ON KNOTS WITH NONTRIVIAL INTERPOLATING MANIFOLDS

BY
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Abstract. If a polygonal knot \( K \) in the 3-sphere \( S^3 \) does not separate an interpolating manifold \( S \) for \( K \), then \( S - K \) does not carry the first homology of either closed component of \( S^3 - S \). It follows that most knots \( K \) with nontrivial interpolating manifolds have the property that a simply connected manifold cannot be obtained by removing a regular neighborhood of \( K \) from \( S^3 \) and sewing it back differently.

0. Introduction. A polygonal knot \( K \) in the 3-sphere \( S^3 \) is said to have Property P [1] if it is impossible to obtain a simply connected manifold by removing a regular neighborhood of \( K \) from \( S^3 \) and sewing it back differently. It has been conjectured that all nontrivial knots have Property P, and large classes of knots with this property have been described by Hempel [5], Bing and Martin [1], Noga [10], Connor [2], Gonzales [4], and the author [12]. If \( K \) has Property P, then the knot type of \( K \) is determined by the topological type of \( S^3 - K \). Furthermore, it would be interesting to know that no fake 3-sphere could be constructed by surgery along a knot, since [7] any closed, orientable 3-manifold can be realized by surgery along some finite link in \( S^3 \).

In [12], a Property Q is defined for knots and it is shown there (Theorem 5) that Property Q, along with an additional technical requirement, implies Property P. Property Q and Neuwirth's notion of an interpolating manifold [9] for a knot are similar in that both require the knot to be contained in a closed 2-manifold in a "sufficiently complicated" manner. It is conjectured in [12] that a knot \( K \) has Property Q iff \( K \) has a nontrivial interpolating manifold. This conjecture is established by the following theorem, which is the main result of this paper:

THEOREM. If \( S \) is a polyhedral, closed 2-manifold in \( S^3 \), \( K \) a nonseparating simple closed curve in \( S \), and \( A \) is either closed complementary domain of \( S \), then \( K \) generates a free factor of \( H_1(A) \) iff \( H_1(A, S - K) = 0 \). It follows that most knots with nontrivial interpolating manifolds have Property P.
§1 contains the proof of the above theorem and its application to Property P. §2 considers generalizations of the main result: Theorem 2 extends Theorem 1 to knots in manifolds other than $S^3$; in Lemma 3.1 necessary criteria are found for an endomorphism of the first homology group of a closed, orientable 2-manifold to be induced by a map. These are used in Theorem 3 to obtain results analogous to Theorem 1 for loops, rather than simple closed curves.

Conventions. All topological spaces, subspaces, and maps considered here are polyhedral, and all manifolds are orientable. A knot is a simple closed curve in $S^3$ that does not bound a disk. A manifold is closed if it is compact, connected, and has no boundary. Homology groups are taken with integer coefficients unless otherwise specified. An interpolating manifold for a knot $K$ is a closed 2-manifold $S \subseteq S^3$ such that $K \subseteq S$ and $K$ does not generate a free factor of the first homology group of either closed complementary domain of $S$; since, as noted in [9], every knot $K$ has an interpolating manifold $S$ such that $K$ separates $S$, we call $S$ nontrivial if $S - K$ is connected. If $K$ is contained in a closed 2-manifold $S$ such that $S - K$ is connected and $S - K$ does not carry the first homology of either closed complementary domain of $S$, then $K$ is said to have Property Q. If $x, y$ are elements of a group $G$, the commutator of $x$ and $y$, denoted $[x, y]$, is $x^{-1}y^{-1}xy$; the commutator subgroup, denoted $G'$, of $G$ is the subgroup generated by $\{[x, y] : x, y \in G\}$. If $R_1, R_2, \ldots$ are elements of the free group $F$ generated by $a_1, a_2, \ldots$, the symbol $P = (a_1, a_2, \ldots | R_1, R_2, \ldots)$ will denote the quotient group $G$ of $F$ by its smallest normal subgroup containing $R_1, R_2, \ldots$; if $H$ is a group isomorphic to $G$, then $P$ is called a presentation of $H$ with generators $\{a_i\}$ and (defining) relators $\{R_i\}$.

