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# A GENERALIZATION OF THE STRICT TOPOLOGY(1)

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**Abstract.** The strict topology $\beta$ on the space $C(X)$ of bounded real-valued continuous functions on a topological space $X$ was defined, for locally compact $X$, by Buck (Michigan Math. J. 5 (1958), 95–104). Among other things he showed that (a) $C(X)$ is $\beta$-complete, (b) the dual of $C(X)$ under the strict topology is the space of all finite signed regular Borel measures on $X$, and (c) a Stone-Weierstrass theorem holds for $\beta$-closed subalgebras of $C(X)$. In this paper the definition of the strict topology is generalized to cover the case of an arbitrary topological space and these results are established under the following conditions on $X$: for (a) $X$ is a $k$-space; for (b) $X$ is completely regular; for (c) $X$ is unrestricted.

1. **Introduction and notation.** Let $X$ be a topological space. We denote by $B(X)$ the algebra of all bounded real-valued functions on $X$ and by $C(X)$ the subalgebra of $B(X)$ consisting of continuous functions. $B_0(X)$ denotes the ideal in $B(X)$ consisting of functions *vanishing at infinity*, in that for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$, and $C_0(X) = B_0(X) \cap C(X)$. Note that $B_0(X)$ contains

(a) the characteristic function $\chi(K)$ of each compact set $K \subset X$;

(b) every function $\psi$ of the form $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(K_n)$, where $\alpha_n \geqq 0$ for all $n$, $\alpha_n \to 0$, and the sets $K_n$ are compact and disjoint; and so, in particular,

(c) the function $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(\{x_n\})$, where $(x_n)$ is any sequence of distinct points.

We shall need the following lemma:

LEMMA 1.1. *If $f$ is a real-valued function on $X$ and $f\psi$ is bounded for every $\psi$ in $B_0(X)$ then $f$ is bounded.*

**Proof.** Suppose $f$ is not bounded. Choose a sequence $(x_n)$ in $X$ with $|f(x_n)| \to \infty$ and put $\psi = \sum |f(x_n)|^{-1/2} \chi(\{x_n\})$. Then $\psi \in B_0(X)$ but $f\psi$ is not bounded.

The strict topology on $C(X)$ was defined for locally compact $X$ by Buck [1] by means of a set of seminorms determined by the elements of $C_0(X)$. If $X$ is completely regular but not locally compact, $C_0(X)$ may be very small (for instance, if $X$ is the rationals [3, p. 109]) and does not yield a useful topology for $C(X)$. We claim that in this case the natural generalization of the strict topology is obtained by letting the role of $C_0(X)$ be played by $B_0(X)$; the change makes no difference if

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$X$ is locally compact. Indeed, under mild conditions on $X$, we shall prove that, with this generalized strict topology, $C(X)$ is complete (Theorem 2.4) and its dual is the space of bounded signed regular Borel measures on $X$ (Theorem 4.6), and we shall establish a Stone-Weierstrass theorem for $C(X)$ (Theorem 3.1).

For the first of these results we assume that $X$ is a $k$-space [6], [7], [10], [12], [13], i.e. a space in which a set is closed if its intersection with every closed compact set is closed. The limitation to $k$-spaces is not a serious restriction. Every locally compact space is a $k$-space; so is every metrisable space. Although there do exist [6] completely regular spaces which are not $k$-spaces, such spaces do not seem to be important. Indeed Steenrod [10] has made a strong case for formulating topology entirely within the category of Hausdorff $k$-spaces.

At the same time as making the change from $C_0(X)$ to $B_0(X)$ it is natural to define the strict topology, in the first instance, on $B(X)$. In this form, our strict topology is a special case of generalizations introduced independently by Busby [2] and Sentilles and Taylor [9] in the context of Banach algebras.

**Note added in proof.** Recently, and independently, Gulick [14] and Sentilles [15] have also discussed the strict topology for $C(X)$, for completely regular $X$.

