INTERPOLATION THEOREMS FOR THE PAIRS OF SPACES \((L^p, L^\infty)\) AND \((L^1, L^q)\)

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Abstract. A Banach space \(Z\) has the interpolation property with respect to the pair \((X, Y)\) if each \(T\), which is a bounded linear operator from \(X\) to \(X\) and from \(Y\) to \(Y\), can be extended to a bounded linear operator from \(Z\) to \(Z\). If \(X=L^p, Y=L^\infty\) we give a necessary and sufficient condition for a Banach function space \(Z\) on \((0, I)\), \(0<\lambda \leq +\infty\), to have this property. The condition is that \(g \prec \ast f\) and \(f \in Z\) should imply \(g \in Z\); here \(g \prec \ast f\) means that \(g \prec \ast f \prec \ast \infty\) in the Hardy-Littlewood-Pólya sense, while \(h^*\) denotes the decreasing rearrangement of the function \(|h|\).

If the norms \(\|T\|_X, \|T\|_Y\) are given, we can estimate \(\|T\|_Z\). However, there is a gap between the necessary and the sufficient conditions, consisting of an unknown factor not exceeding \(h_\lambda, h_\lambda \leq 2^{1/\lambda}, 1/p + 1/q = 1\).

Similar results hold if \(X=L^1, Y=L^q\). For all these theorems, the complete continuity of \(T\) on \(Z\) is assured if \(T\) has this property on \(X\) or on \(Y\), and if \(Z\) satisfies a certain additional necessary and sufficient condition, expressed in terms of \(\|\sigma_\alpha\|_Z, \alpha > 0\), where \(\sigma_\alpha\) is the compression operator \(\sigma_\alpha f(t) = f(at), 0 \leq t < 1\).

1. Introduction. Let \(X, Y\) and \(Z\) be Banach spaces, and let \(\mathcal{B}(X)\) denote the totality of bounded linear operators acting on \(X\), let \(\mathcal{B}(X, Y) = \mathcal{B}(X) \cap \mathcal{B}(Y)\). Also, let \(\mathcal{B}(X, Y; K_1, K_2)\) denote the set of all operators in \(\mathcal{B}(X, Y)\) satisfying \(\|T\|_X \leq K_1\) and \(\|T\|_Y \leq K_2\). The space \(Z\) is said to have the interpolation property for the pair \((X, Y)\), if for every \(T \in \mathcal{B}(X, Y)\), \(T\) (or its unique extension \(\hat{T}\) to \(Z\)) belongs to \(\mathcal{B}(Z)\).

The space \(Z\) has the interpolation property for the pair \((X, Y)\) in the strong sense, if \(T\) has the interpolation property for \((X, Y)\) and if \(\|\|T\|_Z\|\) (or \(\|\|\hat{T}\|_Z\|\)) is majorized by a positive constant depending only on \(\|T\|_X\) and \(\|T\|_Y\). In the sequel, \(I = (0, I)\) will be a (finite or infinite) interval of the real line, and \((X, \|\cdot\|_X)\) will be a Banach function space of locally Lebesgue integrable functions on \(I\) satisfying the following conditions:

(1.1) \(\|g\| \leq \|f\|, f \in X\) implies \(g \in X\) and \(\|g\|_X \leq \|f\|_X\);
(1.2) The norm \(\|\cdot\|_X\) is semicontinuous:

\[0 \leq f_n \uparrow f, \alpha = \sup_{n \geq 1} \|f_n\|_X < \infty \text{ imply } f = \bigcup_{n=1}^\infty f_n \in X \text{ and } \|f\|_X = \alpha.\]
For a positive measurable function $f$, $d_f(y) = m[t : f(t) > y]$, $y \geq 0$, is the distribution function of $f$. Two positive functions $f, g$ are equimeasurable, $f \sim g$, if they have the same distribution function. The space $X$ is called weakly rearrangement invariant (rearrangement invariant), if $0 \leq f \in X$ and $f \sim g$ imply $g \in X$ (resp. $\|g\|_X \leq \gamma \|f\|_X$, where $\gamma$ is a fixed constant independent upon $f$ and $g$). We write $L^p$ for $L^p(I)$, $1 \leq p \leq \infty$, and $\| \cdot \|_p$ for the $L^p$-norm on $I$. In his paper [2] A. P. Calderón showed that $X$ has the interpolation property for the pair $(L^1, L^\infty)$ if and only if $X$ is rearrangement invariant. In §3 and §4 we shall study the interpolation property for the pairs $(L^p, L^{\infty})$, $1 \leq p < \infty$, and $(L^1, L^q)$, $1 < q < \infty$, respectively. We characterize the Banach function spaces having the interpolation property for these pairs (Theorems 2 and 3), extending the results of [2], [11]. In §5 the complete continuity of operators acting on interpolated spaces will be dealt with. Results similar to those of [14] will be obtained, and a special case when $X$ is an Orlicz space will be discussed in the last section.

Let $X$ and $Y$ be Banach function spaces consisting of locally integrable functions. By $X + Y$ we denote the set of all functions $f$ of the form $f = f_1 + f_2$, where $f_1 \in X$ and $f_2 \in Y$. If $Z \subseteq X + Y$, then each operator $T \in \mathfrak{B}(X, Y)$ has a natural extension onto $Z$. For $f \in Z$, we write $f = f_1 + f_2$, and define $Tf = Tf_1 + Tf_2$. Since $T$ is linear, the value of $Tf$ does not depend on the choice of $f_1$ and $f_2$. An extension of $T$ in this sense will be again denoted by $T$. 2. Quasi-orders. For a measurable function $f$ on $I(0,1)$, $f^*$ will denote the decreasing rearrangement of $|f|$, that is, the inverse function of $d_f(y)$, whenever it is finite. By $S$ we denote the set of all positive simple functions, vanishing outside of a set of finite measure. It is easy to see that $f^*$ is defined if $f$ is locally integrable.

The main tool of this paper is different quasi-order relations between measurable functions $f, g$. One of them is the Hardy-Littlewood-Pólya relation $g < f$ for locally integrable $f, g$, which means that

$$\int_0^x g^*(t) \, dt \leq \int_0^x f^*(t) \, dt, \quad x \geq 0. \tag{2.1}$$

Although this relation is classical, some new properties of it were found in [10]. Here is a further property:

**THEOREM 1.** Let $g_1 + g_2 < f$, all these functions being locally integrable and positive. Then there exist positive $f_1, f_2$ for which $f = f_1 + f_2$, $g_i < f_i$, $i = 1, 2$.

