distinctions will not aid in finding the p^α-injectives. The situation in the torsion case is more unusual, as we will show later. For now, we prove only the following:

Definition. A torsion group G is called generally p^α-torsion complete if $T_\alpha(G)/\delta(G)$ is reduced.

THEOREM 2. *Let α be a limit ordinal. Let G be a torsion, p^α-torsion injective group. Then G is generally p^α-torsion complete.*

Proof. There is an exact sequence

$$0 \longrightarrow G/p^\alpha G \xrightarrow{\delta_*} L_\alpha(G) \longrightarrow L_\alpha(G)/\delta(G) \longrightarrow 0.$$

By Proposition 2, this sequence is p^α-pure. By Theorem 18(a) of [11], $0 \to G/p^\alpha G \to T_\alpha(G) \to T_\alpha(G)/\delta(G)$ is also p^α-pure. $T_\alpha(G)/\delta(G)$ is torsion, so this sequence splits. Then $T_\alpha(G) = \delta(G) \oplus T_\alpha(G)/\delta(G)$, and since $T_\alpha(G) \subseteq \prod_{\beta<\alpha} G/p^\beta G$ is reduced, so is $T_\alpha(G)/\delta(G)$.

3. In order to further explore the relationship between generally p^α-torsion complete groups and p^α-torsion injectivity we employ the device of the injectively closed class due to Maranda [13]. We restrict ourselves to the category of p-primary abelian groups unless otherwise mentioned.

DEFINITION. Let α be a limit ordinal. We define

(a) $\mathscr{G}_\alpha$ to be the class of all short exact sequences $\mu: 0 \to A \to B \to C \to 0$ such that μ is p^α-pure and C is divisible; and

(b) $\mathscr{D}_\alpha$ to be the class of all groups G such that G is generally p^α-torsion complete and $p^\alpha G$ is divisible.

REMARKS. (1) Any member of $\mathscr{D}_\alpha$ can be written $D \oplus R$ where D is divisible, $p^\alpha R = 0$ and R is generally p^α-torsion complete.

(2) If $p^\omega G$ is the torsion subgroup of $\prod_{i=1}^{\infty} Z(p^i)$ then G is an example of a member of $\mathscr{D}_{\omega 2}$.

Given an abelian category A we define two maps between classes of objects in A and classes of short exact sequences in A. Recall that if $G \in \mathrm{Ob}\,(A)$ and $\mu: 0 \to A \to B \to C \to 0$ is a short exact sequence in A, G is called injective for μ if every morphism $f: A \to G$ can be lifted to $g: B \to G$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[dr, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & 0 \\ & & G & & \end{tikzcd}$$

is a commutative diagram. We also say μ is injective with respect to G.

DEFINITION. (a) If $\mathscr{D}$ is a class of objects of A, define $\Psi(\mathscr{D})$ to be the class of *all* short exact sequences which are injective with respect to *every* member of $\mathscr{D}$.

(b) If $\mathscr{G}$ is a class of short exact sequences in A, define $\Phi(\mathscr{G})$ to be the class of *all* objects of A which are injective with respect to *every* member of $\mathscr{G}$.

These definitions are slight corruptions of those of Maranda in [13], and are dual to those of [20].

One should note that $\mathscr{D} \subseteq \Phi\Psi(\mathscr{D})$ and $\mathscr{G} \subseteq \Psi\Phi(\mathscr{G})$; further, $\Phi\Psi\Phi(\mathscr{G}) = \Phi(\mathscr{G})$ and $\Psi\Phi\Psi(\mathscr{D}) = \Psi(\mathscr{D})$. If $\mathscr{D} = \Phi\Psi(\mathscr{D})$ we call $\mathscr{D}$ an injectively closed class. If $\mathscr{D}$ is an injectively closed class, $\Psi(\mathcal{D})$ is called the "corresponding class of sequences". The maps Φ and Ψ are inclusion reversing.

