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REMOVABLE SETS FOR POINTWISE SUBHARMONIC FUNCTIONS

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Abstract. Pointwise subharmonic is defined in terms of the pointwise L^1 total derivative of order 2. The class $\mathscr{A}(x^*, r_*)$ is introduced for the ball $B(x^*, r_*)$, and the following theorem is established: *Let Q be a Borel set of Lebesgue measure zero contained in $B(x^*, r_*)$. Then a necessary and sufficient condition that Q be removable for pointwise subharmonic functions with respect to the class $\mathscr{A}(x^*, r_*)$ is that Q be countable.* It is also shown that the class $\mathscr{A}(x^*, r_*)$ is in a certain sense best possible for the sufficiency of the above theorem.

1. **Introduction.** We shall operate in N-dimensional Euclidean space, E_N, $N \geqq 2$, and use the notation $x=(x_1, \ldots, x_N)$, $|x|^2 = x_1^2 + \cdots + x_N^2$, and $B(x, r)$ = the open N-ball with center x and radius r. Also, we shall adopt the convention that if $P(x)$ is a polynomial of degree less than zero, then $P(x)$ is identically zero. We say the function $u(x)$ is in $T_\alpha^1(x^0)$, $\alpha \geqq -N$, if $u(x)$ is in L^1 in a neighborhood of the point x^0, and if there is a polynomial $P(x)$ of degree strictly less than α such that

$$r^{-N} \int_{B(0,r)} |u(x^0+x) - P(x)|\, dx = O(r^\alpha) \quad \text{as } r \to 0.$$

We say $u(x)$ is in $t_\alpha^1(x^0)$ if $u(x)$ is in $T_\alpha^1(x^0)$ and if there is a polynomial $P(x)$ of degree $\leqq \alpha$ such that

$$r^{-N} \int_{B(0,r)} |u(x^0+x) - P(x)|\, dx = o(r^\alpha) \quad \text{as } r \to 0.$$

If $u(x)$ is in $t_2^1(x^0)$, i.e., u has an L^1 total differential at x^0 of order 2, and

$$P(x) = \beta_0 + \sum_{j=1}^N \beta_j x_j + 2^{-1} \sum_{j=1}^N \sum_{k=1}^N \beta_{jk} x_j x_k$$

with $\beta_{jk} = \beta_{kj}$ for $j, k = 1, \ldots, N$, we shall set $D_j u(x^0) = \beta_j$ and $D_{jk} u(x^0) = \beta_{jk}$. It is clear that $D_j u(x^0)$, $D_{jk} u(x^0)$ represent respectively generalizations of the usual $\partial u(x^0)/\partial x_j$ and $\partial^2 u(x^0)/\partial x_j\, \partial x_k$. (The classes $T_2^1(x^0)$ and $t_2^1(x^0)$ were originally defined in [1].)

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The function u will be said to be pointwise subharmonic at x^0, if u is in $t_2^1(x^0)$ and $\sum_{j=1}^N D_{jj}u(x^0) \geqq 0$.

Let G be a domain in E_N. Then following Radó [3, 1.1], we shall say u is subharmonic in G if u meets conditions (a), (b), (c), and (d) where

(a) $-\infty \leqq u(x) < +\infty$ for x in G;

(b) u is not identically $-\infty$ in G;

(c) u is upper semicontinuous in G; and

(d) if G' is a domain such that $\bar{G}'$ (=the closure of G') is contained in G and if H is continuous in $\bar{G}'$, harmonic in G', and $\geqq u$ on $\partial G'$, then $H \geqq u$ in G'.

Let $B(x^*, r_*)$, $r_* > 0$, be a given open ball in E_N. We shall say u is in the class $\mathscr{A}(x^*, r_*)$ if u meets conditions (1.1) and (1.2) where

(1.1) u is in L^1 on compact subsets of $B(x^*, r_*)$ and

(1.2) for every x in $B(x^*, r_*)$,

$$u \text{ is in } t^1_{2-N}(x) \quad \text{for } N \geqq 3$$

and

$$\lim_{r\to 0} [r^2 \log(1/r)]^{-1} \int_{B(x,r)} |u(y)|\, dy = 0 \quad \text{for } N = 2.$$

We shall say u is in the class $\mathscr{B}(x^*, r_*)$ if u meets condition (1.1) and (1.3) where

(1.3) for every x in $B(x^*, r_*)$,

$$u \text{ is in } T^1_{-(N-2)}(x) \quad \text{for } N \geqq 3$$

and

$$\limsup_{r\to 0} [r^2 \log(1/r)]^{-1} \int_{B(x,r)} |u(y)|\, dy < \infty \quad \text{for } N = 2.$$

$\mathscr{B}(x^*, r_*)$ is the natural widening of the class $\mathscr{A}(x^*, r_*)$; so if one proves a theorem which is true for $\mathscr{A}(x^*, r_*)$ and false for $\mathscr{B}(x^*, r_*)$ the result is then in a certain sense best possible.