**LEMMA 1.** Let $g < f$, where $g, f$ are positive and $g$ a decreasing function in $S$:

$$g = \sum_{v=1}^n \alpha_v x_{(c_v - 1, c_v)}, \quad 0 = c_0 < \cdots < c_n \leq l, \quad \alpha_1 \geq \cdots \geq \alpha_n \geq 0.$$  

Then there exist mutually disjoint sets $e_v, v = 1, \ldots, n$, with the following properties:

$$m e_v = c_v - c_{v-1}, \tag{2.2}$$

$$\alpha_v m e_v \leq \int f_{x_v} \, dt. \tag{2.3}$$

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Proof. First we assume that $f$ is decreasing. In this case, we shall also have the following:

(2.4) Each set $e_v$ is a finite union of intervals.

For $n=1$ the assertion holds trivially. Suppose that it holds for $n=k$. Let $n=k+1$. Putting

$$a = \sup \left\{ c : \int_0^{c+c_1} f \, dt \geq \alpha_1 c_1, c \leq c_n - c_1 \right\},$$

we have $\int_0^{c+c_1} f \, dt = \alpha_1 c_1$, unless $a = c_n - c_1$. Let $\tau_c h$ denote the translation operator, defined by

$$\tau_c h(t) = h(t+c) \quad \text{if } t+c \in I,$$

$$= 0 \quad \text{otherwise.}$$

We put

$$f_1 = (f \chi_{(0,a)} + f \chi_{(a+c_1,1)})^\ast = f \chi_{(0,a)} + \tau_{c_1} (f \chi_{(a+c_1,1)}),$$

$$g_1 = \tau_{c_1} \left( \sum_{v=2}^n \alpha_v \chi_{(c_{v-1},c_v)} \right) = \sum_{v=2}^n \alpha_v \chi_{(c_{v-1} - c_1, c_v - c_1)}.$$

We can exclude the possibility that $a = c_n - c_1$, for then $g_x(t) < f_x(t)$ for all $t$. Since

$$\int_0^x g_1 \, dt \leq \int_0^x f \, dt \leq \int_0^x f_1 \, dt \quad \text{if } 0 < x \leq a,$$

$$\int_0^x g_1 \, dt = \int_0^{c_1+x} g \, dt - \alpha_1 c_1 \leq \int_0^{c_1+x} f \, dt - \int_0^{c_1} f \, dt$$

$$= \int_0^x f_1 \, dt \quad \text{if } a < x \leq 1,$$

we see that $g_1 < f_1$. By the assumption, there exist mutually disjoint sets $e_v, 2 \leq v \leq k+1$, such that (2.2)–(2.4) hold for $f_1$ and $g_1$. Setting $e_1 = (a, a+c_1)$ and $e_v = (e_v \cap (0,a)) \cup \{ t : t - c_1 \in e_v \cap (a,1) \}, 2 \leq v \leq k+1$, we obtain mutually disjoint sets $e_v, 1 \leq v \leq k+1$, for which all the required conditions hold for $f$ and $g$.

If $f$ is positive but not decreasing, then, since $g < f^\ast$, we can find mutually disjoint measurable sets $e_v, 1 \leq v \leq n$, such that (2.2)–(2.4) hold for $g$ and $f^\ast$. As each $e_v$ is a finite sum of intervals, we can easily find mutually disjoint sets $e_v, 1 \leq v \leq n$, such that $m e_v = m e_v$ and $\int f \chi_{e_v} \, dt = \alpha_v m e_v$. Measurable sets $e_v, 1 \leq v \leq n$, thus obtained, satisfy the requirements of Lemma 1.

We can now prove Theorem 1 when $g_1$ and $g_2$, and consequently $g = g_1 + g_2$ belong to $S$. Let $e_v, v = 1, \ldots, n$, be sets of constancy of each of the three functions, with $g_1 = \alpha_{v_1}, g_2 = \alpha_{v_2}$ on $e_v$. By means of the decreasing rearrangement of $g$ and Lemma 1, we find disjoint sets $e_v$ with $m e_v = m e_v$, $\int f \chi_{e_v} \, dt \geq \alpha_{v_1} + \alpha_{v_2})m e_v$. Then it is possible to decompose each $e_v$ into disjoint $e_{v_1}, e_{v_2}$ such that $\int f \chi_{e_{v_1}} \, dt \geq \alpha_v m e_v, i = 1, 2$. We shall have $f \chi_{e_{v_i}} > g \chi_{e_{v_i}}, i = 1, 2, v = 1, \ldots, n$. Adding these relations, we obtain $g_1 < f = \sum_{v=1}^n f \chi_{e_v}$, $f_1 + f_2 \leq g$. It is now sufficient to replace $f_2$ by $f - f_1$ to obtain the result.
If \( g_1, g_2 \) are arbitrary positive functions, one finds increasing sequences \( g_{1n} \uparrow g_1, \ g_{2n} \uparrow g_2 \) from \( S \). For the corresponding \( f_{1n}, f_{2n} \) one can use weak \(*\)-compactness on each set \( A \) where \( f \) is bounded, and the absolute continuity of the integrals \( \int f \ dt \) to complete the proof.

**Remark.** It is not difficult to show that the functions \( f_1, f_2 \) of Theorem 1 can be always assumed to be orthogonal (that is, with disjoint supports). However, one cannot, in general, assume that they are decreasing, even if \( g_1, g_2 \) and \( f \) are decreasing step functions with just one step.

In [10] another quasi-order \( g \prec_f \) has been used. With respect to two Banach function spaces this relation means the following. One must have \( g, f \in X_1 + X_2 \), and for each decomposition \( f = f_1 + f_2, \ f_i \in X_i, \ i = 1, 2 \) of \( f \) there should exist a decomposition \( g = g_1 + g_2 \) of \( g, \ g_i \in X_i, \ i = 1, 2 \), with the property that \( \|g_i\|_{X_i} \leq \|f_i\|_{X_i}, \ i = 1, 2 \). We are interested here in the case \( X_1 = L^p, \ X_2 = L^\infty \). Then it is easy to see (compare also [10, p. 38]) that \( g \prec_f \) holds if and only if

\[
\|(g^* - y)_+\|_p \leq \|(f^* - y)_+\|_p, \ y \geq 0.
\]

The quasi-order used in this paper, \( g \prec^p f \), where \( p \geq 1 \), is defined, for two locally \( p \)-integrable functions, by the inequality

\[
\int_0^x g^{*p} \ dt \leq \int_0^x f^{*p} \ dt, \quad x \geq 0,
\]

that is, by \( g^{*p} \prec f^{*p} \). If one writes \( (2.6) \) as \( \|g^* \chi_{(0,x)}\|_p \leq \|f^* \chi_{(0,x)}\|_p \), there is an obvious similarity to \( (2.5) \).