THEOREM 3. *Let α be a limit ordinal. Then $\mathcal{D}_\alpha=\Phi(\mathcal{G}_\alpha)$. Consequently $\mathcal{D}_\alpha$ is an injectively closed class.*

Proof. Let $\mu: 0\to A\to B\to C\to 0$ be an element of $\mathcal{G}_\alpha$ and $G\in\mathcal{D}_\alpha$. Suppose $f: A\to G$. Then $f\mu\in p^\alpha\operatorname{Ext}(C, G)$. (This notation is explained in [14].) It suffices to show $f\mu=0$ in order to show $G\in\Phi(\mathcal{G}_\alpha)$. If $G=R\oplus D$, where D is divisible, and π_R and π_D are the natural projections, $f\mu=0$ if $(\pi_R f\oplus\pi_D f)=0$, which in turn happens iff $\pi_R f\mu\oplus\pi_D f\mu=0$; hence iff $\pi_R f\mu=0$. Thus we may assume G is reduced. We shall tacitly assume in several other proofs that in order to check injectivity of members of $\mathcal{D}_\alpha$ with respect to a sequence we need only check reduced members of $\mathcal{D}_\alpha$.

By assumption, D is divisible, so it suffices to show that G is p^α-divisible injective. Let $C=\sum Z(p^\infty)$. Then $p^\alpha\operatorname{Ext}(C, G)=\prod p^\alpha\operatorname{Ext}(Z(p^\infty), G)$. Since every p-primary divisible group can be so written, we need only to show $p^\alpha\operatorname{Ext}(Z(p^\infty), G)=0$.

Suppose $0\to G\to X\to Z(p^\infty)\to 0$ is an element of $p^\alpha\operatorname{Ext}(Z(p^\infty), G)$. Applying Proposition 3, and the Noether isomorphisms,

$$X/p^\beta X=\{G, p^\beta X\}/p^\beta X\simeq G/(p^\beta X\cap G)=G/p^\beta G \quad \text{for each } \beta<\alpha.$$

We may thus consider $X/p^\alpha X$ as a subgroup of $T_\alpha(G)$. Making this identification we find

$$T_\alpha(G)/\delta(G)\supseteq(X/p^\alpha X)/(\{G, p^\alpha X\}/p^\alpha X)=(X/G)/(\{p^\alpha X, G\}/G).$$

Since $T_\alpha(G)/\delta(G)$ is reduced and X/G is divisible, we must have $X=\{p^\alpha X, G\}$. Since $p^\alpha X\cap G=p^\alpha G=0$, $X=G\oplus Z(p^\infty)$. Thus $p^\alpha\operatorname{Ext}(Z(p^\infty), G)=0$ and so $\mathcal{D}_\alpha\subseteq\Phi(\mathcal{G}_\alpha)$.

Now suppose $G\in\Phi(\mathcal{G})$. Let $\gamma: 0\to G\to X\to C\to 0$ be p^α-pure with C divisible. Then $1_G: G\to G$ can be extended to $\phi: X\to G$. Hence γ splits. This shows G is p^α-divisible injective. Assuming G is reduced, we have $p^\alpha G=0$ by an easy argument, and an exact p^α-pure sequence $0\to G\to T_\alpha(G)\to T_\alpha(G)/\delta(G)\to 0$. Let $T_\alpha(G)/\delta(G)$ $=R\oplus D$ with R reduced, D divisible. Letting $T\supseteq\delta(G)$ be such that $T/\delta(G)=D$, $0\to G\to T\to D\to 0$ is p^α-pure by Theorem 18(a) of [11], and hence it splits. $T\subseteq T_\alpha(G)$ is reduced and so $D=0$. Thus G is generally p^α-torsion complete. Hence $\Phi(\mathcal{G}_\alpha)\subseteq\mathcal{D}_\alpha$.

We are going to investigate $\Psi(\mathcal{D}_\alpha)$ but first we prove the following theorem so that an important example can be presented.

THEOREM 4. (a) *Let n be an integer. If $G/p^\beta G$ is p^β-injective then $G/p^{\beta+n}G$ is $p^{\beta+n}$-injective. The same is true if we replace the words "injective" by "torsion injective" or by "divisible injective".*

(b) *Let G be a $p^{\alpha+n}$-divisible injective, α a limit ordinal. Then $G/p^\alpha G\in\mathcal{D}_\alpha$.*

Proof. (a) The exact sequence

$$0 \longrightarrow p^{\beta}G/p^{\beta+n}G \xrightarrow{i} G/p^{\beta+n}G \xrightarrow{\sigma} G/p^{\alpha}G \longrightarrow 0$$

yields the exact sequence

$$\operatorname{Ext}(A, p^{\beta}G/p^{\beta+n}G) \xrightarrow{i_*} \operatorname{Ext}(A, G/p^{\beta+n}G) \xrightarrow{\sigma_*} \operatorname{Ext}(A, G/p^{\beta}G) \longrightarrow 0.$$

Now $p^{\beta} \operatorname{Ext}(A, G/p^{\beta+n}G) \subseteq \ker \sigma_* = \operatorname{im} i_*$. But $p^{\beta}G/p^{\beta+n}G$ is p^n-bounded and hence so is $\operatorname{Ext}(A, p^{\beta}G/p^{\beta+n}G)$ [4]. Thus $p^{\beta+n} \operatorname{Ext}(A, G/p^{\beta+n}G) \subseteq p^n(\operatorname{im} i_*) = 0$.

