

A GENERALIZATION OF THE STRICT TOPOLOGY⁽¹⁾

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Abstract. The strict topology β on the space $C(X)$ of bounded real-valued continuous functions on a topological space X was defined, for locally compact X , by Buck (Michigan Math. J. 5 (1958), 95–104). Among other things he showed that (a) $C(X)$ is β -complete, (b) the dual of $C(X)$ under the strict topology is the space of all finite signed regular Borel measures on X , and (c) a Stone-Weierstrass theorem holds for β -closed subalgebras of $C(X)$. In this paper the definition of the strict topology is generalized to cover the case of an arbitrary topological space and these results are established under the following conditions on X : for (a) X is a k -space; for (b) X is completely regular; for (c) X is unrestricted.

1. Introduction and notation. Let X be a topological space. We denote by $B(X)$ the algebra of all bounded real-valued functions on X and by $C(X)$ the subalgebra of $B(X)$ consisting of continuous functions. $B_0(X)$ denotes the ideal in $B(X)$ consisting of functions *vanishing at infinity*, in that for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$, and $C_0(X) = B_0(X) \cap C(X)$. Note that $B_0(X)$ contains

- (a) the characteristic function $\chi(K)$ of each compact set $K \subset X$;
 - (b) every function ψ of the form $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(K_n)$, where $\alpha_n \geq 0$ for all n , $\alpha_n \rightarrow 0$, and the sets K_n are compact and disjoint; and so, in particular,
 - (c) the function $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(\{x_n\})$, where (x_n) is any sequence of distinct points.
- We shall need the following lemma:

LEMMA 1.1. *If f is a real-valued function on X and $f\psi$ is bounded for every ψ in $B_0(X)$ then f is bounded.*

Proof. Suppose f is not bounded. Choose a sequence (x_n) in X with $|f(x_n)| \rightarrow \infty$ and put $\psi = \sum |f(x_n)|^{-1/2} \chi(\{x_n\})$. Then $\psi \in B_0(X)$ but $f\psi$ is not bounded.

The strict topology on $C(X)$ was defined for locally compact X by Buck [1] by means of a set of seminorms determined by the elements of $C_0(X)$. If X is completely regular but not locally compact, $C_0(X)$ may be very small (for instance, if X is the rationals [3, p. 109]) and does not yield a useful topology for $C(X)$. We claim that in this case the natural generalization of the strict topology is obtained by letting the role of $C_0(X)$ be played by $B_0(X)$; the change makes no difference if

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X is locally compact. Indeed, under mild conditions on X , we shall prove that, with this generalized strict topology, $C(X)$ is complete (Theorem 2.4) and its dual is the space of bounded signed regular Borel measures on X (Theorem 4.6), and we shall establish a Stone-Weierstrass theorem for $C(X)$ (Theorem 3.1).

For the first of these results we assume that X is a k -space [6], [7], [10], [12], [13], i.e. a space in which a set is closed if its intersection with every closed compact set is closed. The limitation to k -spaces is not a serious restriction. Every locally compact space is a k -space; so is every metrisable space. Although there do exist [6] completely regular spaces which are not k -spaces, such spaces do not seem to be important. Indeed Steenrod [10] has made a strong case for formulating topology entirely within the category of Hausdorff k -spaces.

At the same time as making the change from $C_0(X)$ to $B_0(X)$ it is natural to define the strict topology, in the first instance, on $B(X)$. In this form, our strict topology is a special case of generalizations introduced independently by Busby [2] and Sentilles and Taylor [9] in the context of Banach algebras.

Note added in proof. Recently, and independently, Gulick [14] and Sentilles [15] have also discussed the strict topology for $C(X)$, for completely regular X .

2. Topologies on $B(X)$. Let X be a set. Corresponding to each function ψ in $B(X)$ we define a seminorm p_ψ on $B(X)$ by writing, for every f in $B(X)$, $p_\psi(f) = \|\psi f\|$, where $\|\ \ \|$ denotes the sup norm.