From the definition we see that

\[
g \prec^p f \text{ is equivalent to } g^* \prec^p f^*,
\]

By a theorem of Hardy, Littlewood and Pólya, [3], \( g \prec^p f \) implies \( \Phi(|g|) < \Phi(|f|) \), where \( \Phi(u), \ u \geq 0 \), is convex and increasing. In particular,

\[
g \prec f \text{ implies } g \prec^p f.
\]

We also have

\[
g_i \prec^p f, \ i = 1, 2, \ a_1, a_2 \geq 0, \ a_1 + a_2 = 1 \text{ imply } a_1 g_1 + a_2 g_2 \prec^p f.
\]

In fact, for \( x \in I \) we have, because of the inequality \( (f_1 + f_2)^* \prec f_1^* + f_2^* \) and \( (2.8) \),

\[
\int_0^x (a_1 g_1 + a_2 g_2)^{*p} \ dt \leq \int_0^x (a_1 g_1^* + a_2 g_2^*)^p \ dt \leq a_1 \int_0^x g_1^{*p} \ dt + a_2 \int_0^x g_2^{*p} \ dt \leq \int_0^x f^{*p} \ dt.
\]

**Lemma 2.** (i) Relation \( g \prec^p f \) implies \( g \prec f \); (ii) for each \( p > 1 \), there is a smallest constant \( \lambda_p, \ 1 < \lambda_p \leq 2^{1/q} (1/p + 1/q = 1) \), for which \( g \prec f \) implies \( g \prec^p \lambda_p f \).
Proof. (i) For a given \( y \geq 0 \), we consider the function \( \Phi(u) = (u^{1/p} - y)^p \), which is increasing and convex. Thus, by the theorem of Hardy, Littlewood and Pólya mentioned above \( g <^p f \) implies \( \Phi(g^*) < \Phi(f^*) \); relation (2.5) follows from this.

(ii) Assume \( g < f \). If \( e_0 \subset I \) is a given set, with \( me_0 = a > 0 \), let \( \alpha = f^*(a) \), and let \( f_2 = f^{|e_0} \in L^\infty \) be the \( \alpha \)-truncation of \( f \), let \( f_1 = f - f^{|e_0} \in L^p \). There exist \( g_i, i = 1, 2, \) with \( g = g_1 + g_2, \| g_1 \|_p \leq \| f - f^{|e_0} \|_p, \| g_2 \|_\infty \leq \alpha \). Let \( e \subset I, me \leq a \). Then

\[
\| gxe \|_p \leq \| g_1 xe \|_p + \| g_2 xe \|_p \leq \| g_1 \|_p + \alpha d^{1/p} \leq \| f - f^{|e_0} \|_p + \| f^{|e_0} xe_0 \|_p \leq 2^{1/q} \| fxe_0 \|_p.
\]

We have used here the fact that if \( f_1, f_2 \geq 0, \) then \( \| f_1 \|_p + \| f_2 \|_p \leq 2^{1/q} \| f_1 + f_2 \|_p \). From (2.10) it follows that \( g <^p 2^{1/q} f \).

3. An interpolation theorem for the pair \( L^p, L^\infty \). In this section we assume that \( X \) is a Banach function space satisfying \( X \subset L^p + L^\infty \) for some \( p, 1 \leq p < +\infty \). We shall say that \( X \) is monotone with respect to the relation \( <^p \), or that \( X \) belongs to the class \( M^p \) if \( g <^p f \) and \( f \in X \) imply \( g \in X \). For \( A > 0 \), we shall say that \( X \in M^p(A) \) if \( g <^p f \) and \( f \in X \) imply \( g \in X \) and \( \| g \|_X \leq A \| f \|_X \).

Lemma 3. If \( X \in M^p \), then \( X \in M^p(A) \) for some \( A > 0 \); moreover, \( X \) is rearrangement invariant.

Proof. By (2.8) it is clear that \( X \) is weakly rearrangement invariant if \( X \in M^p \). Suppose that \( M^p(A) \) is violated for each \( A > 0 \). Then there exist positive functions \( f_n, g_n, n = 1, 2, \ldots \), such that \( g_n <^p f_n, \| g_n \|_X \geq n \) and \( \| f_n \|_X \leq 2^{-2n} \). Putting \( f = \sum_{n=1}^{\infty} 2^n f_n \), we have \( f \in X \) and \( 2^n g_n <^p f, n \geq 1 \). By (2.9) we get

\[
g = \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} 2^{-n} (2^n g_n) <^p f,
\]

hence \( g \in X \). This, however, contradicts the fact that \( \| g \|_X \geq \| g_n \|_X \geq n \) for all \( n \geq 1 \). Thus, the condition \( M^p(A) \) holds for some \( A > 0 \), and \( X \) is necessarily rearrangement invariant.

(For \( p = 1 \), Lemma 3 was given in [12], [16], but the present proof is simpler.)

A space \( X \subset L^p \) is normally imbedded in \( L^p \) if \( X \) is dense in \( L^p \) and if \( \| f \|_p \leq \| f \|_X \) for all \( f \in X \). Each of the Lorentz spaces \( \Lambda(C, p) \) [9] (where \( C \) is a class of decreasing positive functions \( c \) with \( \int c \, dt = 1 \)) is normally imbedded in \( L^p \) and satisfies \( M^p(1) \). Here is an example in the opposite direction:

Example 1. The space \( \Lambda_{a}, a = p^{-1}, p > 1 \), with the norm \( \| f \|_\Lambda = a \int_0^1 t^{n-1} f^*(t) \, dt \), is normally imbedded in \( L^p \). On the other hand, it does not satisfy \( M^p \). Indeed, let \( \phi(t) = t^{-a} \log^{-1}(1/t), 0 < t \leq e^{-1}; \quad = 0; \quad e^{-1} < t < 1 \). Then \( \phi \in L^p \), but \( \| \phi \|_\Lambda = +\infty \).

We put, for \( 0 < a < e^{-1} \),

\[
g_a(t) = \phi(a), \quad 0 \leq t \leq a, \quad f_a(t) = \phi(a), \quad 0 \leq t \leq b, \]

\[
= \phi(t), \quad a \leq t \leq 1, \quad = 0, \quad b \leq t \leq 1,
\]

selecting \( b \) in such a way that \( \| f_a \|_p = \| g_a \|_p \).
Then for each $\alpha$, $g_\alpha <^p f_\alpha$, but $\|g_\alpha\|_\Lambda \to \infty$, $\|f_\alpha\|_\Lambda = \|f_\alpha\|_p = \|g_\alpha\|_p \to \|f\|_p$ for $\alpha \to 0$. Lemma 3 shows that $\mathcal{M}_p$ is violated.

**Lemma 4.** Assume that $f_0, g_0, g_0 \in S$ are positive and that $g_0 <^p f_0$. Then there exists a positive operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property that $g_0 \leq Tf_0$.