It will also be useful to define a "standard basis" for a closed 2-manifold $S$ of genus $n \geq 1$. Let $a_1, \ldots, a_{2n}$ be a system of simple closed curves in $S$ such that (1) if $|i - j| = n$ then $a_i$ and $a_j$ are transverse, and (2) $a_i \cap a_j = \emptyset$ otherwise. Choose a base point $s \in S$, and, for $i = 1, \ldots, n$, let $t_i$ be an arc in $S$ from $s$ to $a_i \cap a_{i+n}$ such that for $i \neq j$, $t_i \cap t_j = \{s\}$. For $i = n+1, \ldots, 2n$, let $t_i = t_{i-n}$. Orient the curves $a_i$, $i = 1, \ldots, 2n$, and let $a_i$ be the loop obtained from $a_i$ by tracing out $t_i$, $a_i$, and then $t_i^{-1}$. Then $\{a_i\}_{i=1,\ldots,2n}$ generates $\Pi_1(S, s)$, and, with possible changes of orientations and renumbering of curves within the pairs $a_i, a_{n+i}$, the function $\lambda : a_i \mapsto x_i$, $i = 1, \ldots, 2n$, defines an isomorphism of $\Pi_1(S, s)$ onto

$$\left( x_1, \ldots, x_{2n} \bigg| \prod_{i=1}^{n} [x_i, x_{n+i}] \right).$$

With such orientations and numbering, the curves $\{a_i\}_{1=1,\ldots,2n}$ will be called a standard basis for $S$. It may be the case, however, as in §2, that necessary properties of $\{a_i\}$ would be lost by renumbering. It is still possible to orient the curves so that $[\lambda(a_i), \lambda(a_{n+i})]$ is conjugate to $[x_i, x_{n+i}]$. With such orientations, $\{a_i\}$ will be called a prestandard basis for $S$. This choice of orientations is independent of the base point $s$ and the arcs $t_i$. With any orientations, $\{a_i\}_{i=1,\ldots,2n}$ is a basis for $H_1(S)$.
1. **The main result.** Let \( S \) be a closed 2-manifold of genus \( n \geq 1 \) in \( S^3 \) containing a nonseparating simple closed curve \( K \), and let \( A \) be the closure of a complementary domain of \( S \).

**Theorem 1.** \( K \in pH_1(A) \) for some prime \( p \in \mathbb{Z} \) iff there exists a homomorphism of \( H_1(A, S-K) \) onto \( \mathbb{Z}_p \). In particular, \( K \) generates a free factor of \( H_1(A) \) iff \( H_1(A, S-K) = 0 \).

**Corollary 1.** A knot \( K \) has a nontrivial interpolating manifold iff \( K \) has Property Q.

**Corollary 2.** If a knot \( K \) has a nontrivial interpolating manifold \( S \) such that a boundary component \( J \) of a regular neighborhood of \( K \) in \( S \) is not 0, a generator, or twice a generator in \( H_1(S^3-K) \), then \( K \) has Property P.

**Proof of Corollary 2.** By Theorem 1, \( S \) and \( J \) satisfy the requirements of Theorem 5 of [12], and so \( K \) has Property P.

**Proof of Theorem 1.** If \( n=1 \), the result is obvious, so it will be assumed throughout that \( n \geq 2 \). Since \( A \subset S^3 \) and \( \text{bdy}(A) \) is connected, by a theorem of Fox [3], \( A \) can be re-embedded in \( S^3 \) so that \( (S^3-A)^- \) is a regular neighborhood of a finite graph. From a theorem of Papakyriakopoulos (Theorem (4.1) of [11]), it then follows that there is a prestandard basis \( \{a_i\} \) for \( S \) such that

\[
\begin{align*}
\text{(i) } & a_1, \ldots, a_n \text{ is a basis for } H_1(A), \text{ a free abelian group of rank } n, \\
\text{(ii) } & a_i \sim 0 \text{ in } A \text{ for } n+1 \leq i \leq 2n.
\end{align*}
\]

For notational convenience, it will be assumed that no renumbering is necessary to change \( \{a_i\} \) to a standard basis for \( S \); the arguments below will accommodate any such complication by appropriate alteration of subscripts.

Since \( K \) is a nonseparating simple closed curve in \( S \), there exists a homeomorphism \( f: S \rightarrow S \) taking \( a_i \) to \( K \). Since \( f \) is a homeomorphism, \( f \) induces a conjugacy class of automorphisms of \( \Pi_1(S) \), which in turn induces an automorphism \( f_* \) of \( H_1(S) \). Let \( E=(e_{ij}) \) be the \( 2n \times 2n \) integer matrix of \( f_*a_i \), where \( f_*(a_i) = \sum_{j=1}^{2n} e_{ij}a_j \). By a theorem of Magnus (Corollary 5.15 of [8]), since \( f_*a_i \) is induced by a homeomorphism, and \( \{a_i\} \) is a standard basis, the matrix \( E \) must be symplectic. That is, if \( I \) is the \( n \times n \) identity matrix, \( J \) is the \( 2n \times 2n \) matrix with block diagram \((-J \ I) \), and \( E' \) is the transpose of \( E \), then \( EJE' = \pm J \) (although \( EJE' \neq E'JE \) in general, \( E \) is symplectic iff \( E' \) is symplectic). Thus, in particular, for all \( s, t \) such that \( 1 \leq s < t \leq 2n \) and \( |s-t| \neq n \), it must be the case that

\[
\sum_{i=1}^{n} \begin{vmatrix} e_{i,s} & e_{i,t} \\ e_{i+n,s} & e_{i+n,t} \end{vmatrix} = 0.
\]