(b) Let

$$0 \longrightarrow G/p^{\alpha}G \xrightarrow{\delta_*} T_{\alpha}(G) \longrightarrow T_{\alpha}(G)/\delta(G) \longrightarrow 0$$

be the usual exact sequence and suppose $T_{\alpha}(G)/\delta(G) = D \oplus R$ where R is reduced, D is divisible, and $r(D) \geqq 1$. Then

$$\begin{aligned} 0 = \operatorname{Hom}(Q/Z, T_{\alpha}(G)) &\to \operatorname{Hom}(Q/Z, T_{\alpha}(G)/\delta(G)) \\ &\to p^{\alpha} \operatorname{Ext}(Q/Z, G/p^{\alpha}G) \to p^{\alpha} \operatorname{Ext}(Q/Z, T_{\alpha}(G)) \end{aligned}$$

is exact by [11, Theorem 17]. Now $T_{\alpha}(G) \in \mathscr{D}_{\alpha}$ by [22, Proposition 5], and so $p^{\alpha} \operatorname{Ext}(Q/Z, T_{\alpha}(G)) = 0$ by Theorem 3. Letting I_p denote the p-adic integers or $\operatorname{Hom}(Z(p^{\infty}), Z(p^{\infty}))$, we have $\operatorname{Hom}(Q/Z, T_{\alpha}(G)/\delta(G)) = \operatorname{Hom}(Q/Z, D) = \prod_{r(D)} I_p$, and also $\operatorname{Hom}(Q/Z, T_{\alpha}(G)/\delta(G)) = p^{\alpha} \operatorname{Ext}(Q/Z, G/p^{\alpha}G)$. By Lemma 1.1 of [17], we have an exact sequence

$$0 \to \operatorname{Ext}(Q/Z, p^{\alpha}G) \to p^{\alpha} \operatorname{Ext}(Q/Z, G) \to p^{\alpha} \operatorname{Ext}(Q/Z, G/p^{\alpha}G) \to 0.$$

From above the last term is isomorphic to a product of p-adic integers and hence is torsion free. Then the sequence is pure exact [12]. Since $p^{\alpha}G$ is a p^n-bounded group direct sum with a divisible group, $\operatorname{Ext}(Q/Z, p^{\alpha}G)$ is p^n-bounded. By Kulikov's Theorem, a bounded pure subgroup is a summand. Hence $p^n p^{\alpha} \operatorname{Ext}(Q/Z, G) = 0$ implies $p^n p^{\alpha} \operatorname{Ext}(Q/Z, G/p^{\alpha}G) = 0$ and thus $p^n \prod_{r(D)} I_p = 0$. This implies $r(D) = 0$ and hence $G \in \mathscr{D}_{\alpha}$.

COROLLARY 1. *$G/p^{\omega+n}G$ is $p^{\omega+n}$-injective iff $G/p^{\omega}G$ is p^{ω}-injective. Hence a reduced $p^{\omega+n}$-injective is exactly a group G such that $p^{\omega+n}G = 0$ and G/G^1 is complete in its p-adic topology.*

COROLLARY 2. *The same statements hold for "torsion injective" and "torsion complete" respectively.*

The proof of the corollaries follows from Theorem 5 and from Theorem 26.1 of [3], and a modified form of that theorem for torsion groups which is easy to prove.

The following example was developed by Doyle O. Cutler, who has kindly permitted its presentation in this paper.

EXAMPLE. A $p^{\omega 2}$-torsion injective group such that $G/p^{\omega}G$ is *not* torsion complete in the p^{ω}-topology.

For each $i \in \omega$ let G_i be a group such that $G_i/p^{\omega}G_i \simeq \bar{B}$ and $p^{\omega}G_i \simeq Z(p^i)$, where $\bar{B}$ is the torsion completion of the direct sum of one cyclic group $Z(p^i)$ for each $i \in \omega$.