**Proof.** Let $g_0$ be given by

$$g_0 = \sum_{i=1}^n a_i \chi_{e_i}, \quad \alpha_1 \geq \cdots \geq \alpha_n \geq 0, \quad e_i \cap e_\mu = \emptyset, \quad \nu \neq \mu.$$  

By Lemma 1 (applied to $g_0^{<p}$) there exist disjoint subsets $e_\nu, \nu = 1, \ldots, n$, of $I$ for which $m_{e_\nu} = m_{e_\nu}$ and

$$\int_0^\alpha f_0^{<p} \chi_{e_\nu} \, dt \geq \alpha^p m_{e_\nu}, \quad \nu = 1, \ldots, n.$$  

We define an operator $T$ on the set of all locally $p$-integrable functions by

$$Tf = \sum_{\nu=1}^n \left\langle f \chi_{e_\nu}, h_\nu \right\rangle \chi_{e_\nu}.$$  

Clearly $T$ is positive and linear and

$$\|Tf\|_p \leq \sum_{\nu=1}^n \|f \chi_{e_\nu}\|_p \leq \|f\|_p$$  

for all $f \in L^p$. On the other hand, for any $f \in L^\infty$,

$$\left\langle f \chi_{e_\nu}, h_\nu \right\rangle \leq \|f \chi_{e_\nu}\|_p \leq \|f\|_\infty \|X_{e_\nu}\|_p, \quad 1 \leq \nu \leq n.$$  

Consequently, $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$. Furthermore by (3.2),

$$Tf_0 = \sum_{\nu=1}^n \|f_0 \chi_{e_\nu}\|_p \chi_{e_\nu} \geq \sum_{\nu=1}^n \alpha_\nu \chi_{e_\nu} = g_0.$$  

Now we can prove

**Theorem 2.** Let $X$ be a Banach function space over $I$ with $X \subset L^p + L^\infty$. The necessary and sufficient condition for $X$ to have the interpolation property for the pair $(L^p, L^\infty)$ or, equivalently, this property in the strong sense for $(L^p, L^\infty)$ is that $X \in \mathcal{M}_p$.

**Proof.** First let $X \in \mathcal{M}_p$. By Lemma 3, $X \in \mathcal{M}_p(A)$ for some $A > 0$. Let $g, f \in X$ and $g <^p f$ (with respect to $L^p, L^\infty$). Then by Lemma 2, $g <^p \lambda_A f$, and so $\|g\|_X \leq \lambda_A \|f\|_X$.
Now if \( T \in \mathcal{B}(L^p, L^\infty; 1, 1) \), then for each \( f \in X \), \( Tf \prec f \). It follows that \( T \) maps \( X \) into itself and that \( \| T \|_X \leq A\lambda_p \). And if \( 0 \neq T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty) \), then \( \alpha T \in \mathcal{B}(L^p, L^\infty; 1, 1) \), where \( \alpha^{-1} = \text{Max} (K_p, K_\infty) \). This shows that \( X \) has the interpolation property in the strong sense.

Conversely, suppose that \( X \) has the interpolation property for the pair \((L^p, L^\infty)\), but fails to satisfy \( M^p \). Then there exist positive functions \( f \) and \( g \) such that \( f \in X \), \( \| f \|_X = 1 \), \( g \prec f \), but \( g \notin X \). Let \( 0 \leq g_n \in S \) and \( g_n \uparrow g \). As \( g_n \prec f \), there exist, by Lemma 4, positive operators \( T_n \in \mathcal{B}(L^p, L^\infty; 1, 1) \) such that \( g_n \leq T_n f \) for each \( n \geq 1 \). This implies that \( g_n \in X \) for each \( n \geq 1 \). Since \( \cdot \|_X \) satisfies (1.2), \( \| g_n \|_X \uparrow \infty \) holds. We may therefore assume without loss of generality that \( \| g_n \|_X > n \cdot 2^n \), \( n \geq 1 \). It follows that \( \| T_n \|_X \geq \| T_n f \|_X > n \cdot 2^n \), \( n \geq 1 \). Putting \( T = \sum_{n=1}^{\infty} 2^{-n} T_n \), we obtain a positive operator belonging to \( \mathcal{B}(L^p, L^\infty; 1, 1) \). On the other hand, \( \| Tf \|_X \geq \| 2^{-n} T_n f \|_X > n \) holds for each \( n \), since \( T_n \) is a positive operator. This contradicts the fact that \( Tf \in X \), and shows that the condition is necessary.

From the proof above, we have immediately

**Corollary 1.** If \( X \) satisfies the condition \( M^p(A) \) and \( T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty) \), then

\[
Tf \prec^p \lambda_p \text{Max} (K_p, K_\infty) f \quad \text{for each } f \in L^p;
\]

\[
\| T \|_X \leq A\lambda_p \text{Max} (K_p, K_\infty).
\]

In the last inequalities, \( \lambda_p \leq 2^{1/p} \leq 2 \). We shall show that \( \lambda_p \) cannot be here replaced by 1.

**Example 2.** For each \( p \), \( 1 < p < +\infty \), there exists an operator \( T \in \mathcal{B}(L^p, L^\infty; 1, 1) \), for which \( T \prec_1 f \) is not true for some \( f \).

Let \( \alpha > 1 \) be chosen so that \( c = (a^p - 1 + 1)/(a^p + 1) < 1 \) (actually, this is true for any \( \alpha > 1 \)). We define

\[
f_0(x) = \alpha \quad \text{on } (0, \frac{1}{2}), \quad g_0(x) = \beta \quad \text{on } (0, c),
\]

\[
= 1 \quad \text{on } (\frac{1}{2}, 1), \quad = 0 \quad \text{on } (c, 1),
\]

where \( \beta = (a^p + 1)/(a^p - 1 + 1) \). An easy calculation shows that

\[
\| g_0 \|_p = \| f_0 \|_p,
\]

\[
\| f_0 \|_1 \| g_0 \|_\infty = \| f_0 \|_p.
\]

We define the positive operator

\[
Tf = \frac{1}{\| f_0 \|_p^p} \langle f, f_0^p \rangle g_0.
\]

Since \( \| h \|_q = \| h \|_p \) for \( h \geq 0 \), it follows from (3.5) that, if \( f \in L^p \),

\[
\| Tf \|_p \leq \frac{1}{\| f_0 \|_p^p} \| f \|_p \| f_0 \|_1 \| g_0 \|_p = \| f \|_p,
\]

and from (3.6) that, if \( f \in L^\infty \),

\[
\| Tf \|_\infty \leq \frac{1}{\| f_0 \|_p^p} \| f \|_\infty \| f_0 \|_1 \| g_0 \|_\infty = \| f \|_\infty.
\]
Also, $Tf_o = g_0$. However, $g_0 <^p f_0$ is incorrect, since

$\int_0^c g_0^p \, dt = \|f_0\|_p^p > \int_0^c f_0^p \, dt$. 

**Example 3.** There exists a space $X \in \mathcal{M}^p(1)$, and an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, for which $\|T\|_X > 1$.