2. **Topologies on $B(X)$.** Let $X$ be a set. Corresponding to each function $\psi$ in $B(X)$ we define a seminorm $p_\psi$ on $B(X)$ by writing, for every $f$ in $B(X)$, $p_\psi(f) = \|\psi f\|$, where $\| \ \|$ denotes the sup norm.

Let $\Psi \subset B(X)$ be any subset. By the *$\Psi$-topology* on $B(X)$ we mean the topology determined by the set of seminorms $\{p_\psi : \psi \in \Psi\}$. A basis of open neighbourhoods for the $\Psi$-topology is $\{U_\psi : \psi \in \Psi\}$, where $U_\psi = \{f \in B(X) : p_\psi(f) < 1\}$.

Let $\Psi \subset B(X)$, $\Psi' \subset B(X)$, $\psi \in B(X)$, $\psi' \in B(X)$. If, for some constant $\lambda$, $\lambda|\psi'| \geqq |\psi|$ we say $\psi'$ *dominates* $\psi$; if $\psi'$ dominates every element of $\Psi$ we say $\psi'$ dominates $\Psi$. The proof of the following lemma is easy.

LEMMA 2.1. *If every element of $\Psi$ is dominated by some element of $\Psi'$ then the $\Psi'$-topology on $B(X)$ is finer than the $\Psi$-topology.*

LEMMA 2.2. *If $\psi'$ dominates $\Psi$ then $U_{\psi'}$ is a $\Psi$-bounded set (i.e. bounded in the $\Psi$-topology). Moreover, given any $\Psi$-bounded set $B \subset B(X)$ there is a $\psi'$ dominating $\Psi$ with $B \subset U_{\psi'}$.*

**Proof.** Given $\psi$ in $\Psi$ choose $\lambda$ so that $\lambda|\psi'| \geqq |\psi|$. Then, for any $f$ in $B(X)$, $p_\psi(f) \leqq \lambda p_{\psi'}(f)$, so that $p_\psi$ is bounded on $U_{\psi'}$. Since this is true for every $\psi$, $U_{\psi'}$ is $\Psi$-bounded.

Now suppose $B \subset B(X)$ is $\Psi$-bounded. Then, for each $\psi$ in $\Psi$ we can choose $\lambda_\psi$ such that $B \subset \lambda_\psi U_\psi$; clearly, we may also assume $\lambda_\psi \geqq \|\psi\|$. But then, for all $\psi$ in $\Psi$,

$$f \in B \Rightarrow \|f\psi/\lambda_\psi\| < 1.$$

Thus $B \subset U_{\psi_0}$, where $\psi_0 = \sup\{|\psi|/\lambda_\psi : \psi \in \Psi\} \in B(X)$. Clearly, $\psi_0$ dominates each $\psi$ in $\Psi$.

COROLLARY 2.3. *If every function which dominates $\Psi$ dominates $\Psi'$ then every $\Psi$-bounded set is $\Psi'$-bounded.*

Now let $X$ be a topological space. We introduce four topologies on $B(X)$:

(a) $\sigma$, the uniform (or sup norm) topology, is the $B(X)$-topology or, equivalently (by Lemma 2.1), the $\{\mathbf{1}\}$-topology. (We denote by $\mathbf{1}$ the unit function on $X$.)

(b) $\beta$, our generalized strict topology, is the $B_0(X)$-topology.

(c) $\kappa$, the topology of compact convergence, is the $\Psi_\kappa$-topology, where $\Psi_\kappa=\{\psi \in B(X) : \psi \text{ has compact support}\}$.

(d) $\rho$, the topology of pointwise convergence, is the $\Psi_\rho$-topology, where $\Psi_\rho=\{\psi \in B(X) : \psi \text{ has finite support}\}$.

The following theorem gives the main properties of the strict topology $\beta$. The proofs are similar to those of Buck [1].