Next, let Q be a Borel set of Lebesgue measure zero contained in $B(x^*, r_*)$. We introduce two conditions which will be used in defining the concept of a removable set for pointwise subharmonic functions in $B(x^*, r_*)$; namely

(i) u is in $T_2^1(x)$ for x in $B(x^*, r_*) - Q$; and

(ii) u is pointwise subharmonic almost everywhere in $B(x^*, r_*)$.

It is clear that if u is in $T_2^1(x)$ then

$$\lim_{r\to 0} |B(x, r)|^{-1} \int_{B(x,r)} u(y)\, dy \tag{1.4}$$

exists and is finite where $|B(x, r)|$ designates the N-volume of $B(x, r)$. In the sequel it will always be assumed that if u is $T_2^1(x)$ then u is defined at x by the limit in (1.4). In particular, if u meets condition (i) above, then u is well defined in $B(x^*, r_*) - Q$.

We shall say that the Borel set Q of Lebesgue measure zero is a removable set for pointwise subharmonic functions with respect to the class $\mathscr{A}(x^*, r_*)$ provided the following holds:

If u meets conditions (i) and (ii) stated above and u is also in class $\mathscr{A}(x^*, r_*)$, then u can be defined at the points of Q so that u is subharmonic in $B(x^*, r_*)$.

We intend to establish the following result:

THEOREM. *Let Q be a Borel set of Lebesgue measure zero contained in $B(x^*, r_*)$, $r_* > 0$. Then a necessary and sufficient condition that Q be removable for pointwise subharmonic functions with respect to the class $\mathscr{A}(x^*, r_*)$ is that Q be countable.*

Before proceeding with the proof of the above theorem we first observe that the sufficiency is in a certain sense best possible, i.e., it is false for the class $\mathscr{B}(x^*, r_*)$. To see this fact set $u(x) = |x-x^*|^{2-N}$ for $N \geqq 3$ and $u(x) = \log |x-x^*|^{-1}$ for $N=2$. Then u is in L^1 on every compact subset of E_N, and an easy computation shows that u is indeed in class $\mathscr{B}(x^*, r_*)$ for any $r_* > 0$, i.e., (1.3) holds for x^* (it obviously holds for $x \neq x^*$). Also, it is clear that u meets conditions (i) and (ii) above with $Q = \{x^*\}$. If Q were a removable set for pointwise subharmonic functions with respect to the class $\mathscr{B}(x^*, r_*)$, then in particular according to conditions (a) and (c) above, u could be assigned a value at x^* different from $+\infty$ and still be upper semicontinuous at x^*. This is manifestly impossible. Our observation is therefore established.

Also, before proceeding with the proof of the above theorem, we would like to point out that this paper is motivated by and is (in certain respects) a sequel to our paper [5]. However, unlike our paper [5], we shall not make any explicit use of the theory of multiple trigonometric series.

2. **Proof of the necessary condition.** Let $Q \subset B(x^*, r_*)$ be an uncountable Borel set of Lebesgue measure zero. As is well known there is a perfect subset $Q_1 \subset Q$ [2, p. 205], and there exists a nonnegative Borel measure μ of total mass one having its support contained in Q_1, possessing the additional property

$$\mu[B(x, r)] \to 0 \quad \text{as } r \to 0 \text{ for } x \text{ in } Q_1. \tag{2.1}$$

For x in $B(x^*, r_*) - Q_1$, we define

$$u(x) = \int_{Q_1} |x-y|^{-N+2}\, d\mu(y) \quad \text{for } N \geqq 3$$

and

$$u(x) = \int_{Q_1} \log |x-y|^{-1}\, d\mu(y) \quad \text{for } N = 2$$

and observe that u has the following properties:

(2.2) u is in $L^1[B(x^*, r_*)]$;

(2.3) u is harmonic $B(x^*, r_*) - Q_1$;

(2.4) $-u$ can be defined at the points of Q_1 so that $-u$ is subharmonic in $B(x^*, r_*)$ [3, 4.26].

Next, we show that u meets condition (1.2) so that u is indeed in class $\mathscr{A}(x^*, r_*)$. We do this for $N=2$; a similar proof will prevail for $N \geqq 3$, and we leave the details to the reader.