In the notations of the last example, we take $X = L^p$ with the norm $\|f\|_X = \|f \cdot \chi(0, 1)\|_p$. It is immediately clear that $g <^p f$ implies $\|g\|_X \leq \|f\|_X$, so that $X \in \mathcal{M}^p(1)$. For the operator $(3.7)$ we have $g_0 = Tf_0$, but $\|g_0\|_X > \|f_0\|_X > 0$ by $(3.8)$.

If $g = Tf$ and $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, then we have $g <^p f$. We shall show that the converse is not true, in general. This will also show that one cannot replace the relation $<^p$ by $\ll$ in Lemma 4.

**Example 4.** Let $p > 1$ be an integer, and let $f_0$ and $g_0$ be the functions of the Example 2. We put $f_1 = f_0 + 1$ and $g_1 = g_0 + 1$, where $1$ denotes the characteristic function of $(0, 1)$. Let

$G(t) = \|f_0 + t1\|_p^p - \|g_0 + t1\|_p^p, \quad t \geq 0.$

Using (3.5), (3.6) and elementary calculations (for instance, with induction in $k$) we can show that

$G(0) = G'(0) = 0, \quad G^{(k)}(0) \geq 0, \quad 2 \leq k \leq p.$

It follows that $G(t) \geq 0$ for all $0 \leq t \leq 1$, hence we have $g_1 <^p f_1$ on account of (2.5).

Now suppose that there exists an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ such that $Tf_1 = g_1$. Since $\|T1\|_\infty \leq 1$, we have $0 \leq g_1 - 1 \leq Tf_1 - T1$, hence

$\|g_0\|_p = \|g_1 - 1\|_p \leq \|Tf_1 - T1\|_p \leq \|f_1 - 1\|_p = \|f_0\|_p.$

From (3.5) it follows that $T1 = 1$ and $Tf_0 = g_0$. Since $\chi(1/2, 1) = (\alpha 1 - f_0)/(\alpha - 1)$, we have

$T\chi(1/2, 1) = (\alpha - 1)^{-1}(\alpha 1 - g_0).$

The last function has values $\alpha/(\alpha - 1) > 1$ on the interval $(c, 1)$, hence $\|T\chi(1/2, 1)\|_\infty > 1$, a contradiction. Consequently, there does not exist an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property $Tf_1 = g_1$.

**4. Interpolation theorems for the pair $L^1, L^\infty$.** In this section we assume that $X$ is a Banach function space for which $X \subset L^1 + L^\infty$, $1 < q < +\infty$, and that $p$ is the conjugate exponent, $1/p + 1/q = 1$. We define a quasi-order relation $<^q$. We write $f_1 <^q f_2$ if for every $g_1 \in L^p$, a $g_2 \in L^p$ such that both $g_2 <^p g_1$ and $\langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle$. For example, $0 \leq f_1 \leq f_2$ implies $f_1 <^q f_2$, for here we can take $g_2 = |g_1|$. We begin with some properties of the relation $<^q$. For given $f, g \geq 0$, there exists a $g \geq 0$ with the properties $g \sim \tilde{g}$ and $\langle f^*, g \rangle = \langle f, \tilde{g} \rangle$, [8, p. 61]. From this, using (2.7), it is not difficult to derive

$\langle f_1, g \rangle <^q \langle f_2, g \rangle$ if and only if $f_1 <^q f_2^*.$
If $f_1 < f_2$, then for each $g_1$ we have $\langle f_1^*, g_1 \rangle \leq \langle f_2^*, g_1 \rangle \leq \langle f_2^*, g_1 \rangle$. Hence, by (4.1),

\begin{equation}
(4.2) \quad f_1 < f_2 \text{ implies } f_1 <_q f_2.
\end{equation}

Similar to (2.9) is the property

\begin{equation}
(4.3) \quad f_i <_q f_i, \quad i = 1, 2, \text{ and } a_1, a_2 \geq 0, \quad a_1 + a_2 = 1 \quad \text{imply } a_1 f_1 + a_2 f_2 <_q f.
\end{equation}

In fact, for each $g \in L^p$, we can find $g_1$ and $g_2$ such that $g <^g g$ and $\langle f_i, g \rangle \leq \langle f, g \rangle$, $i = 1, 2$. Hence,

\[ \langle a_1 f_1 + a_2 f_2, g \rangle \leq \langle f, a_1 g_1 \rangle + \langle f, a_2 g_2 \rangle = \langle f, a_1 g_1 + a_2 g_2 \rangle, \]

where $a_1 g_1 + a_2 g_2 <^g g$ by (2.9). Since $g$ is arbitrary, we get (4.3).

For a Banach function space $X$, $X'$ will denote the conjugate space of $X$, that is, the space of all measurable functions $g$ such that

\[ \|g\|_{X'} = \sup \{ |\langle f, g \rangle|; f \in X, \|f\|_X \leq 1 \} < \infty. \]

For any operator $T$ acting on $X$, $T'$ will denote the conjugate operator of $T$ acting on the conjugate space $X'$. Note that $T \in \mathcal{B}(L^1, L^1; K_1, K_2)$ implies $T' \in \mathcal{B}(L^p, L^{q'}; K_q, K_1)$.

**Lemma 5.** If $T \in \mathcal{B}(L^1, L^1; 1, 1)$, then

\begin{equation}
(4.4) \quad Tf <_q \lambda f \quad \text{for each } f \in L^1.
\end{equation}

**Proof.** We have $T' \in \mathcal{B}(L^{p'}, L^{q'}; 1, 1)$, hence $T'g <^p \lambda g$ holds for every $g \in L^{p'}$, by (3.3). If $f \in L^p$ and $g_1 \in L^p$ are given, we select $g_2 = (1/\lambda f)T'g_1$. Then $g_2 <^p g_1$ and $\langle Tf, g_1 \rangle = \langle f, T'g_1 \rangle = \langle \lambda f, g_2 \rangle$, and we have proven (4.4).

We shall use the following monotony conditions for a Banach function space $X$:

- $X \in \mathcal{M}_q$, if $g <_q f, f \in X$ imply $g \in X$;
- $X \in \mathcal{M}_q(A)$, if $g <_q f, f \in X$ imply $g \in X$ and $\|g\|_X \leq A\|f\|_X$.

With the same proof as for Lemma 3 we have

**Lemma 6.** If $X \in \mathcal{M}_q$, then $X \in \mathcal{M}_q(A)$ for some $A > 0$; moreover, $X$ is rearrangement invariant.

**Lemma 7.** If the space $X$ does not satisfy the condition $\mathcal{M}_q(A)$, then there exists a positive operator $T \in \mathcal{B}(L^1, L^q; 1, 1)$ and a function $0 \leq f \in X$ for which $\|Tf\|_X > A\|f\|_X$.