Since it has been assumed that no renumbering among \( a_i, a_{i+n} \) was necessary, the remainder of the proof will only make use of equations (*) for \( 1 \leq s < t \leq n \).
Let \( f_A : H_1(S) \to H_1(A) \) be the homomorphism induced by \( f \) followed by inclusion of \( S \) into \( A \). Then in \( H_1(A) \), \( K = f_A(a_i) \) and
\[
H_1(A, S-K) = H_1(A)/f_AH_1(S-a_i)
\]
\[
= H_1(A)/\{f_A(a_i) : i = 1, \ldots, 2n, i \neq n + 1\}.
\]

Case 1. Assume that for some prime \( p \in \mathbb{Z} \), \( K \in pH_1(A) \). To map \( H_1(A, S-K) \) onto \( \mathbb{Z}_p \), it suffices to find a homomorphism of \( H_1(A) \) onto \( \mathbb{Z}_p \) annihilating \( \{f_A(a_i) : i = 2, \ldots, 2n, i \neq n + 1\} \), since any homomorphism of \( H_1(A) \to \mathbb{Z}_p \) will annihilate \( f_A(a_i) \).

First define \( \sigma : H_1(A) \to H_1(A; \mathbb{Z}_p) \) by \( \sigma(a_i) = a_i \). Then \( \sigma f_A(a_i) = \sum_{j=1}^n e_j a_j \), where now \( e_j \) is a residue class mod \( p \). The equations \( \ast \) remain valid over \( \mathbb{Z}_p \); in particular, since \( \sigma f_A(a_1) = 0 \), we have for \( 1 \leq s < t \leq n \),
\[
\sum_{i=2}^n \begin{vmatrix} e_{i,s} & e_{i,t} \\ e_{i+n,s} & e_{i+n,t} \end{vmatrix} = 0 \in \mathbb{Z}_p.
\]

By Lemma 1.1 below, the \((2n-2) \times n\) matrix \((e_{ij})_{i=2}^{2n}; i \neq n + 1; j = 1, \ldots, n\) has rank at most \( n-1 \) over \( \mathbb{Z}_p \). Thus there exists an epimorphism \( \theta : H_1(A; \mathbb{Z}_p) \to \mathbb{Z}_p \) annihilating \( \{f_A(a_i) : i = 2, \ldots, 2n, i \neq n + 1\} \), and so \( \theta \circ \sigma \) induces a map of \( H_1(A, S-K) \) onto \( \mathbb{Z}_p \).

Case 2. Assume now that for some prime \( p \in \mathbb{Z} \), there exists an epimorphism \( \rho : H_1(A; \mathbb{Z}_p) \to \mathbb{Z}_p \). We wish to show that \( f_A(a_i) \in pH_1(A) \). Let \( \Pi \) be the natural projection of \( H_1(A) \) onto \( H_1(A)/f_AH_1(S-a_i) \), and let \( \sigma \) be as in Case 1. Since \( \mathbb{Z}_p \) has characteristic \( p \), \( \theta = \rho \circ \Pi \circ \sigma^{-1} \) is a well-defined homomorphism of \( H_1(A; \mathbb{Z}_p) \) onto \( \mathbb{Z}_p \). Since \( S \) contains generators for \( H_1(A) \), specifically \( a_1, \ldots, a_n \), and \( f \) is a homeomorphism, \( f_A \) must be surjective, and so \( \{ \sigma f_A(a_i) : i = 1, \ldots, 2n \} \) generates \( H_1(A; \mathbb{Z}_p) \). Thus it must be the case that \( \theta \sigma f_A(a_{n+1}) \) generates \( \mathbb{Z}_p \). On the other hand, if \( \sigma f_A(a_1) \neq 0 \), then by Lemma 1.2 below, equations \( \ast \) imply that \( \sigma f_A(a_{n+1}) \) is a linear combination of \( \{ \sigma f_A(a_i) : i = 1, \ldots, 2n, i \neq n + 1 \} \), and so \( \theta \sigma f_A(a_{n+1}) = 0 \). We conclude that \( \sigma f_A(a_1) = 0 \), i.e., \( K \in pH_1(A) \).