Let x^0 be in $B(x^*, r_*)$ and let $\varepsilon > 0$ be given. If we can show

$$\limsup_{r\to 0}\, [r^2 \log(1/r)]^{-1} \int_{B(0,r)} |u(x^0+x)|\, dx < \pi\varepsilon \tag{2.5}$$

then condition (1.2) will follow for u.

Since the support of μ is contained in Q_1, it follows from (2.1) that $\mu[B(x^0, r)] \to 0$ as $r \to 0$. Consequently there exists a δ with $0 < \delta < \frac{1}{2}$ such that $\mu[B(x^0, \delta)] < \varepsilon$.

Setting $Q_\delta = Q_1 \cap [E_2 - B(x^0, \delta)]$, we observe that

$$\int_{B(0,r)} dx \left| \int_{Q_\delta} \log|x^0+x-y|^{-1}\, d\mu(y) \right| = o[r^2 \log(1/r)] \quad \text{as } r \to 0.$$

Consequently, it follows from the definition of u and this last fact that (2.5) will follow if we show that

$$\limsup_{r\to 0}\, [r^2 \log(1/r)]^{-1} I_1(r) = 0 \tag{2.6}$$

and

$$\limsup_{r\to 0}\, [r^2 \log(1/r)]^{-1} I_2(r) \leqq \pi\varepsilon, \tag{2.7}$$

where, for $0 < 2r < \delta$, we define

$$g(y, r) = \int_{B(0,r)} \log|x^0+x-y|^{-1}\, dx,$$

$$I_1(r) = \int_{B(x^0,2r)} g(y, r)\, d\mu(y),$$

and

$$I_2(r) = \int_{B(x^0,\delta)-B(x^0,2r)} g(y, r)\, d\mu(y).$$

Now, for $|x^0-y| < 2r$,

$$\begin{aligned} g(y, r) &\leqq \int_{B(0,3r)} \log|x|^{-1}\, dx \\ &\leqq O[r^2 \log(1/r)] \quad \text{as } r \to 0. \end{aligned}$$

Since $\mu[B(x^0, 2r)] \to 0$ as $r \to 0$, we conclude from this last fact that (2.6) does indeed hold.

On the other hand, for $|x^0-y| \geqq 2r$, $g(y, r) = \pi r^2 \log|x^0-y|^{-1}$, and we consequently obtain that

$$I_2(r) \leqq \pi r^2 \log(2r)^{-1} \mu[B(x^0, \delta)].$$

Since by assumption, $\mu[B(x^0, \delta)] < \varepsilon$, we see that (2.7) is established and therefore that u is in class $\mathscr{A}(x^*, r_*)$.

Suppose that Q is a removable set for pointwise subharmonic functions with respect to the class $\mathscr{A}(x^*, r_*)$. Then since $Q_1 \subset Q$, it follows from (2.3) and the fact that u is in $\mathscr{A}(x^*, r_*)$ that u can be defined at the points of Q_1 so that u is subharmonic in $B(x^*, r_*)$. But then it follows [3, p. 8] that

$$\lim_{r\to 0} |B(0, r)|^{-1} \int_{B(0,r)} u(x+y)\, dy = u(x)$$

for every x in $B(x^*, r_*)$. We consequently conclude from (2.4) that u can be defined at the points of Q_1 so that u is harmonic in $B(x^*, r_*)$. We shall show that this fact leads to a contradiction.

Let $S(x, r)$ designate the $(N-1)$-sphere which is the boundary of $B(x, r)$, and let $dS(x)$ designate its natural $(N-1)$-dimensional volume element.

Since Q_1 is a compact subset of $B(x^*, r_*)$, there exists r_1 with $0 < r_1 < r_*$ such that $Q_1 \subset B(x^*, r_1)$. Therefore it follows from the divergence theorem and the harmonicity of u that

$$\int_{S(x^*,r_*)} (\text{grad } u, n)\, dS(x) = 0 \tag{2.8}$$

where $n(x)$ represents the outward pointing unit normal at x.

On the other hand, it follows from the definition of u, Fubini's theorem and an easy computation that $\int_{S(x^*,r_*)} (\text{grad } u, n)\, dS(x)$ equals a nonzero multiple of $\mu(Q_1)$. But then it follows from (2.8) that $\mu(Q_1) = 0$. This is a contradiction to the fact that μ was chosen so that $\mu(Q_1) = 1$. Consequently, Q is not a removable set for pointwise subharmonic functions with respect to the class $\mathscr{A}(x^*, r_*)$, and the necessary condition of the theorem is established.