**Proof.** We shall first show that under the assumptions of Lemma 7, the conjugate space $X'$ of $X$ does not satisfy $\mathcal{M}_p(A)$. There exist functions $f_1, f_2 \in X$ such that $f_1 <_q f_2$ and $\|f_1\|_X > A\|f_2\|_X$. For any $\epsilon > 0$ satisfying $(1 - \epsilon)\|f_1\|_X > A\|f_2\|_X$, we can find, by virtue of the reflexivity of the semicontinuous norm $\|\cdot\|_X$, a function $g_1 \in X' \cap L^p$ such that $\|g_1\|_X = 1$ and $(1 - \epsilon)\|f_1\|_X \leq \langle f_1, g_1 \rangle$. Since $f_1 <_q f_2$, there exists a function $g_2 \in L^p$ for which $g_2 <^p g_1$ and $\langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle$. This implies

\[ A\|f_2\|_X < (1 - \epsilon)\|f_1\|_X \leq \|f_2\|_X \|g_2\|_X. \]
Thus, we have obtained two functions \( g_1, g_2 \in X' \), for which \( g_2 < p g_1 \), but \( \|g_2\|_X > A\|g_1\|_X \), contradicting the condition \( M^p(\mathcal{A}) \).

For \( g_1 \) and \( g_2 \), obtained above, we may assume \( g_1, g_2 \geq 0 \). Since \( \cdot \) is also semicontinuous, we can select an \( h \in S \cap X' \) such that \( 0 \leq h \leq g_2 \) and \( \|h\|_X > A\|g_1\|_X \). By Lemma 4 there exists a positive operator \( T \in \mathscr{B}(L^p, L^{\infty}; 1, 1) \) for which \( TG_1 \geq h \). Choose an \( \varepsilon > 0 \) such that \((1 - \varepsilon)\|h\|_X \geq A\|g_1\|_X \). There exists a function \( 0 \leq f \in X \), \( \|f\|_X = 1 \) with the property \( \langle f, h \rangle \geq (1 - \varepsilon)\|h\|_X \). It follows that \((1 - \varepsilon)\|h\|_X \leq \langle f, h \rangle \leq \langle f, Tg_1 \rangle \leq \|Tf\|_X \|g_1\|_X \). Consequently, we get \( \|Tf\|_X > A \), for the positive operator \( T' \in \mathscr{B}(L^1, L^\infty; 1, 1) \).

Now we can state our interpolation theorem for the pair \((L^1, L^\infty)\).

**Theorem 3.** Let \( X \) be a Banach function space over \( I \) with \( X \subseteq L^1 + L^\infty \). The necessary and sufficient condition for \( X \) to have the interpolation property for the pair \((L^1, L^\infty)\), or, equivalently, this property for \((L^1, L^\infty)\) in the strong sense, is that \( X \in \mathscr{M}_q \).

**Proof.** First let \( X \in \mathscr{M}_q \). By Lemma 6, \( X \in \mathscr{M}_q(\mathcal{A}) \) for some \( A > 0 \). Let \( 0 \neq T \in \mathscr{B}(L^1, L^q; K_1, K_q) \), we put \( \alpha^{-1} = \text{Max}(K_1, K_q) \). Then \( \alpha T \in \mathscr{B}(L^1, L^q; 1, 1) \) and so \( \alpha T \langle f, \rangle < \lambda_p \phi \) holds for all \( f \in L^q \) by Lemma 5. Thus, \( f \in L^q \cap X \) implies \( Tf \in X \) and \( \|Tf\|_X \leq \lambda_p A \alpha^{-1} \|f\|_X \). We extend this relation to all \( f \in X \). Since \( f \in L^1 + L^q \), all truncations \( f^{(n)} \) belong to \( L^q \), and all differences \( f - f^{(n)} \) belong to \( L^1 \) for \( n = 1, 2, \ldots \). Since \( |f - f^{(n)}| \to 0 \) a.e. and \( \|T(f - f^{(n)})\|_1 \leq K_1 \|f - f^{(n)}\|_1 \), we have \( \|Tf - T^{(n)}\|_1 \to 0 \). Taking, if necessary, a subsequence, we can assume that the sequence \( T^{(n)} \), \( n = 1, 2, \ldots \), converges a.e. to \( Tf \). By (1.2) and the semicontinuity of \( \cdot \) \( X \) we have

\[
\|Tf\|_X \leq \liminf_{n \to \infty} \|T^{(n)}\|_X \\
\leq \liminf_{n \to \infty} \lambda_p A \alpha^{-1} \|f^{(n)}\|_X \leq \lambda_p A \alpha^{-1} \|f\|_X.
\]

This shows that \( T \in \mathscr{B}(X) \), and that \( \|T\|_X \leq \lambda_p A \alpha^{-1} \).

The necessity of the condition \( \mathscr{M}_q(\mathcal{A}) \) follows exactly as in the proof of Theorem 2.

**Corollary 2.** If \( X \in \mathscr{M}_q(\mathcal{A}) \) and \( T \in \mathscr{B}(L^1, L^q; K_1, K_q) \), then \( T \in \mathscr{B}(X) \) and

\[
\|T\|_X \leq \lambda_p A \text{ Max } (K_1, K_q).
\]

5. **Complete continuity of operators in interpolation theorems.** In this section we give necessary and sufficient conditions for the space \( X \) in order that every operator \( T \) in \( \mathscr{B}(L^p, L^\infty) \) (or in \( \mathscr{B}(L^1, L^\infty) \)) should be completely continuous on \( X \) if \( T \) is completely continuous on one of the spaces of the pair. We assume that \( X \subseteq L^p + L^\infty \) (or \( X \subseteq L^1 + L^\infty \)) for the pair \((L^p, L^\infty)\) (respectively, \((L^1, L^\infty)\)). The basic idea of the arguments below is due to the paper \([4]\), and the setting and the proofs follow the lines of \([11]\), \([14]\). The conditions are given in terms of the norms of compression operators. We denote by \( \sigma_a, a > 0 \), the compression operator:

\[
\sigma_a f(t) = f(at) \quad \text{if } 0 < at < l,
\]
\[
= 0 \quad \text{otherwise.}
\]
For any rearrangement invariant space \((X, \| \cdot \|_X)\) with \(\gamma = 1\), we have \(\sigma_a \in \mathcal{B}(X)\) and (see [13])
\[
\|\sigma_a\|_X \leq 1 \quad \text{if } a \geq 1; \quad 1 \leq \|\sigma_a\|_X \leq a^{-1} \quad \text{if } 0 < a \leq 1.
\]

It is clear that \(\sigma_{ab} = \sigma_a \sigma_b\) if \(b \geq 1\), or if \(0 < a, b < 1\). It follows from this and (5.2) that
\[
\|\sigma_a\|_X \leq (c/a)\|\sigma_c\|_X \quad \text{if } 0 < a \leq c, c > 1.
\]

The norms \(\|\sigma_a\|_X\), which play an important role in the theory of function spaces, have been discussed in [1], [13], [14]. We improve the inequality (3.4) of Corollary 1.