Let $\Psi \subset B(X)$ be any subset. By the Ψ -topology on $B(X)$ we mean the topology determined by the set of seminorms $\{p_\psi : \psi \in \Psi\}$. A basis of open neighbourhoods for the Ψ -topology is $\{U_\psi : \psi \in \Psi\}$, where $U_\psi = \{f \in B(X) : p_\psi(f) < 1\}$.

Let $\Psi \subset B(X)$, $\Psi' \subset B(X)$, $\psi \in B(X)$, $\psi' \in B(X)$. If, for some constant λ , $\lambda|\psi'| \geq |\psi|$ we say ψ' dominates ψ ; if ψ' dominates every element of Ψ we say ψ' dominates Ψ . The proof of the following lemma is easy.

LEMMA 2.1. *If every element of Ψ is dominated by some element of Ψ' then the Ψ' -topology on $B(X)$ is finer than the Ψ -topology.*

LEMMA 2.2. *If ψ' dominates Ψ then $U_{\psi'}$ is a Ψ -bounded set (i.e. bounded in the Ψ -topology). Moreover, given any Ψ -bounded set $B \subset B(X)$ there is a ψ' dominating Ψ with $B \subset U_{\psi'}$.*

Proof. Given ψ in Ψ choose λ so that $\lambda|\psi'| \geq |\psi|$. Then, for any f in $B(X)$, $p_\psi(f) \leq \lambda p_{\psi'}(f)$, so that p_ψ is bounded on $U_{\psi'}$. Since this is true for every ψ , $U_{\psi'}$ is Ψ -bounded.

Now suppose $B \subset B(X)$ is Ψ -bounded. Then, for each ψ in Ψ we can choose λ_ψ such that $B \subset \lambda_\psi U_\psi$; clearly, we may also assume $\lambda_\psi \geq \|\psi\|$. But then, for all ψ in Ψ ,

$$f \in B \Rightarrow \|f\psi/\lambda_\psi\| < 1.$$

Thus $B \subset U_{\psi_0}$, where $\psi_0 = \sup \{|\psi|/\lambda_\psi : \psi \in \Psi\} \in B(X)$. Clearly, ψ_0 dominates each ψ in Ψ .

COROLLARY 2.3. *If every function which dominates Ψ dominates Ψ' then every Ψ -bounded set is Ψ' -bounded.*

Now let X be a topological space. We introduce four topologies on $B(X)$:

(a) σ , the uniform (or sup norm) topology, is the $B(X)$ -topology or, equivalently (by Lemma 2.1), the $\{\mathbf{1}\}$ -topology. (We denote by $\mathbf{1}$ the unit function on X .)

(b) β , our generalized strict topology, is the $B_0(X)$ -topology.

(c) κ , the topology of compact convergence, is the Ψ_κ -topology, where $\Psi_\kappa = \{\psi \in B(X) : \psi \text{ has compact support}\}$.

(d) ρ , the topology of pointwise convergence, is the Ψ_ρ -topology, where $\Psi_\rho = \{\psi \in B(X) : \psi \text{ has finite support}\}$.

The following theorem gives the main properties of the strict topology β . The proofs are similar to those of Buck [1].

THEOREM 2.4. *Let X be any topological space. Let the topologies σ , β , κ , ρ on $B(X)$ be defined as above. Then*

- (i) $\sigma \supset \beta \supset \kappa \supset \rho$.
- (ii) *If X is locally compact then β coincides with the strict topology as defined by Buck [1].*
- (iii) β and σ have the same bounded sets.
- (iv) *On any σ -bounded set B the topologies β and κ coincide.*
- (v) *If X is a k -space then $C(X)$ is β -complete.*

Proof. (i) follows at once from Lemma 2.1.