3. **Fundamental lemmas.** We first prove the following lemma:

LEMMA 1. *Let $v(x)$ be subharmonic in the punctured ball $B(x^0, r_0) - \{x^0\}$. Suppose furthermore that $v(x)$ is in class $\mathscr{A}(x^0, r_0)$. Then $v(x)$ can be defined at x^0 so that it is subharmonic in $B(x^0, r_0)$.*

With no loss in generality, we suppose $x^0 = 0$. Next, we set $v^+(x) = \max [v(x), 0]$ and observe that $v^+(x)$ is subharmonic in $B(0, r_0) - \{0\}$. Also, we obtain from [3, 1.12] that $v^+(x)$ is in $L^1[S(0, r)]$ with respect to the natural $(N-1)$-measure on $S(0, r)$ for $0 < r < r_0$. With $|S(0, r)|$ designating the $(N-1)$-volume of the $(N-1)$-sphere $S(0, r)$, we furthermore obtain from [3, 1.13] that

$$\mathscr{L}(v^+, r) = |S(0, 1)|^{-1} \int_{S(0,1)} v^+(rx)\, dS(x) \tag{3.1}$$

is a convex function of $\Phi(r)$ for $0 < r < r_0$ where

$$\begin{aligned} \Phi(r) &= \log r && \text{for } N = 2, \text{ and} \\ &= r^{-N+2}/(-N+2) && \text{for } N \geqq 3. \end{aligned} \tag{3.2}$$

From elementary properties of convex functions, it follows from this fact that the following limit exists:

$$\lim_{r\to 0} \mathscr{L}(v^+, r)/[-\Phi(r)]. \tag{3.3}$$

If this last limit is zero, it follows from [3, 7.17] that v can be defined at 0 so that it is subharmonic in $B(0, r_0)$.

Let us suppose that the limit in (3.3) is not equal to zero. Then there is an $\varepsilon>0$ and an r_1 with $0<r_1<\min(1, r_0)$ such that

$$\mathscr{L}(v^+, r) \geqq \varepsilon[-\Phi(r)] \quad \text{for } 0 < r < r_1.$$

But then it follows from this last fact that, for $0<r<r_1$,

$$\begin{aligned} \int_{B(0,r)} |v(x)|\, dx &\geqq \int_0^r \rho^{N-1}\mathscr{L}(v^+, \rho)|S(0, 1)|\, d\rho \\ &\geqq -\varepsilon|S(0, 1)| \int_0^r \rho^{N-1}\Phi(\rho)\, d\rho. \end{aligned} \tag{3.4}$$

We conclude from (3.2) and (3.4) that respectively for $N\geqq 3$ and $N=2$,

$$\begin{aligned} \liminf_{r\to 0} r^{-2} \int_{B(0,r)} |v(x)|\, dx &\geqq \varepsilon|S(0, 1)|/2(N-2) \quad \text{and} \\ \liminf_{r\to 0} [r^2 \log(1/r)]^{-1} \int_{B(0,r)} |v(x)|\, dx &\geqq \varepsilon|S(0, 1)|/2. \end{aligned} \tag{3.5}$$

On the other hand, by assumption, v is in $\mathscr{A}(0, r_0)$; so in particular, for $N\geqq 3$, v is in $t^1_{2-N}(0)$ and for $N=2$,

$$\lim_{r\to 0} [r^2 \log(1/r)]^{-2} \int_{B(0,r)} |v(x)|\, dx = 0.$$

In either case, therefore, the fact that v is in $\mathscr{A}(0, r_0)$ constitutes a contradiction to (3.5). We conclude, consequently, that the limit in (3.3) must be zero. Therefore, v can be defined at 0 so that it is subharmonic in $B(0, r_0)$, and the proof to the lemma is complete.

(Before proceeding, with reference to the last paragraph in the Introduction, we would like to point out that one dimensional trigonometric series and the theory of spherical harmonics appear implicitly in the proof of the above lemma.)

If v is in $L^1(G)$ where G is a domain and if $B(x, h_0)\subset G$, we shall set for $0<h<h_0$

$$v_h(x) = |B(0, h)|^{-1} \int_{B(0,h)} v(x+y)\, dy \tag{3.6}$$

and define respectively the upper and lower generalized Laplacian of v at x, $\Delta^*v(x)$ and $\Delta_*v(x)$, to be

$$\Delta^*v(x) = 2(N+2) \limsup_{h\to 0} [v_h(x)-v(x)]/h^2, \tag{3.7}$$

and

$$\Delta_*v(x) = 2(N+2) \liminf_{h\to 0} [v_h(x)-v(x)]/h^2. \tag{3.8}$$

We next prove the following lemma:

LEMMA 2. *Let v be a function defined in $B(x^0, r_0)$, $r_0>0$, with the following properties: v is upper semicontinuous in $B(x^0, r_0)$; v is in $L^1[B(x^0, r_0)]$; $-\infty \leqq v(x) < +\infty$ for x in $B(x^0, r_0)$. Also, let g be a function defined in $B(x^0, r_0)$ with the following properties: there exists a finite negative constant M such that $M \leqq g(x) \leqq 0$ for x in $B(x^0, r_0)$; $g(x)=0$ almost everywhere in $B(x^0, r_0)$. Suppose $g(x) \leqq \Delta^* v(x)$ for x in $B(x^0, r_0)$. Then v is subharmonic in $B(x^0, r_0)$.*

The proof of the above lemma is similar to the proof of [5, Lemma 5], but different enough, nevertheless, to warrant that it be given in complete detail here. (Incidentally, in the proof of [5, Lemma 5], $0<r_0<1$ should read $0<r_0<\frac{1}{2}$ and in [5, (3.36)] for x in $B(0, r_0)$ should read for almost every x in $B(0, r_0)$.)

With no loss in generality, we suppose $x^0=0$ and $0<r_0<\frac{1}{2}$. Next, we invoke the Vitali-Caratheodory theorem [4, p. 75] to obtain a sequence $\{g_k\}_{k=1}^{\infty}$ with the following properties:

(3.9) $g_k(x) \leqq g_{k+1}(x) \leqq g(x)$ for x in $B(0, r_0)$ and $k=1, 2, \ldots$;

(3.10) g_k is upper semicontinuous in $B(0, r_0)$;

(3.11) g_k is in $L^1[B(0, r_0)]$, $k=1, 2, \ldots$;

(3.12) $\lim_{k\to\infty} g_k(x)=0$ for almost every x in $B(0, r_0)$.

Since the supremum of two upper semicontinuous functions is upper semicontinuous, we can also suppose that

(3.13) $M \leqq g_k(x) \leqq g(x)$ for $k=1, 2, \ldots$.

We set, for x in $B(0, r_0)$,

$$
\begin{aligned}
v_k(x) &= [(N-2)|S(0,1)|]^{-1} \int_{B(0,r_0)} g_k(y)|x-y|^{2-N}\, dy \quad \text{for } N \geqq 3, \\
&= (2\pi)^{-1} \int_{B(0,r_0)} g_k(y) \log |x-y|^{-1}\, dy \qquad \text{for } N = 2,
\end{aligned} \tag{3.14}
$$

and observe using (3.9) through (3.13) that v_k has the following properties:

(3.15) there is a finite negative constant M' such that $M' \leqq v_k(x) \leqq 0$ for x in $B(0, r_0)$ and $k=1, 2, \ldots$;

(3.16) v_k is upper semicontinuous in $B(0, r_0)$ for $k=1, 2, \ldots$;

(3.17) $\lim_{k\to\infty} v_k(x)=0$ for every x in $B(0, r_0)$;

(3.18) $\Delta_* v_k(x) \geqq -g_k(x)$ for x in $B(0, r_0)$.

From the hypothesis of the lemma, (3.15), and (3.16) we observe that, for each positive integer k, $-\infty \leqq v(x)+v_k(x) < +\infty$ for x in $B(0, r_0)$ and $v+v_k$ is upper semicontinuous in $B(0, r_0)$. Also from the hypothesis of the lemma, (3.9), and (3.18), we have for each positive integer k that

$$
\Delta^*[v+v_k](x) \geqq \Delta^* v(x) + \Delta_* v_k(x) \geqq 0 \quad \text{for } x \text{ in } B(0, r_0). \tag{3.19}
$$

We conclude from [3, 3.7] that

(3.20) $v+v_k$ is subharmonic in $B(0, r_0)$ for $k=1, 2, \ldots$.

Let x' be a point in $B(0, r_0)$ with the property that $v(x') > -\infty$. Let h' designate the distance of x' to $S(0, r_0)$. Then from (3.15), (3.20) and [3, 1.7] we have that

(3.21) $v(x') + v_k(x') \leqq v_h(x') + [v_k]_h(x')$ for $0 < h < h'$ and $k = 1, 2, \ldots$.

On the other hand it follows from (3.15) and (3.17) that $\lim_{k\to\infty} [v_k]_h(x') = 0$ for $0 < h < h'$. We conclude from this fact, (3.17) and (3.21) that

(3.22) $v(x') \leqq v_h(x')$ for $0 < h < h'$.

Consequently, it follows from (3.22) and the hypothesis of the lemma that v is in the class K_3 with respect to $B(0, r_0)$, where the class K_3 is defined in [3, 2.2]. But then we conclude from [3, 2.3] that v is indeed subharmonic in $B(0, r_0)$. The proof to the lemma is therefore complete.