Lemma 1.1. Let
\[
E = (e_{ij})_{i=2}^{2n}; i \neq n + 1; j = 1, \ldots, n
\]
be a \((2n-2) \times n\) matrix over a field subject to equations \( \ast \). Then rank \((E) \leq n-1 \).

Proof. If \( n = 2 \), the result is obvious. We proceed by induction on \( n \). The rank of \( E \) and equations \( \ast \) are preserved under the following transformations: (i) permute columns, (ii) divide a column by a nonzero scalar, (iii) add to one column a multiple of another, and (iv) permute rows in pairs: row \((i) \geq row (i') \) and row \((i+n) \geq row (i'+n) \). If rank \((E) = n \), then with finitely many transformations of types (i)–(iv), we can obtain a new matrix \( E' \), satisfying equations \( \ast \), such that \( e_{2,1} = e_{2+n,2} = 1 \), all the other terms in rows \((i=2) \) and \((i=2+n) \) are 0, and the submatrix
\[
\tilde{E} = (e_{ij})_{i=3}^{2n}; i \neq 1+n; j = 2, \ldots, n
\]
has \((n-2)\) linearly independent rows, the first component of each being 0. But if rows \((i=2)\) and \((i=2+n)\) are as specified, then we have, for \(2 \leq s < t < n\),

\[
\sum_{i=3}^{\infty} \begin{vmatrix} e_{i, s} & e_{i, t} \\ e_{i+n, s} & e_{i+n, t} \end{vmatrix} = 0,
\]

so inductively, \(\text{rank}(E) \leq n-2\). Thus the first component of each row of \(E\) must be 0, and so \(e_{2+n, 2} = 1\) and all the other terms in column \((j=2)\) of \(E\) are 0. Since \(e_{21} = 1\), this contradicts the equation

\[
0 = \sum_{i=2}^{\infty} \begin{vmatrix} e_{i, 1} & e_{i, 2} \\ e_{i+n, 1} & e_{i+n, 2} \end{vmatrix}.
\]

**Lemma 1.2.** Let \(E=(e_{ij})_{i=1,2n;j=1,...,n}\) be a \(2n \times n\) matrix over a field subject to equations \((\ast)\) (for \(1 \leq s < t \leq n\)). If row \((1)\) is not identically 0, then row \((n+1)\) is a linear combination of the other rows of \(E\).

**Proof.** In addition to preserving equations \((\ast)\), the transformations (i)-(iii) described in the preceding proof do not alter the fact of whether or not row \((n+1)\) of \(E\) is a linear combination of the other rows. Since row \((1)\) has some nonzero component, we can therefore assume that row \((1)\) = \((1, 0, \ldots, 0)\). Equations \((\ast)\), for \(s=1\), then become

\[
i_{1} + n.t = \sum_{i=2}^{n} \begin{vmatrix} e_{i, 1} & e_{i, 2} \\ e_{i+n, 1} & e_{i+n, 2} \end{vmatrix}.
\]

It follows easily that if \(\alpha_{j} = e_{j+n, 1} - e_{j-n, 1}\) according as \(1 \leq j \leq n\) or \(n+2 \leq j \leq 2n\), then

\[
\text{row (n+1)} = \sum_{j=1,2n;j\neq n+1} \alpha_{j} \cdot \text{row (j)}.
\]

2. **Generalizations.** Several of the hypotheses of Theorem 1 can be weakened, while maintaining the same or appropriately modified conclusions. First, \(p\) need not be prime; Theorem 1 easily extends to the case that \(p\) is a product of distinct primes. It is not clear, however, how one might establish a duality theorem of the following sort:

**Conjecture.** If \(K, A, S\) are as in Theorem 1, then the torsion subgroups of \(H_{1}(A, K)\) and \(H_{1}(A, S-K)\) are isomorphic.

It is also unnecessary to require the ambient space to be \(S^{3}\).

**Definition.** A compact, connected, 3-manifold \(A\) with boundary a closed 2-manifold \(S\) of genus \(n\) is called a homology cube-with-holes (HCWH) if \(H_{1}(A)\) is a free abelian group of rank \(n\) and \(S\) has a prestandard basis \(\{a_{i}\}_{i=1,2n}\) satisfying conditions \((1)\). From the proof of Theorem 1 we have immediately

**Theorem 2.** If \(A\) is a HCWH and, otherwise, \(K, S, p, A\) are as in Theorem 1, then \(K \in \pi H_{1}(A)\) iff \(\exists p : H_{1}(A, S-K) \to Z_{p}\).
If $A$ can be embedded in a homology 3-sphere as the closure of the complement of a regular neighborhood of a finite graph, then, from Theorem (4.1) of [11] and the Mayer-Vietoris sequence, it follows that $A$ is a HCWH. If $A$ can be embedded in a homotopy 3-sphere, then, modifying Fox's proof in [3], $A$ can be embedded in a (possibly different) homotopy 3-sphere as the closed complement of a cube-with-handles, so, again, $A$ is a HCWH.