Then $\operatorname{Ext}(Z(p^{\infty}), G_i)$ is a $p^{\omega+i}$-injective. It follows that $K = \prod_{i=1}^{\infty} \operatorname{Ext}(Z(p^{\infty}), G_i)$ is a $p^{\omega 2}$-injective. Hence K_t is a $p^{\omega 2}$-torsion injective. Now we note that

$$K_t = \left(\prod_{i=1}^{\infty} \operatorname{Ext}(Z(p^{\infty}), G_i)\right)_t \simeq \left(\prod_{i=1}^{\infty} \operatorname{Ext}(Z(p^{\infty}), G_i)_t\right)_t \simeq \left(\prod_{i=1}^{\infty} G_i\right)_t.$$

Let $x_i \in G_i$ be such that $i \leqq h_{G_i}(x_i) < \infty$ and $\langle px_i \rangle = p^{\omega}G_i$. Let $y_i \in \prod_{i=1}^{\infty} G_i$ be such that $y_i(j) = x_i$ if $j = i$ and $y_i(j) = 0$ if $i \neq j$. We let $z_n = \sum_{i=1}^{n} y_i$.

Suppose $n < m$. Then $z_m - z_n = \sum_{i=n+1}^{m} y_i$. Since for $i \geqq n$, $y_i \in \prod p^n G_i = p^n \prod G_i$, we see that $\{z_n\}_{n<\omega}$ is a Cauchy sequence in the p-adic topology on $\prod_{i=1}^{\infty} G_i$. It converges to an element $y \in \prod_{i=1}^{\infty} G_i$ such that $y(i) = y_i$. Note that y is not a torsion element, but each z_n is in K_t. If $K_t/p^{\omega}K_t$ is complete, then $\{z_n + p^{\omega}K_t\}_{n<\omega}$ must converge to an element $x + p^{\omega}K_t$. Thus $\{z_n + p^{\omega} \prod G_i\}$ converges to $x + p^{\omega} \prod G_i$ in $\prod G_i/p^{\omega} \prod G_i$. But $\{z_n + p^{\omega} \prod G_i\}$ converges to $y + p^{\omega} \prod G_i$. So $x - y \in p^{\omega} \prod G_i$. Note that $py \in p^{\omega} \prod G_i$ and $h_{p^{\omega}\Pi G_i}(py) = 0$. Thus $px \in p^{\omega} \prod G_i$. Because

$$h_{p^{\omega}\Pi G_i}(py - px) > 0$$

we have $h_{p^{\omega}\Pi G_i}(px) = 0$ also. Let n_0 be such that $p^{n_0}(px) = 0$. Thus for $i > n_0$, $h_{p^{\omega}G_i}(px(i)) > 0$. Hence for $i > n_0$, $h_{p^{\omega}G_i}(py(i) - px(i)) = 0$, and thus $h_{p^{\omega}\Pi G_i}(py - px) = 0$, which contradicts the above. Hence $\{z_n + p^{\omega}K_t\}_{n<\omega}$ does not converge to an element of $K_t/p^{\omega}K_t$.

This example shows that the groups with K/K^1 and K^1 torsion complete are not exactly the $p^{\omega 2}$-torsion injectives, which the author believes was widely regarded as a possibility.

Beginning at the ordinal $\omega 2$ the properties of p^{α}-divisible injective and p^{α}-torsion injective are *not* equivalent. We show this using an analogous theorem of [19].

THEOREM 5. *The following are equivalent*:

(1) *For torsion groups G, if G is p^{α}-divisible injective then G is p^{α}-torsion injective.*

(2) $\alpha < \omega 2$.

Proof. Assume the first property, and suppose $\alpha \geqq \omega 2$. Then let C be any cotorsion group with $p^{\alpha}C = 0$. Then $C = \operatorname{Ext}(Q/Z, C)$ and hence $p^{\alpha} \operatorname{Ext}(Q/Z, C) = 0$. C is therefore a p^{α}-divisible injective. We have an exact sequence

$$\operatorname{Hom}(A, C/C_t) \to \operatorname{Ext}(A, C_t) \to \operatorname{Ext}(A, C).$$

For divisible torsion groups A, $\operatorname{Hom}(A, C/C_t) = 0$. Hence C_t is p^{α}-divisible injective and by hypothesis is p^{α}-torsion injective. By Theorem 1, C is p^{α}-injective. By Theorem 6.8 and Theorem 5.1 of [19], every cotorsion group C with $p^{\alpha}C = 0$ is p^{α}-injective iff $\alpha < \omega 2$.