THEOREM 2.4. *Let $X$ be any topological space. Let the topologies $\sigma$, $\beta$, $\kappa$, $\rho$ on $B(X)$ be defined as above. Then*

(i) $\sigma\supset\beta\supset\kappa\supset\rho$.

(ii) *If $X$ is locally compact then $\beta$ coincides with the strict topology as defined by Buck* [1].

(iii) *$\beta$ and $\sigma$ have the same bounded sets.*

(iv) *On any $\sigma$-bounded set $B$ the topologies $\beta$ and $\kappa$ coincide.*

(v) *If $X$ is a k-space then $C(X)$ is $\beta$-complete.*

**Proof.** (i) follows at once from Lemma 2.1.

(ii) Let $K$ be locally compact. By Lemma 2.2, it is sufficient to show that each $\psi$ in $B_0(X)$ is dominated by some $\psi'$ in $C_0(X)$. We may clearly assume $\|\psi\|<1$. Choose compact sets $K_n$ with $\varnothing=K_0\subset K_1\subset K_2\subset\cdots$ such that $|\psi(x)|<2^{-n}$ for $x\notin K_n$. Choose $\psi_n$ in $C_0(X)$ with $\psi_n(x)=2^{-n}$ for $x\in K_n$ and $0\leqq\psi_n\leqq 2^{-n}\mathbf{1}$. Let $\psi'=\sum_{n=0}^{\infty}\psi_n$. Then $\psi'\in C_0(X)$ and $\psi'$ dominates $\psi$.

(iii) Since $\sigma\supset\beta$ and since $\sigma$ is the $\{\mathbf{1}\}$-topology on $B(X)$ it suffices, by Corollary 2.3, to show that if $\psi_0$ in $B(X)$ dominates $B_0(X)$ then $\psi_0$ dominates $\mathbf{1}$. Suppose, then, that $\psi_0$ does not dominate $\mathbf{1}$. Then there is a sequence $(x_n)$ in $X$ with $|\psi_0(x_n)|\to 0$. Let $\psi=\sum_{n=1}^{\infty}|\psi_0(x_n)|^{1/2}\chi(\{x_n\})$. Then $\psi\in B_0(X)$ but $\psi_0$ does not dominate $\psi$.

(iv) Choose $M$ so that $\|f\|<M$ whenever $f\in B$. By (i), it suffices to show that the $\beta$-closure $B_\beta$ of $B$ contains the $\kappa$-closure $B_\kappa$ of $B$. Suppose $g\in B_\kappa$. Given any $\psi$ in $B_0(X)$ and any $\varepsilon>0$ choose a compact set $K\subset X$ with $|\psi(x)|<\varepsilon$ for $x\notin K$. Let $\psi'=\psi\chi(K)$. Then $\psi'\in\Psi_\kappa$ and, for every $f$ in $B$, $p_\psi(f-g)=\|(f-g)\psi\|\leqq\|(f-g)\psi'\|+\|(f-g)(\psi-\psi')\|\leqq p_{\psi'}(f-g)+(M+\|g\|)\varepsilon$. Since $g\in B_\kappa$ and $\varepsilon$ is arbitrary, this gives $\inf\{p_\psi(f-g) : f\in B\}=0$. Since this holds for all $\psi$ in $B_0(X)$, $g\in B_\beta$.

(v) Let $\{f_\alpha\}$ be a $\beta$-Cauchy net in $C(X)$. By (i), $\{f_\alpha\}$ is $\kappa$-Cauchy, and hence [6], since $X$ is a $k$-space, $f_\alpha \xrightarrow{\kappa} f$ where $f$ is a continuous function on $X$. It remains to show that $f$ is bounded and that $f_\alpha \xrightarrow{\beta} f$.

Now, for each $\psi$ in $B_0(X)$, $\{\psi f_\alpha\}$ is a $\sigma$-Cauchy net in $B(X)$. Since $B(X)$ is $\sigma$-

complete, $\psi f_\alpha \xrightarrow{\sigma} g$ for some $g$ in $B(X)$. But then $\psi f_\alpha \xrightarrow{\rho} g$ whereas also $\psi f_\alpha \xrightarrow{\rho} \psi f$ so that $g=\psi f$.