4. **Proof of the sufficiency condition of the theorem.** With no loss in generality we assume that $x^* = 0$ and $0 < r_* < \frac{1}{2}$. Also with no loss in generality, we assume that Q is countably infinite and write

$$Q = \bigcup_{j=1}^{\infty} \{q_j\} \tag{4.1}$$

where $q_j \neq q_k$ for $j \neq k$, $j, k = 1, 2, \ldots$.

By assumption u is in $T_2^1(x)$ for x in $B(0, r_*) - Q$. Consequently, $\lim_{h\to 0} u_h(x)$ exists and is finite in $B(0, r_*) - Q$, and as mentioned in the introduction we assume that $u(x)$ is defined for x in $B(0, r_*) - Q$ by this limit. For x in Q we define $u(x)$ as follows:

$$u(x) = \limsup_{h\to 0} u_h(x). \tag{4.2}$$

The function u is now defined at every point in $B(0, r_*)$. We intend to show that u so defined is actually subharmonic in $B(0, r_*)$.

In order to do this, we observe that $\int_{B(0,h)} y_j \, dy = 0$ for $j = 1, \ldots, N$, and obtain from the fact that u is in $T_2^1(x)$ for x in $B(0, r_*) - Q$ that

$$u_h(x) - u(x) = O(h^2) \quad \text{as } h \to 0 \text{ for } x \text{ in } B(0, r_*) - Q. \tag{4.3}$$

Also, observing that $\int_{B(0,h)} y_j y_k \, dy = 0$ for $j \neq k$, $j, k = 1, \ldots, N$, and that

$$\int_{B(0,h)} y_j^2 \, dy = |B(0, h)| h^2/(N+2) \quad \text{for } j = 1, \ldots, N,$$

we obtain from (3.7), (3.8), and the fact that u is pointwise subharmonic almost everywhere in $B(0, r_*)$ that

$$0 \leqq \Delta_* u(x) = \Delta^* u(x) < +\infty \quad \text{almost everywhere in } B(0, r_*). \tag{4.4}$$

Next, we define the set Z as follows:

$$Z = \{x : \text{for every } r \text{ such that } 0 < r < r_* - |x|, \; u \text{ is not subharmonic in } B(x, r)\}. \tag{4.5}$$

In other words, y is in $B(0, r_*)-Z$ if there is some open neighborhood containing y in which u is actually subharmonic. If Z is empty the theorem is proved. We shall assume, therefore, that Z is nonempty and arrive at a contradiction. Before proceeding, we observe that

$$Z \text{ is closed relative to } B(0, r_*). \tag{4.6}$$

We now proceed with the proof and let k_0 designate the first integer strictly greater than $3/r_*$. Then for $k \geqq k_0$, we define

$$A_k = \{x : |u_h(x)-u_{h'}(x)| \leqq kh^2 \text{ for } 0 < h' \leqq h < k^{-1} \text{ and } |x| \leqq r_*-2k^{-1}\} \tag{4.7}$$

and observe that for $k \geqq k_0$, A_k is a closed set. Also, we observe from (4.3) that

(4.8) if x is in $B(0, r_*)-Q$, then there exists an integer k such that x is in A_k.

We furthermore observe that

$$A_k \subset A_{k+1} \quad \text{for } k \geqq k_0. \tag{4.9}$$

Next for $k \geqq k_0$, we define

$$F_k = \{x : |x| \leqq r_*-3/k, \text{ and, for all } h \text{ such that } 0 < h < k^{-1}, |u_h(x+y)-u_h(x)| \leqq kh \text{ for } |y| \leqq h\} \tag{4.10}$$

and observe that for $k \geqq k_0$, F_k is a closed set.

Let x^0 be in $B(0, r_*)$ and suppose that u is in $T_2^1(x^0)$. Then u is in $T_1^1(x^0)$ and therefore, for h sufficiently small, and $|y| \leqq h$

$$\begin{aligned} |u_h(x^0+y)-u_h(x^0)| &\leqq |B(0,h)|^{-1}\left|\int_{B(0,h)} [u(x^0+y+p)-u(x^0+p)]\,dp\right| \\ &\leqq |B(0,h)|^{-1}\int_{B(0,h)} |u(x^0+y+p)-u(x^0)|\,dp+O(h) \\ &\leqq |B(0,h)|^{-1}\int_{B(0,2h)} |u(x^0+p)-u(x^0)|\,dp+O(h) \\ &\leqq O(h) \quad \text{as } h \to 0. \end{aligned}$$

We consequently conclude that

(4.11) if x is in $B(0, r_*)-Q$, then there exists an integer k such that x is in F_k.