**Lemma 8.** If \(X\) satisfies the condition \(\mathcal{M}^p(A), 1 \leq p < \infty\), then, for every \(0 \neq T \in \mathcal{B}(L^p, L^\infty; K_\rho, K_{\infty})\),
\[
\|T\|_X \leq A\lambda_p K_{\infty} \|\sigma_a\|_X,
\]
where \(a = K_{\rho} \cdot K_{\rho}^{-p}\).

**Proof.** In the assumptions of the lemma, both operators \(T' = K_{\infty}^{-1} T \sigma_{a^{-1}}\) and \(T'' = K_{\infty}^{-1} T \sigma_{a^{-1}}\) belong to \(\mathcal{B}(L^p, L^\infty; 1, 1)\), as can be easily seen. If \(a \geq 1\), \(\sigma_{a^{-1}} T a^{-1} = I\) on \(X\), hence \(T = K_{\infty} T' \sigma_{a^{-1}}\); if \(0 < a < 1\), \(\sigma_{a^{-1}} = I\) on \(X\), hence \(T = K_{\infty} T''\). In both cases, (5.4) follows from Corollary 1.

An operator \(A\) on the set of locally integrable functions is called an averaging operator if \(A\) is defined by
\[
Af = \sum_{v=1}^{n} (m_{e_v})^{-1} \langle f, \chi_{e_v} \rangle \chi_{e_v}
\]
where \(m_{e_v} < \infty\), \(e_v \subset e_\mu = \emptyset\), if \(v \neq \mu\) and \(n \geq 1\). For convenience, we sometimes denote the operator (5.5) by \(A_g\), where \(g = \sum_{v=1}^{n} \sigma_{e_v} \chi_{e_v}\), is any function in \(S\) corresponding to the sets \(e_1, \ldots, e_n\). It is clear that \(A_g g = g\) for all \(g \in S\). An averaging operator \(A\) belongs to \(\mathcal{B}(L^1, L^\infty)\). If \(X\) is rearrangement invariant, then, because of the relation \(Af < f\), \(A\) and \(I - A\) belong to \(\mathcal{B}(X)\). Moreover, \(A\) is completely continuous. For each \(p, 1 \leq p < \infty\), there exists a sequence of averaging operators \(A_n, n = 1, 2, \ldots\), which converges in \(L^p\) strongly to the identity operator \(I_n\) [5, p. 21].

We have

**Theorem 4.** Let \(X\) satisfy \(\mathcal{M}^p(A), 1 \leq p < \infty\). In order that every operator \(T \in \mathcal{B}(L^p, L^\infty)\), which is completely continuous on \(L^p\), should be also completely continuous on \(X\), it is necessary and sufficient that
\[
\lim_{a \to \infty} \|\sigma_a\|_X = 0.
\]

**Proof.** Assume that (5.6) is satisfied. The image \(TV\) of the unit ball \(V\) in \(L^p\) has compact closure in \(L^p\). We select a sequence \(A_n, n = 1, 2, \ldots\), of averaging operators converging strongly to \(I\) in \(L^p\). Then
\[
\lim_{n \to \infty} \left\{ \sup_{f \in V} \| (I - A_n)Tf \|_p \right\} = 0,
\]
hence \( \lim_{a \to 0} \| (I - A_n) T \|_{p} = 0 \). Since \( \| (I - A_n) T \|_{\infty} \leq \| T \|_{\infty} \), putting \( a_n = \left( \| (I - A_n) T \|_{\infty} \right)^p \) and \( c_n = \left( \| T \|_{\infty} \right)^p \), we have \( a_n \leq c_n \) and \( c_n \to \infty \). Using (5.4) and (5.5) we obtain

\[
\| (I - A_n) T \|_X \leq A \lambda_p \| (I - A_n) T \|_{\infty} \sigma_n \|_X
\]

\[
\leq A \lambda_p \| T \|_{\infty} \| \sigma_n \|_X \to 0.
\]

Since \( T \) is the uniform limit of the operators \( A_n T \), \( n = 1, 2, \ldots \), which are completely continuous on \( X \), \( T \) also has the property.

Conversely, assume that (5.6) is not valid for \( X \). It has been shown in [14] that there exists an operator \( T_0 \in \mathcal{B}(L^1, L^\infty) \) which is completely continuous on \( L^1 \), but fails to be so on \( X \). Such an operator \( T_0 \) is also completely continuous on \( L^p \), \( 1 \leq p < \infty \) [4], [14]. Thus the necessity is proved.

If \( I \) is a finite interval, then for each operator \( T \) which is completely continuous on \( L^\infty \), there exists a sequence of averaging operators \( A_n, n = 1, 2, \ldots \), such that \( \| (I - A_n) T \|_{\infty} \to 0 \) [5, p. 22]. This fact can be used in the proof of the following theorem.

**Theorem 5.** Let \( I \) be a finite interval, and let \( X \) satisfy \( \mathcal{M}^p(A), 1 \leq p < \infty \). In order that every \( T \in \mathcal{B}(L^p, L^\infty) \) which is completely continuous on \( L^\infty \) should also be completely continuous on \( X \), it is necessary and sufficient that

\[
(5.7) \lim_{a \to 0} a^{1/p} \| \sigma_a \|_X = 0.
\]

**Proof.** The sufficiency is derived from (5.4) in a similar manner as in the proof of Theorem 4. Without loss of generality, we prove the necessity for \( l = 1 \). First we note that the condition \( \mathcal{M}^p(A) \) implies

\[
(5.8) a^{1/p} \| \sigma_a \|_X \leq A \quad \text{for all } a, 0 < a \leq 1.
\]

This follows from the relation \( a^{1/p} \sigma_a f \leq p f, 0 < a \leq 1, f \in L^p \), which can be easily verified.

Suppose that (5.7) is not true. Then there exists a \( \delta > 0 \) such that for arbitrarily small \( a > 0 \), \( a^{1/p} \| \sigma_a \|_X > \delta \). For each \( a \) of this kind there exists a function \( g \), which we may assume positive, such that

\[
(5.9) g \in \mathcal{S}; \quad \| g \|_X \leq 1; \quad a^{1/p} \| \sigma_a g \|_X > \delta.
\]

We can replace \( g \) by \( \chi_{(0,a)} g \), since this will not change \( \sigma_a g \). Then, for \( n \to \infty \) we will have \( \chi_{(1/n,a)} \uparrow g, \sigma_a (\chi_{(1/n,a)} g) \uparrow \sigma_a g \). From (1.2) it follows that we can assume that the functions \( g \) in (5.9) have support \( (c, a) \), \( 0 < c < a \). In addition to (5.9) we have

\[
(5.10) a^{1/2p} \| \sigma_{\sqrt{a}} g \|_X > A^{-1} \delta,
\]

since by (5.8) and (5.9)

\[
\delta < a^{1/p} \| \sigma_{\sqrt{a}} \|_X \| \sigma_{\sqrt{a}} g \|_X.
\]
We can select a sequence of functions $g_n$, with supports $(c_n, a_n)$, $n = 1, 2, \ldots$, which satisfy (5.9) and (5.10) and for which, in addition, all intervals $(c_n, a_n)$, $n = 1, 2, \ldots$, are disjoint, all intervals $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$ are disjoint, and $\sum a_n^{1/2} < +\infty$.