(ii) Let K be locally compact. By Lemma 2.2, it is sufficient to show that each ψ in $B_0(X)$ is dominated by some ψ' in $C_0(X)$. We may clearly assume $\|\psi\| < 1$. Choose compact sets K_n with $\emptyset = K_0 \subset K_1 \subset K_2 \subset \dots$ such that $|\psi(x)| < 2^{-n}$ for $x \notin K_n$. Choose ψ_n in $C_0(X)$ with $\psi_n(x) = 2^{-n}$ for $x \in K_n$ and $0 \leq \psi_n \leq 2^{-n}\mathbf{1}$. Let $\psi' = \sum_{n=0}^{\infty} \psi_n$. Then $\psi' \in C_0(X)$ and ψ' dominates ψ .

(iii) Since $\sigma \supset \beta$ and since σ is the $\{\mathbf{1}\}$ -topology on $B(X)$ it suffices, by Corollary 2.3, to show that if ψ_0 in $B(X)$ dominates $B_0(X)$ then ψ_0 dominates $\mathbf{1}$. Suppose, then, that ψ_0 does not dominate $\mathbf{1}$. Then there is a sequence (x_n) in X with $|\psi_0(x_n)| \rightarrow 0$. Let $\psi = \sum_{n=1}^{\infty} |\psi_0(x_n)|^{1/2} \chi(\{x_n\})$. Then $\psi \in B_0(X)$ but ψ_0 does not dominate ψ .

(iv) Choose M so that $\|f\| < M$ whenever $f \in B$. By (i), it suffices to show that the β -closure B_β of B contains the κ -closure B_κ of B . Suppose $g \in B_\kappa$. Given any ψ in $B_0(X)$ and any $\varepsilon > 0$ choose a compact set $K \subset X$ with $|\psi(x)| < \varepsilon$ for $x \notin K$. Let $\psi' = \psi \chi(K)$. Then $\psi' \in \Psi_\kappa$ and, for every f in B , $p_\psi(f-g) = \|(f-g)\psi\| \leq \|(f-g)\psi'\| + \|(f-g)(\psi-\psi')\| \leq p_{\psi'}(f-g) + (M + \|g\|)\varepsilon$. Since $g \in B_\kappa$ and ε is arbitrary, this gives $\inf \{p_\psi(f-g) : f \in B\} = 0$. Since this holds for all ψ in $B_0(X)$, $g \in B_\beta$.

(v) Let $\{f_\alpha\}$ be a β -Cauchy net in $C(X)$. By (i), $\{f_\alpha\}$ is κ -Cauchy, and hence [6], since X is a k -space, $f_\alpha \xrightarrow{\kappa} f$ where f is a continuous function on X . It remains to show that f is bounded and that $f_\alpha \xrightarrow{\beta} f$.

Now, for each ψ in $B_0(X)$, $\{\psi f_\alpha\}$ is a σ -Cauchy net in $B(X)$. Since $B(X)$ is σ -

complete, $\psi f_\alpha \xrightarrow{\sigma} g$ for some g in $B(X)$. But then $\psi f_\alpha \xrightarrow{\rho} g$ whereas also $\psi f_\alpha \xrightarrow{\rho} \psi f$ so that $g = \psi f$.

We have thus shown that, for each ψ in $B_0(X)$, $\psi f_\alpha \xrightarrow{\sigma} \psi f$ and $\psi f \in B(X)$. The first assertion implies that $f_\alpha \xrightarrow{\beta} f$ and, by Lemma 1.1, the second assertion means f is bounded.

3. A Stone-Weierstrass theorem. A Stone-Weierstrass theorem for $C(X)$ with the strict topology was established by Buck [1] subject to the condition that the algebra \mathfrak{A} (see below) contains a function which vanishes nowhere. This condition was removed by Glicksberg [4] and later, in a simpler way, by Todd [11]. In all these cases the underlying space is, of course, locally compact. In this section we avoid the condition in a new way and do not impose any restriction on the space X .