We have thus shown that, for each $\psi$ in $B_0(X)$, $\psi f_\alpha \xrightarrow{\sigma} \psi f$ and $\psi f \in B(X)$. The first assertion implies that $f_\alpha \xrightarrow{\beta} f$ and, by Lemma 1.1, the second assertion means $f$ is bounded.

3. **A Stone-Weierstrass theorem.** A Stone-Weierstrass theorem for $C(X)$ with the strict topology was established by Buck [1] subject to the condition that the algebra $\mathfrak{A}$ (see below) contains a function which vanishes nowhere. This condition was removed by Glicksberg [4] and later, in a simpler way, by Todd [11]. In all these cases the underlying space is, of course, locally compact. In this section we avoid the condition in a new way and do not impose any restriction on the space $X$.

THEOREM 3.1. *Let $X$ be any topological space and $\mathfrak{A}$ be a $\beta$-closed subalgebra of $C(X)$ which separates points and contains, for each $x$ in $X$, a function nonvanishing at $x$. Then $\mathfrak{A}=C(X)$.*

The proof uses

LEMMA 3.2. *Let $\varphi\colon \mathbf{R}\to\mathbf{R}$ be a continuous function with $\varphi(0)=0$. If $f\in\mathfrak{A}$ then $\varphi\circ f\in\mathfrak{A}$, where $\varphi\circ f$ is defined by $(\varphi\circ f)(x)=\varphi(f(x))$.*

**Proof.** Since $\mathfrak{A}$ is $\beta$-closed it is $\sigma$-closed. The lemma now follows from the Gelfand representation theorem for commutative $C^*$-algebras.

**Proof of Theorem 3.1.** Let $f\in C(X)$. We must show that, given any $\psi\in B_0(X)$, there exists $f'$ in $\mathfrak{A}$ with $\|(f-f')\psi\|<1$. For it then follows that $f$ is in the $\beta$-closure of $\mathfrak{A}$.

Choose $M$ so that $M>\|f\|$ and $M>\|\psi\|$. Given $\varepsilon>0$ choose a compact set $K\subset X$ with $\psi(x)<\varepsilon$ for $x\notin K$. Clearly, $\mathfrak{A}$ separates points of $K$ and does not vanish identically at any point of $K$. Hence, by the ordinary Stone-Weierstrass theorem, $\mathfrak{A}|K$ is $\sigma$-dense in $C(K)$. Choose $f''\in\mathfrak{A}$ so that $\|(f-f'')|K\|<\varepsilon$.

Now assume $\varepsilon<M$ and let $\varphi\colon\mathbf{R}\to\mathbf{R}$ be defined by $\varphi(\lambda)=\lambda$ for $|\lambda|\leqq 2M$, $\varphi(\lambda)=2M$ for $\lambda>2M$, $\varphi(\lambda)=-2M$ for $\lambda<-2M$. Let $f'=\varphi\circ f''$. Then $\|f'\|\leqq 2M$ and, by Lemma 3.1, $f'\in\mathfrak{A}$. We now have $|[f(x)-f'(x)]\psi(x)|<\varepsilon M$ for $x\in K$, and $|[f(x)-f'(x)]\psi(x)|<3M\varepsilon$ for $x\notin K$, so that, by choosing $\varepsilon<1/3M$, we ensure $\|(f-f')\psi\|<1$. This completes the proof.

From Theorem 2.4(v) we have

COROLLARY 3.3. *If $X$ is a k-space and $\mathfrak{A}$ is a subalgebra of $C(X)$ which separates points and vanishes identically nowhere then the $\beta$-closure of $\mathfrak{A}$ in $B(X)$ is $C(X)$.*

4. **The strict dual of $C(X)$.** For a locally compact space $X$, the dual space of $C(X)$ under the strict topology is the space of all finite regular signed Borel measures on $X$. We here extend this result to an arbitrary (not necessarily Hausdorff) completely regular space. Our measure-theoretic terminology is a simple generalization of that generally used [5] in the locally compact case.