Suppose $\alpha<\omega 2$ and A is torsion. Let D be a torsion divisible hull of A. By Theorem 5.1 of [19], $i: A \to D$ induces an *epimorphism*

$$p^\alpha \operatorname{Ext}(D, G) \to p^\alpha \operatorname{Ext}(A, G) \to 0.$$

Thus condition (1) holds.

REMARK. This theorem together with Theorem 3 shows that $\Psi(\mathscr{D}_\alpha)$ will not be every p^α-pure sequence of torsion groups when $\alpha \geqq \omega 2$. Hence we will investigate $\Psi(\mathscr{D}_\alpha)$ and characterize the sequences in it in the case that α is accessible.

4. We begin with some general properties of $\Psi(\mathscr{D})$ for any class $\mathscr{D}$. We call a short exact sequence $\Psi(\mathscr{D})$-pure if it is a member of $\Psi(\mathscr{D})$. Let $\mu: 0 \to A \to B \to C \to 0$ be exact.

P1. If μ is $\Psi(\mathscr{D})$-pure and $f: D \to C$ then μf is $\Psi(\mathscr{D})$-pure.

Proof. If $g: A \to E \in \mathscr{D}$ then $g(\mu f)=(g\mu)f=0f=0$.

P2. If E is $\Psi(\mathscr{D})$-pure in F and F is $\Psi(\mathscr{D})$-pure in G then E is $\Psi(\mathscr{D})$-pure in G.

P3. If E is $\Psi(\mathscr{D})$-pure in G, $E \subseteq F \subseteq G$ and F/E is $\Psi(\mathscr{D})$-pure in G/E then F is $\Psi(\mathscr{D})$-pure in G.

Proof. Let $f: F \to D \in \mathscr{D}$. Extend $f|_E$ to $g: G \to D$. Since $(g-f)|_E=0$, we can define $(g-f)_*: F/E \to D$ by $(g-f)_*(x+E)=g(x)-f(x)$. We extend this homomorphism to $h: G/E \to D$. Letting $\pi: G \to G/E$ be the natural map, let $\alpha=g-h\pi$. Then $\alpha: G \to D$ and $\alpha|_F=f$.

There are many other such properties, but the reader should note that the union of $\Psi(\mathscr{D})$-pure subgroups need not be $\Psi(\mathscr{D})$-pure.

For the sequel it is important to keep in mind that $\mathscr{G}_\alpha \subseteq \Psi\Phi(\mathscr{G}_\alpha)=\Psi(\mathscr{D}_\alpha)$.

THEOREM 6. *Let α be a limit ordinal. If $\mu \in \Psi(\mathscr{D}_\alpha)$, μ is p^α-pure.*

Proof. Suppose $\beta<\alpha$ and let $\mu: 0 \to A \to B \to C \to 0$ be such that $\mu \in \Psi(\mathscr{D}_\alpha)$. Then $A/p^\beta A$ is discrete in the α-topology and hence $A/p^\beta A$ is an element of $\mathscr{D}_\alpha$. Let $\pi: A \to A/p^\beta A$ be the natural map. Then $\pi\mu$ is exactly $0 \to A/p^\beta A \to B/p^\beta A \to C \to 0$ and must split. By Proposition 3 of [7], μ is then p^β-pure since $\pi\mu$ is p^β-pure. Hence $\mu \in \bigcap_{\beta<\alpha} p^\beta \operatorname{Ext}(C, A)=p^\alpha \operatorname{Ext}(C, A)$.

COROLLARY. *If α is a limit ordinal and $0 \to A \to B \to C \to 0 \in \Psi(\mathscr{D}_\alpha)$, $p^\beta B \cap A=p^\beta A$ for all $\beta<\alpha$.*

Proof. This follows directly from Theorem 6 and Proposition 3. A direct proof is an amusing exercise.