THEOREM 3.1. *Let X be any topological space and \mathfrak{A} be a β -closed subalgebra of $C(X)$ which separates points and contains, for each x in X , a function nonvanishing at x . Then $\mathfrak{A} = C(X)$.*

The proof uses

LEMMA 3.2. *Let $\varphi: \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function with $\varphi(0) = 0$. If $f \in \mathfrak{A}$ then $\varphi \circ f \in \mathfrak{A}$, where $\varphi \circ f$ is defined by $(\varphi \circ f)(x) = \varphi(f(x))$.*

Proof. Since \mathfrak{A} is β -closed it is σ -closed. The lemma now follows from the Gelfand representation theorem for commutative C^* -algebras.

Proof of Theorem 3.1. Let $f \in C(X)$. We must show that, given any $\psi \in B_0(X)$, there exists f' in \mathfrak{A} with $\|(f - f')\psi\| < 1$. For it then follows that f is in the β -closure of \mathfrak{A} .

Choose M so that $M > \|f\|$ and $M > \|\psi\|$. Given $\varepsilon > 0$ choose a compact set $K \subset X$ with $\psi(x) < \varepsilon$ for $x \notin K$. Clearly, \mathfrak{A} separates points of K and does not vanish identically at any point of K . Hence, by the ordinary Stone-Weierstrass theorem, $\mathfrak{A}|_K$ is σ -dense in $C(K)$. Choose $f'' \in \mathfrak{A}$ so that $\|(f - f'')|_K\| < \varepsilon$.

Now assume $\varepsilon < M$ and let $\varphi: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $\varphi(\lambda) = \lambda$ for $|\lambda| \leq 2M$, $\varphi(\lambda) = 2M$ for $\lambda > 2M$, $\varphi(\lambda) = -2M$ for $\lambda < -2M$. Let $f' = \varphi \circ f''$. Then $\|f'\| \leq 2M$ and, by Lemma 3.1, $f' \in \mathfrak{A}$. We now have $|[f(x) - f'(x)]\psi(x)| < \varepsilon M$ for $x \in K$, and $|[f(x) - f'(x)]\psi(x)| < 3M\varepsilon$ for $x \notin K$, so that, by choosing $\varepsilon < 1/3M$, we ensure $\|(f - f')\psi\| < 1$. This completes the proof.

From Theorem 2.4(v) we have

COROLLARY 3.3. *If X is a k -space and \mathfrak{A} is a subalgebra of $C(X)$ which separates points and vanishes identically nowhere then the β -closure of \mathfrak{A} in $B(X)$ is $C(X)$.*

4. The strict dual of $C(X)$. For a locally compact space X , the dual space of $C(X)$ under the strict topology is the space of all finite regular signed Borel measures on X . We here extend this result to an arbitrary (not necessarily Hausdorff) completely regular space. Our measure-theoretic terminology is a simple generalization of that generally used [5] in the locally compact case.

DEFINITION 4.1. Let X be any topological space. By the *Borel sets* in X we mean the elements of the σ -algebra $\mathcal{B}(X)$ generated by the open sets. A *regular measure* on X is a (positive countably additive) measure on a σ -algebra $\mathcal{A} \subset \mathcal{B}(X)$ such that every A in \mathcal{A} is

(a) *inner regular*, i.e.

$$\mu(A) = \sup \{ \mu(K) : K \subset A, K \text{ closed compact} \},$$

and

(b) *outer regular*, i.e.

$$\mu(A) = \inf \{ \mu(U) : A \subset U, U \text{ open} \}.$$

A signed measure μ is regular iff its total variation is regular.

If $\mu = \mu^+ - \mu^-$ is a Hahn decomposition of μ then μ is regular iff both μ^+ and μ^- are regular. Using this fact, properties of a regular signed measure can often be deduced from the case $\mu \geq 0$.

LEMMA 4.2. Let μ be a finite regular signed measure on a topological space X . For each function f in $C(X)$ let $L(f) = \int f d\mu$. Then L is a β -continuous linear functional on $C(X)$.