DEFINITION 4.1. Let $X$ be any topological space. By the *Borel sets* in $X$ we mean the elements of the $\sigma$-algebra $\mathscr{B}(X)$ generated by the open sets. A *regular measure* on $X$ is a (positive countably additive) measure on a $\sigma$-algebra $\mathscr{A}\subset\mathscr{B}(X)$ such that every $A$ in $\mathscr{A}$ is

(a) *inner regular*, i.e.

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ closed compact}\},$$

and

(b) *outer regular*, i.e.

$$\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ open}\}.$$

A signed measure $\mu$ is regular iff its total variation is regular.

If $\mu=\mu^+-\mu^-$ is a Hahn decomposition of $\mu$ then $\mu$ is regular iff both $\mu^+$ and $\mu^-$ are regular. Using this fact, properties of a regular signed measure can often be deduced from the case $\mu\geqq 0$.

LEMMA 4.2. *Let $\mu$ be a finite regular signed measure on a topological space $X$. For each function $f$ in $C(X)$ let $L(f)=\int f\,d\mu$. Then $L$ is a $\beta$-continuous linear functional on $C(X)$.*

**Proof.** It is sufficient to treat the case $\mu\geqq 0$. Since every element of $C(X)$ is a bounded Borel function and $\mu$ is finite, $L(f)$ is always defined and is linear. Assume for simplicity that $\mu(X)=1$. Choose closed compact sets $K_n$, with $\varnothing=K_0\subset K_1\subset K_2\subset\cdots$, such that $\mu(K_n)\geqq 1-2^{-2n}$.

Let $\psi=\sum 2^{-n}\chi(K_n)$. Then, for $x\in K_{n+1}-K_n$, $2^{-n-1}\leqq\psi(x)\leqq 2^{-n}$. The extended real-valued Borel measurable function $1/\psi$ is $\mu$-integrable, indeed,

$$\int(1/\psi)\,d\mu = \sum_{n=0}^{\infty}[\mu(K_{n+1})-\mu(K_n)]2^n \leqq \sum_{n=0}^{\infty}2^{-n} = 2.$$

Now suppose $\varepsilon>0$. Let $f\in C(X)$. Then, if $f\in U_{2\psi/\varepsilon}$, $\|2f\psi/\varepsilon\|<1$ so that $|f|<\varepsilon/2\psi$ whence $|\int f\,d\mu|\leqq\int|f|\,d\mu\leqq\int(\varepsilon/2\psi)\,d\mu\leqq\varepsilon$. This proves the $\beta$-continuity of $L$.

In order to apply the Riesz representation theorem we now relate the regular Borel measures on a completely regular Hausdorff space to those on its Stone-Čech compactification.

LEMMA 4.3. *Let $X$ be a completely regular Hausdorff space. We denote by $\beta X$ its Stone-Čech compactification. For any regular signed Borel measure $\nu$ on $\beta X$ we say $\nu$ satisfies the condition* (1) *iff*

$$|\nu|(\beta X) = \sup\{|\nu|(K) : K\subset X, K \text{ compact}\} \tag{1}$$

*or, equivalently, iff there is a $\sigma$-compact set $J\subset X$ such that $|\nu|(J)=|\nu|(\beta X)$. Then*

(a) *For any finite regular signed Borel measure $\mu$ on $X$ let $\mu'$ denote the set function defined for each $A$ in $\mathscr{B}(\beta X)$ by $\mu'(A)=\mu(A\cap X)$. Then $\mu'$ is a finite regular signed Borel measure on $\beta X$ satisfying the condition* (1).