Also it follows from (4.10) that

$$F_k \subset F_{k+1} \quad \text{for } k \geqq k_0. \tag{4.12}$$

It consequently follows from (4.1), (4.8), (4.9), (4.10), and (4.11) that

$$Z \subset \bigcup_{k=k_0}^{\infty} (A_k \cap F_k) \cup \bigcup_{j=1}^{\infty} \{q_j\}. \tag{4.13}$$

From (4.6), Z is closed relative to $B(0, r_*)$. It follows, therefore, from Baire's theorem [4, p. 54] and (4.13) that if Z is nonempty, at least one of the sets $A_k \cap F_k$ or $\{q_j\}$, $j, k=1, 2, \ldots$, must be dense in a portion of Z (= a nonempty intersection of Z with an open ball). Recalling that $A_k \cap F_k$ is a closed set, we see then that if Z is nonempty we are faced with one (or both) of two cases:

Case I. There exists $B(q_{j_0}, r_0)$, $r_0>0$, with $\bar{B}(q_{j_0}, r_0)\subset B(0, r_*)$, such that

(4.14) u is subharmonic in $B(q_{j_0}, r_0)-\{q_{j_0}\}$, and

(4.15) q_{j_0} is in Z.

Case II. There exists $B(x^0, r_0)$, $r_0>0$, with $\bar{B}(x^0, r_0)\subset B(0, r_*)$ and there exists $A_{k_1} \cap F_{k_1}$ such that

(4.16) $Z \cap B(x^0, r_0)$ is nonempty, and

(4.17) $Z \cap \bar{B}(x^0, r_0)\subset A_{k_1} \cap F_{k_1}$.

If we show that each case leads to a contradiction (and therefore neither Case I nor Case II holds), the proof to the theorem will be complete, for then Z must be empty.

To show that Case I leads to a contradiction, we observe, from (4.14), the fact that u is in $\mathscr{A}(q_{j_0}, r_0)$, and, from Lemma 1, that u can be assigned a value at q_{j_0} such that u is subharmonic in $B(q_{j_0}, r_0)$. From well-known properties of subharmonic functions [3, p. 8], the value that we assign to u at q_{j_0} is the following limit which we now know exists: $\lim_{h\to 0} u_h(q_{j_0})$. But by (4.2), this is the original way we defined u at q_{j_0}. We conclude that q_{j_0} is not in Z, which is a contradiction to (4.15).

To show that Case II leads to a contradiction, we first observe from (4.7) and (4.17) that

$$\text{(4.18)} \qquad \lim_{h\to 0} u_h(x) \text{ exists and is finite for } x \text{ in } Z \cap \bar{B}(x^0, r_0).$$

Consequently, from (4.2) and the definition of u in $B(0, r_*)-Q$, we conclude that

$$\text{(4.19)} \qquad u(x) \text{ is finite valued for } x \text{ in } Z \cap \bar{B}(x^0, r_0).$$

Observing that we now have that $\lim_{h\to 0} u_h(x)=u(x)$ for x in $Z \cap \bar{B}(x^0, r_0)$, we obtain from (4.7) and (4.17) that

$$\text{(4.20)} \qquad |u_h(x)-u(x)| \leqq k_1h^2 \quad \text{for } 0 < h < k_1^{-1} \text{ and } x \text{ in } Z \cap \bar{B}(x^0, r_0),$$

and consequently that

$$\text{(4.21)} \qquad \begin{aligned} -2(N+2)k_1 &\leqq \Delta_* u(x) \leqq \Delta^* u(x) \\ &\leqq 2(N+2)k_1 \quad \text{for } x \text{ in } Z \cap \bar{B}(x^0, r_0). \end{aligned}$$

Now if x is in $B(x^0, r_0)-[Z \cap B(x^0, r_0)]$, it follows that u is subharmonic in an open ball centered at x. We consequently have that

$$\text{(4.22)} \qquad -\infty \leqq u(x) < +\infty \quad \text{for } x \text{ in } B(x^0, r_0)-[Z \cap B(x^0, r_0)],$$

$$\text{(4.23)} \qquad u \text{ is upper semicontinuous in } B(x^0, r_0)-[Z \cap B(x^0, r_0)],$$

and furthermore from [3, 1.7] that

$$0 \leqq \Delta^* u(x) \quad \text{for } x \text{ in } B(x^0, r_0) - [Z \cap B(x^0, r_0)]. \tag{4.24}$$

We conclude first of all from (4.19) and (4.22) that

$$-\infty \leqq u(x) < +\infty \quad \text{for } x \text{ in } B(x^0, r_0). \tag{4.25}$$

Next we set $g(x) = \min [0, \Delta^* u(x)]$ and observe from (4.4), (4.21) and (4.24) that

(4.26) there is a positive integer k_1 such that $-2(N+2)k_1 \leqq g(x) \leqq 0$ for all x in $B(x^0, r_0)$, and $g(x) = 0$ almost everywhere in $B(x^0, r_0)$.