We now proceed to characterize $\Psi(\mathscr{D}_\alpha)$ in the case that α is accessible. Let α be a limit ordinal and let $\mu: 0 \to A \xrightarrow{i} B \to B/A \to 0$ be p^α-pure exact. We define $\bar{\mu}^\alpha$ as follows: let $i_\beta: A/p^\beta A \to \{A, p^\beta B\}/p^\beta B \to B/p^\beta B$ be the natural inclusions for each $\beta<\alpha$. We define $\iota: T_\alpha(A) \to T_\alpha(B)$ by $\iota((a_\beta)_{\beta<\alpha})=(i_\beta a_\beta)_{\beta<\alpha}$. Using Proposition 3 we see ι is a monomorphism and we let $\bar{\mu}^\alpha$ be the sequence

$$\bar{\mu}^\alpha: 0 \longrightarrow T_\alpha(A) \xrightarrow{\ \iota\ } T_\alpha(B) \longrightarrow T_\alpha(B)/\iota(T_\alpha(A)) \longrightarrow 0.$$

THEOREM 7. *Let α be an accessible ordinal. Then $\Psi(\mathscr{D}_\alpha)$ is the class of all p^α-pure sequences μ such that $\bar{\mu}^\alpha$ splits.*

Proof. Let $\mu: 0 \to A \xrightarrow{\iota} B \to C \to 0$ be p^α-pure such that $\bar{\mu}^\alpha$ splits, and suppose G is a reduced member of $\mathscr{D}_\alpha$. Let $f: A \to G$ be given. We define $f_*: T_\alpha(A) \to T_\alpha(G)$ by $f_*((a_j+p^jA)_{j<\alpha}) = (f(a_j)+p^jG)_{j<\alpha}$. Since if $i<j$, $a_j+p^iA = a_i+p^iA$, $f(a_j)-f(a_i) = f(a_j-a_i) \in f(p^iA) \subseteq p^iG$. Hence f_* maps into $T_\alpha(G)$ as claimed. We let $\Psi: T_\alpha(B) \to T_\alpha(A)$ be such that $\Psi\iota = 1_{T_\alpha(A)}$. Since $G \in \mathscr{D}_\alpha$ and α is accessible, the natural map $\delta_G: G \to T_\alpha(G)$ is an isomorphism. Then defining $g: B \to G$ by $g = \delta_G^{-1} f_* \Psi \delta_B$, we have the property desired since

$$g_\iota = \delta_G^{-1} f_* \Psi \delta_B i = \delta_G^{-1} f_* \Psi \iota \delta_A = \delta_G^{-1} f_* \delta_A = \delta_G^{-1} \delta_G f = f.$$

Hence $\mu \in \Psi(\mathscr{D}_\alpha)$.

Next suppose $\mu \in \Psi(\mathscr{D}_\alpha)$. Then μ is p^α-pure by Theorem 6. We have a commutative diagram

$$\begin{array}{ccccccccc}
\mu: 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_B} & & \downarrow & & \\
\bar{\mu}^\alpha: 0 & \longrightarrow & T_\alpha(A) & \xrightarrow{\iota} & T_\alpha(B) & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Since α is accessible, $T_\alpha(A) \in \mathscr{D}_\alpha$. Thus we have a map $\theta: B \to T_\alpha(A)$ such that $\theta i = \delta_A$. We now define a map $\phi: T_\alpha(B) \to T_\alpha(A)$ such that $\phi\iota = 1_{T_\alpha(A)}$.

If $x = (b_j+p^jB)_{j<\alpha} \in T_\alpha(B)$, let ϕ be defined by $\phi(x) = (\pi_j\theta(b_j))_{j<\alpha}$ where

$$\pi_j: \prod_{i<\alpha} A/p^iA \to A/p^jA$$

is the natural projection. Note that if $b_j - b_j' \in p^jB$ then $\theta(b_j-b_j') \in p^jT_\alpha(A)$. Therefore $\pi_j\theta(b_j-b_j') \in p^j(A/p^jA) = 0$. Hence ϕ is well defined.

Let f_{ij} be the natural map $f_{ij}: A/p^iA \to A/p^jA$ for each pair of ordinals $j<i<\alpha$. For each such pair,

$$\begin{aligned}
f_{ij}((\phi(x))_i) &= f_{ij}(\pi_i\theta(b_i)) = \pi_j\theta(b_i) \qquad (\text{range } \theta \subseteq T_\alpha(A)) \\
&= \pi_j\theta(b_j) = (\phi(x))_j.
\end{aligned}$$

Hence the range of ϕ is contained in $T_\alpha(A)$ as desired.