(b) *Conversely, if $\nu$ is any finite regular signed Borel measure on $\beta X$ satisfying the condition* (1) *then there is a unique finite regular signed Borel measure $\mu$ on $X$ such that $\nu=\mu'$.*

**Proof.** (a) Assume first that $\mu \geqq 0$. It is clear that $\mu'$ agrees with $\mu$ on $\mathscr{B}(X)$ and satisfies condition (1) by the regularity of $\mu$. It remains to establish the regularity of $\mu'$.

Let $A' \in \mathscr{B}(\beta X)$ and $A=A' \cap X$. Then $A \in \mathscr{B}(X)$ and

$$\begin{aligned}\mu'(A') &= \mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\} \\ &\leqq \sup\{\mu'(K) : K \subset A', K \text{ compact}\} \leqq \mu'(A').\end{aligned}$$

Thus $A'$ is inner regular. On the other hand, since $A$ is outer regular, given $\varepsilon>0$ there is a set $V$, relatively open in $X$, with $A\subset V\subset X$ and $\mu(V)<\mu(A)+\varepsilon$. Let $V'$ be an open set in $\beta X$ with $V' \cap X=V$, so that $\mu'(V')=\mu(V)$. By condition (1) there is a compact set $K\subset X$ with $\mu(X)-\mu(K)<\varepsilon$. Putting $W'=\beta X-K$, $W'$ is open in $\beta X$ and $\mu'(W')<\varepsilon$. Let $U'=V' \cup W'$. Since $W'\supset\beta X-X$, $U'\supset A \cup (\beta X-X)\supset A'$. Also $U'$ is open and $\mu'(U')\leqq\mu'(V')+\mu'(W')<\mu(A)+2\varepsilon=\mu'(A')+2\varepsilon$. Thus $A'$ is outer regular. This completes the proof in the case $\mu\geqq 0$. The general case follows easily by using a Hahn decomposition for $\mu$.

(b) Any regular Borel measure is determined by its values on closed compact sets. The uniqueness of $\mu$ thus follows from the fact that if $K$ is any compact subset of $X$ then $K\in\mathscr{B}(\beta X)$ so that $\mu(K)=\nu(K)$. We establish the existence of $\mu$. By condition (1) there is a $\sigma$-compact set $J\subset X$ with $|\nu|(\beta X-J)=0$. For the rest of the proof we consider first the case $\nu\geqq 0$. Let $\hat{\nu}$ be the completion of $\nu$. Then $X$, and hence every Borel set in $X$, is $\hat{\nu}$-measurable. Let $\mu$ be the restriction of $\hat{\nu}$ to $\mathscr{B}(X)$. Clearly, $\hat{\nu}$ is regular and it follows easily from this that $\mu$ is regular. Lastly, for each $A'$ in $\mathscr{B}(\beta X)$, $\nu(A')=\hat{\nu}(X \cap A')=\mu(X \cap A')$, which shows that $\mu'=\nu$.

For the general case ($\nu\geqq 0$) it is sufficient to observe that if $\nu=\nu^+-\nu^-$ is a Hahn decomposition of $\nu$, so that $|\nu|=\nu^+ +\nu^-$, then $\nu^+(\beta X-J)=\nu^-(\beta X-J)=0$. The above argument can then be applied to $\nu^+$ and $\nu^-$ to obtain the positive and negative parts of $\mu$.

Corollary 4.4. *The mapping $\mu\mapsto\mu'$ is a bijection between the finite regular signed Borel measures on $X$ and those finite regular signed Borel measures on $\beta X$ that satisfy the condition* (1). *Moreover, for every $f$ in $C(X)$, $\int f\,d\mu=\int f'\,d\mu'$, where $f'\in C(\beta X)$ is the unique continuous extension of $f$.*

**Proof.** Let $J\subset X$ be a $\sigma$-compact set with $|\mu|(X-J)=0$. Then $|\mu'|(\beta X-J)=0$ too, while on $J$ both the functions $f$ and $f'$ and the measures $\mu$ and $\mu'$ coincide.