Also, we have that

$$g(x) \leqq \Delta^* u(x) \quad \text{for } x \text{ in } B(x^0, r_0). \tag{4.27}$$

We next establish the following fact:

(4.28) u is upper semicontinuous in $B(x^0, r_0)$.

In order to establish (4.28), it follows from (4.23) that we need only establish that u is upper semicontinuous at points x in $Z \cap B(x^0, r_0)$. Consequently, let x' be a point in $Z \cap B(x^0, r_0)$. If we show that

(4.29) u is upper semicontinuous at x',

then (4.28) will be established.

Since for each h with $0 < h < k_1^{-1}$, $u_h(x)$ is a continuous function for x restricted to the closed set $Z \cap \bar{B}(x^0, r_0)$, we have from (4.20) that $u(x)$ is a continuous function for x restricted to the set $Z \cap B(x^0, r_0)$.

Consequently, since x' is in $Z \cap B(x^0, r_0)$, given an $\varepsilon > 0$, there exists $r_1 > 0$, with the following properties:

$$\bar{B}(x', 4r_1) \subset B(x^0, r_0) \tag{4.30}$$

and

$$|u(x) - u(y)| < \varepsilon \quad \text{for } x \text{ and } y \text{ in } Z \cap \bar{B}(x', 4r_1). \tag{4.31}$$

Also from (4.20), we have the existence of an $h_0 > 0$ such that

$$0 < h_0 < \min [r_1, k_1^{-1}] \tag{4.32}$$

and

$$|u_h(x) - u(x)| < \varepsilon \quad \text{for } 0 < h \leqq h_0 \text{ and } x \text{ in } Z \cap \bar{B}(x', 4r_1). \tag{4.33}$$

Next, we observe from (4.17) that $Z \cap \bar{B}(x', 4r_1) \subset F_{k_1}$. Consequently from (4.10), we have the existence of an $h_1 > 0$ such that

$$0 < h_1 \leqq h_0 \tag{4.34}$$

and

$$|u_h(x+y) - u_h(x)| < \varepsilon \quad \text{for } 0 < h \leqq h_1,\ |y| \leqq h, \text{ and } x \text{ in } Z \cap \bar{B}(x', 4r_1). \tag{4.35}$$

Observing from (4.32) and (4.34) that $0<h_1<r_1$, we propose to show now that

$$u(x) < u(x')+3\varepsilon \quad \text{for } x \text{ in } B(x', h_1), \tag{4.36}$$

which fact will establish (4.29) and therefore (4.28).

If x is in $Z \cap B(x', h_1)$, we conclude from (4.34), (4.32), and (4.31) that (4.36) holds. If x is in $B(x', h_1)-[Z \cap B(x', h_1)]$, let h_2 designate the distance from x to Z, and let z be a point in Z such that $|x-z|=h_2$. Then $0<h_2\leqq h_1$, and u is subharmonic in $B(x, h_2)$. Therefore, we obtain from [3, 1.7] that $u(x)\leqq u_{h_2}(x)$. Since z is in $B(x', 2r_1) \cap Z$, we have from (4.35), (4.33) and (4.31) that

$$u(x) \leqq u_{h_2}(x) \leqq u_{h_2}(z)+\varepsilon \leqq u(z)+2\varepsilon \leqq u(x')+3\varepsilon.$$

Consequently, (4.36) is established. Therefore (4.29) is established, and we conclude that (4.28) holds.

Before proceeding with the establishment of the fact that Case II leads to a contradiction, we observe from (1.1) [since u is in class $\mathscr{A}(0, r_*)$] that

$$u \text{ is in } L^1[\bar{B}(x^0, r_0)]. \tag{4.37}$$

To show that Case II does indeed lead to a contradiction, we see from (4.25), (4.26), (4.27), (4.28), (4.37), and Lemma 2 that u is subharmonic in $B(x^0, r_0)$. But then $Z \cap B(x^0, r_0)$ is empty, and we have a contradiction to (4.16). Therefore Case II does not hold.

Since both Case I and Case II do not hold, Z must be empty. We conclude, therefore, that u is indeed subharmonic in $B(0, r_*)$, and the proof of the theorem is complete.

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