We define the operators

\[ T = \sum_{n=1}^{\infty} T_n; \quad T_n = a_n^{1/2a} \sigma_{(a_n)^{1/2}} A g_n, \quad n = 1, 2, \ldots, \]

where $A g_n$ are the averaging operators corresponding to the functions $g_n$. Then $\|T_n\|_\infty \leq a_n^{1/2p};$ the $T_n$ are completely continuous on $L^\infty$. It follows that also $T$ is completely continuous in $L^\infty$.

For any $f$, $T_n f = T_n (f_{X(c_n, a_n)})$. Also, $T_n f$ has support $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$. Thus, all $T_n f$ are disjoint. It is easy to see that $\|\sigma_a\|_p \leq a^{-1/p}, 0 < a < 1$. From this it follows that $\|T_n\|_p \leq 1$. Therefore

\[ \|Tf\|_p = \sum_{n=1}^{\infty} \|T_n f\|_p \leq \sum_{n=1}^{\infty} \|f_{X(c_n, a_n)}\|_p \leq \|f\|_p, \quad f \in L^p, \]

and we see that $T \in \mathcal{B}(L^p)$.

It remains to show that $T$ is not completely continuous on $X$. For the sequence of functions $g_n$, bounded in norm in $X$, we have $T g_n = T_n g_n$, and by (5.10), $\|T_n g_n\|_x \geq A^{-1} \delta > 0$, also $T_n g_n(t) \to 0$ everywhere. If $T g_n$ would have a convergent subsequence in $X$, it could converge only to 0, and this is impossible.

We turn now to the pair $(L^1, L^q)$, $1 < q < \infty$. Applying similar arguments (or considering the conjugate spaces) we obtain

**Lemma 9.** If $X \in \mathcal{M}_q(A)$, $1 < q < \infty$, then, for every $0 \neq T \in \mathcal{B}(L^1, L^q; K_1, K_q)$, we have

\[ \|T\|_X \leq \lambda_p A (K_q^q K_1^{-1})^{1/(q-1)} \|\sigma_a\|_x, \]

where $a = (K_q^q K_1^{-1})^{q/(q-1)}$.

We also have

**Theorem 6.** Let $X \in \mathcal{M}_q(A)$, $1 < q < \infty$. In order that every operator $T \in \mathcal{B}(L^1, L^q)$ which is completely continuous on $L^q$ (or $L^1$) should be also completely continuous on $X$, it is necessary and sufficient that the following condition (5.13) (resp. (5.14)) hold:

\[ \lim_{a \to 0} a \|\sigma_a\|_x = 0; \]

\[ \lim_{a \to \infty} a^{1/a} \|\sigma_a\|_x = 0. \]

**6. Orlicz spaces.** In view of Examples 2 and 3 it appears to be worthwhile to give examples of classes of function spaces $X \in \mathcal{M}_p(1)$, $1 \leq p < \infty$, for which $\|T\|_x \leq 1$ holds for every $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$. 

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We consider $N$-functions (compare [6]) $M$ having the expression

\[(6.1) \quad M(u) = \int_0^u (u-t)^p \, d\phi(t), \quad u > 0,\]

where $1 \leq p < \infty$ and $\phi$ is a positive nondecreasing left continuous function with $\phi(0) = 0$. For example, for $r$ with $p \leq r < \infty$, the $N$-function $M(u) = u^r$, $u > 0$, has an expression (6.1). For an $N$-function $M$, let $L_M = L_M(I)$ denote the Orlicz space defined by $M$ with the norm $\| \cdot \|_M$, where,

$$
\|f\|_M = \inf \{ \xi : \rho_M(\xi^{-1}f) \leq 1, \xi > 0 \}
$$

and

$$
\rho_M(f) = \int_I M(|f(t)|) \, dt, \quad f \in L_M.
$$

Then we have

\[\text{THEOREM 7.} \quad \text{Let } M \text{ have the expression (6.1). The Orlicz space } L_M \text{ has the interpolation property for the pair } (L^p, L^\infty) \text{ in the strong sense. In addition, for every } T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty), \]

\[\|T\|_M \leq \max (K_p, K_\infty). \]

\[\text{Proof.} \quad \text{We may assume that } K_p = K_\infty = 1. \text{ Let } f \in L_M \text{ and } T \in \mathcal{B}(L^p, L^\infty; 1, 1). \text{ We have}
\]

$$
\rho_M(Tf) = \int_I M(|Tf(t)|) \, dt = \int_I \left\{ \int_0^{\|Tf(t)\|} (|Tf(t)| - s)^p \, d\phi(s) \right\} \, dt
$$

by (6.1). By Fubini's theorem this implies

$$
\rho_M(Tf) = \int_0^\infty d\phi(s) \int_{E_s} (|Tf(t)| - s)^p \, dt,
$$

where $E_s$, $s > 0$, is the set $\{ t : |Tf(t)| > s, t \in I \}$. In view of the equality $(|Tf| - s)\chi_{Es} = |Tf| - |Tf|^{(s)} = |Tf - T(f)^{(s)}|$, the last term is equal to

$$
\int_0^\infty d\phi(s) \int_I |Tf(t) - (Tf(t))^{(s)}|^p \, dt = \int_0^\infty \|Tf - (Tf)^{(s)}\|_p^p \, d\phi(s).
$$

Since $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, we get

$$
\|Tf - (Tf)^{(s)}\|_p^p \leq \|Tf - T(f)^{(s)}\|_p^p \leq \|f - f^{(s)}\|_p^p,
$$

which, in turn, implies $\rho_M(Tf) \leq \rho_M(f)$. Consequently, on account of the fact that $\|f\|_M \leq 1$ if and only if $\rho_M(f) \leq 1$, we have $\|Tf\|_M \leq \|f\|_M$. As $f$ is arbitrary, we obtain $\|T\|_M \leq 1$.

In the proof of the previous theorem, we see that this theorem is also valid for Lipschitz operators acting on both $L^p$ and $L^\infty$ if the norms of the operators are now interpreted as their bounds. Thus, $\|T\|_X$ is now the smallest number $\gamma$ satisfying $\|Tf - Tg\|_X \leq \gamma \|f - g\|_X$ for all $f, g \in X$. Since every $N$-function $M$ has the expression (6.1) for $p = 1$, Theorem 6 is a generalization of a theorem by W. Orlicz [13].
References


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