We can now prove a converse to Lemma 4.2:

Lemma 4.5. *Let $X$ be a completely regular space* (*not necessarily Hausdorff*) *and let $L$ be a $\beta$-continuous linear functional on $C(X)$. Then there is a unique regular signed Borel measure $\mu$ on $X$ such that $L(f)=\int f\,d\mu$ for all $f$ in $C(X)$.*

**Proof.** First assume that $X$ is Hausdorff. Since $L$ is $\beta$-continuous it is certainly $\sigma$-continuous. Now, the canonical isomorphism of $C(X)$ onto $C(\beta X)$, which assigns to each bounded continuous function $f$ on $X$ its unique continuous extension $f'$ on $\beta X$, is an isometry for the sup norm. So, by the Riesz representation theorem, there is a unique regular signed Borel measure $\nu$ on $\beta X$ such that $L(f)=\int_{\beta X} f'\, d\nu$ for every $f$ in $C(X)$.

We claim that the measure $\nu$ satisfies condition (1) of Lemma 4.3. Indeed, suppose that this condition is not satisfied. Then there is an $\varepsilon>0$ such that $|\nu|(\beta X-K)>\varepsilon$ for every compact set $K\subset X$. Now, since $L$ is $\beta$-continuous there is a function $\psi$ in $B_0(X)$ such that $\int_{\beta X} f'\, d\nu=L(f)<1$ whenever $f\in C(X)$ and $\|f\psi\|\leqq 1$. Let $K\subset X$ be a compact set such that $|\psi(x)|<\varepsilon$ for $x\in X-K$. Then $M=\beta X-K$ is a locally compact space, and the restriction $\nu_M$ of $\nu$ to $\mathscr{B}(M)$ is a bounded signed regular Borel measure on $M$. However, by the Riesz representation theorem, the set of all such measures is the dual $C_0(M)^*$ of the space $C_0(M)$ of all continuous real-valued functions vanishing at infinity on $M$. Regarded as an element of $C_0(M)^*$, $\nu_M$ has the norm $|\nu_M|(M)=|\nu|(\beta X-K)>\varepsilon$. Hence, by the Hahn-Banach theorem, there is a function $f_M$ in $C_0(M)$ with $\|f_M\|<1$ and $\int_M f_M\, d\nu_M>\varepsilon$. Let $f'$ in $C(\beta X)$ be the extension of $f_M$ obtained by setting $f'(x)=0$ for $x\in K$ and let $f=f'|X$. Then $\int_{\beta X} f'\, d\nu>\varepsilon$. On the other hand $\|f\psi\|<\varepsilon$ whence, by the choice of $\psi$, $|\int_{\beta X} f'\, d\nu|<\varepsilon$, which is a contradiction.

It now follows from Lemma 4.3 that there is a bounded signed regular Borel measure $\mu$ on $X$ such that $\mu'=\nu$ and we then have $L(f)=\int_X f\, d\mu$ for every $f$ in $C(X)$.

Now suppose $X$ is completely regular but not Hausdorff. Introduce the quotient space $Y=X/\sim$, where $x\sim y$ means $f(x)=f(y)$ for every $f$ in $C(X)$. Then [8, p. 155] $Y$ is completely regular and Hausdorff and the canonical map of $X$ onto $Y$ establishes a bijection between $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ under which open sets and closed compact sets are preserved. There is thus a natural one-to-one correspondence between the regular signed Borel measures on $X$ and those on $Y$. Moreover, the natural isomorphism [3, p. 41] of $C(Y)$ onto $C(X)$ is a homeomorphism for the strict topology—this follows from the easily established fact that the closure of each compact set in $X$ is compact. The validity of the lemma for the space $X$ is thus an immediate consequence of its validity for $Y$.

From Lemmas 4.2 and 4.5 we obtain

THEOREM 4.6. *For any completely regular space $X$ the dual of $C(X)$ under the strict topology is the space of all bounded signed regular Borel measures on $X$.*

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