Finally, results analogous to Theorem 1 can be obtained in the case that $K$ is the image of a nonseparating simple closed curve under a map of $S$ to $S$.

**Theorem 3.** Let $A$ be a HCWH with boundary $S$, $K$ a nonseparating simple closed curve in $S$, $f$: $S \to S$ a map, and $f_A: H_1(S) \to H_1(S) \to H_1(A)$ the induced homomorphism. If, for some prime $p \in \mathbb{Z}$, $f_A(K) \in pH_1(A)$, then there is a homomorphism of $H_1(A)/f_AH_1(S - K)$ onto $\mathbb{Z}$. The converse holds providing $f_A$ is assumed to be surjective.

**Proof.** Let $E=(e_{ij})$ be as in the proof of Theorem 1. The full strength of the fact that $E$ was symplectic was not required for that proof, but only that equations (*) be valid for $1 \leq s < t \leq 2n$, $|s - t| \neq n$. It thus suffices to verify these equations in the case that $f$ is a map.

Let $\{a_i\}_{i=1}^{2n}$ be a prestandard basis for $S, f_A(a_i) = \sum_{j=1}^{2n} e_{ij}a_j$ the endomorphism of $H_1(S)$ induced by $f,

\[ x_{s,t} = \sum_{i=1}^{n} \begin{vmatrix} e_{i,s} & e_{i,t} \\ e_{i+n,s} & e_{i+n,t} \end{vmatrix}, \]

and

\[ G_{a,n} = \left( a_1, \ldots, a_{2n} \right| \prod_{i=1}^{n} [a_i, a_{n+i}], \left\{ [u, [v, w]] \right\}_{u,v, w \in \omega(a_i)} \]

(isomorphic to the quotient group of $\Pi_1(S)$ by the third term in its lower central series). Since $f$ is a map and $\{a_i\}$ is a prestandard basis, $f$ induces an endomorphism $\#$ of $G_{a,n}$ which induces $f_A$. Thus $\prod_{i=1}^{n} [f_A(a_i), f_A(a_{n+i})] = 1 \in G_{a,n}$. But, using the identity $u^p v^q = v^q u^p [u, v]^{pq}$ in $G_{a,n}$, it is easy to show that

\[ \prod_{i=1}^{n} [f_A(a_i), f_A(a_{n+i})] \]

(\#) \[ = \left( \prod_{1 \leq s \leq 2n, |s - t| \neq n} [a_t, a_s]^{x_{s,t}} \right) \left( \prod_{s=1, \ldots, n-1; t=s+n} [a_t, a_s]^{x_{s,t} - x_{n,2n}} \right). \]

Since $G_{a,n}$ is a free abelian group, generated by $\{[a_t, a_s] \}_{1 \leq s < t \leq 2n}$ and freely generated by

\[ \{[a_t, a_s] \}_{1 \leq s < t \leq 2n, (s,t) \neq (n,2n)} \]

it follows that each exponent in the right side of equation (\#) must be 0.

**Question.** Which endomorphisms of $H_1(S)$ are induced by maps? According to the above calculations, if $E=(e_{ij})$ is the matrix of a map-induced endomorphism
of $H_1(S)$, given in terms of a prestandard basis for $S$, then $E$ must be "nearly symplectic," in the sense that for some integer $\lambda$, $E'JE = \lambda J$. For genus $(S) = 1$, this is no restriction, consonant with the fact that $H_1(S) = \Pi_1(S)$. But if genus $(S) \geq 2$, and $f$ is not (homotopic to) a homeomorphism, then, using Euler characteristic arguments, the fact that $\Pi_1(S)$ is Hopfian, and Lemma 3.2 of [6], it can be shown that there is a homeomorphism $h: S \to S$ such that $f \circ h$ annihilates at least one of $a_i, a_{i+1}$ for each $i = 1, \ldots, n$. It thus follows that $\lambda \neq \pm 1 \Rightarrow \lambda = 0$. But is it true that any $2n \times 2n$ integer matrix $E$ such that $E'JE = 0$ is the matrix of a map $f: S \to S$?

References

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