Now we see that $\bar{\mu}^\alpha$ splits since

$$\begin{aligned}
\phi\iota((a_j+p^jA)_{j<\alpha}) &= \phi((i(a_j)+p^jB)_{j<\alpha}) = (\pi_j\theta(i(a_j)))_{j<\alpha} \\
&= (\pi_j\delta_A(a_j))_{j<\alpha} = (\pi_j((a_j+p^iA)_{i<\alpha}))_{j<\alpha} = (a_j+p^jA)_{j<\alpha}.
\end{aligned}$$

Thus $\mu \in \Psi(\mathscr{D}_\alpha)$ implies $\bar{\mu}^\alpha$ splits.

5. **Applications.** We note that the class $\mathscr{D}_\alpha$ is the solution to two other "injective" questions. We again mean "p-group" when we say "group".

THEOREM 8. *Let α be a limit ordinal and G a group such that $p^\alpha G=0$. Then G is a summand of every group in which it is a p^α-high subgroup iff $G \in \mathscr{D}_\alpha$.*

Proof. Recall that G is N-high for a subgroup N if it is maximal with respect to $G \cap N=0$. G is p^α-high in X if it is $p^\alpha X$-high. (See [9], [10] for a discussion of the properties of N-high subgroups.)

Suppose $G \in \mathscr{D}_\alpha$ and G is p^α-high in X. Then X/G is divisible and G is pure in X. In fact, G is a $p^{\alpha+1}$-pure subgroup of X by Proposition 2 of [7]. By Theorem 3, we have that G is a summand of X.

Assume G has the p^α-high injective property. Let $0 \to G \to T_\alpha(G) \to D \oplus R \to 0$ be exact with D divisible and R reduced. Let $T \subseteq T_\alpha(G)$ be such that $0 \to G \to T \to D \to 0$ is exact. This sequence is p^α-pure and hence $D[p]=\{G+p^\beta T[p]\}/G$ for each $\beta<\alpha$ (Proposition 3). By Proposition 1.7 of [8], there exists a group X with $X/p^\alpha X=T$ and $G \subseteq X$ as a $p^\alpha X$-high subgroup. Then $X=G \oplus p^\alpha X$ and hence $T=G$. Thus $T_\alpha(G)/\delta(G)$ is reduced and $G \in \mathscr{D}_\alpha$.

REMARK. This result was independently arrived at by Charles Megibben in *On p^α-high injectives* (to appear).

Another kind of injective property is the following topologically motivated property: G dense in X implies G is a summand of X. The following theorem shows that $\mathscr{D}_\alpha$ solves this problem for the topology generated by taking $\{p^\beta G\}_{\beta<\alpha}$ as a fundamental neighborhood system of zero.

THEOREM 9. *Let α be a limit ordinal and G a group satisfying $p^\alpha G=0$. Let (*) be the property*

(*) *G is neat in X and $X[p]=\{G[p], (p^\beta X)[p]\}$ for each $\beta<\alpha$.*

Then $G \in \mathscr{D}_\alpha$ iff (*) implies G is a summand of X.

Proof. Note (*) implies $X/\{p^\alpha X, G\}$ is divisible. Suppose $G \in \mathscr{D}_\alpha$ and X is any group for which (*) holds. The following diagram commutes:

$$\begin{array}{ccccccccc}
\mu': 0 & \longrightarrow & \{G, p^\alpha X\}/p^\alpha X & \longrightarrow & X/p^\alpha X & \longrightarrow & X/\{G, p^\alpha X\} & \longrightarrow & 0 \\
 & & \wr\!\wr & & \uparrow & & \uparrow & & \\
\mu: 0 & \longrightarrow & G & \longrightarrow & X & \longrightarrow & X/G & \longrightarrow & 0
\end{array}$$

The divisibility of $X/\{p^\alpha X, G\}$ together with (*) implies μ' is p^α-pure by Proposition 3. Thus μ' splits because $G \in \mathscr{D}_\alpha$. Then μ splits as desired.

Conversely, suppose (*) implies G is a summand of X. We show that G is a divisible injective and hence a member of $\mathscr{D}_\alpha$ by Theorem 3. Let G be p^α-pure in X with X/G divisible. Then (*) holds (use Proposition 3) and hence G is a summand of X. Thus $G \in \Phi(\mathscr{D}_\alpha)=\mathscr{D}_\alpha$.

REMARK. Analogies of these theorems can be proven for the category of g.p. groups.

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