Proof. It is sufficient to treat the case $\mu \geq 0$. Since every element of $C(X)$ is a bounded Borel function and μ is finite, $L(f)$ is always defined and is linear. Assume for simplicity that $\mu(X) = 1$. Choose closed compact sets K_n , with $\emptyset = K_0 \subset K_1 \subset K_2 \subset \dots$, such that $\mu(K_n) \geq 1 - 2^{-2n}$.

Let $\psi = \sum_{n=0}^{\infty} 2^{-n} \chi(K_n)$. Then, for $x \in K_{n+1} - K_n$, $2^{-n-1} \leq \psi(x) \leq 2^{-n}$. The extended real-valued Borel measurable function $1/\psi$ is μ -integrable, indeed,

$$\int (1/\psi) d\mu = \sum_{n=0}^{\infty} [\mu(K_{n+1}) - \mu(K_n)] 2^n \leq \sum_{n=0}^{\infty} 2^{-n} = 2.$$

Now suppose $\varepsilon > 0$. Let $f \in C(X)$. Then, if $f \in U_{2\psi/\varepsilon}$, $\|2f\psi/\varepsilon\| < 1$ so that $|f| < \varepsilon/2\psi$ whence $|\int f d\mu| \leq \int |f| d\mu \leq \int (\varepsilon/2\psi) d\mu \leq \varepsilon$. This proves the β -continuity of L .

In order to apply the Riesz representation theorem we now relate the regular Borel measures on a completely regular Hausdorff space to those on its Stone-Ćech compactification.

LEMMA 4.3. Let X be a completely regular Hausdorff space. We denote by βX its Stone-Ćech compactification. For any regular signed Borel measure ν on βX we say ν satisfies the condition (1) iff

$$(1) \quad |\nu|(\beta X) = \sup \{ |\nu|(K) : K \subset X, K \text{ compact} \}$$

or, equivalently, iff there is a σ -compact set $J \subset X$ such that $|\nu|(J) = |\nu|(\beta X)$. Then

(a) For any finite regular signed Borel measure μ on X let μ' denote the set function defined for each A in $\mathcal{B}(\beta X)$ by $\mu'(A) = \mu(A \cap X)$. Then μ' is a finite regular signed Borel measure on βX satisfying the condition (1).

(b) *Conversely, if ν is any finite regular signed Borel measure on βX satisfying the condition (1) then there is a unique finite regular signed Borel measure μ on X such that $\nu = \mu'$.*

Proof. (a) Assume first that $\mu \geq 0$. It is clear that μ' agrees with μ on $\mathcal{B}(X)$ and satisfies condition (1) by the regularity of μ . It remains to establish the regularity of μ' .

Let $A' \in \mathcal{B}(\beta X)$ and $A = A' \cap X$. Then $A \in \mathcal{B}(X)$ and

$$\begin{aligned} \mu'(A') &= \mu(A) = \sup \{ \mu(K) : K \subset A, K \text{ compact} \} \\ &\leq \sup \{ \mu'(K) : K \subset A', K \text{ compact} \} \leq \mu'(A'). \end{aligned}$$

Thus A' is inner regular. On the other hand, since A is outer regular, given $\varepsilon > 0$ there is a set V , relatively open in X , with $A \subset V \subset X$ and $\mu(V) < \mu(A) + \varepsilon$. Let V' be an open set in βX with $V' \cap X = V$, so that $\mu'(V') = \mu(V)$. By condition (1) there is a compact set $K \subset X$ with $\mu(X) - \mu(K) < \varepsilon$. Putting $W' = \beta X - K$, W' is open in βX and $\mu'(W') < \varepsilon$. Let $U' = V' \cup W'$. Since $W' \supset \beta X - X$, $U' \supset A \cup (\beta X - X) \supset A'$. Also U' is open and $\mu'(U') \leq \mu'(V') + \mu'(W') < \mu(A) + 2\varepsilon = \mu'(A') + 2\varepsilon$. Thus A' is outer regular. This completes the proof in the case $\mu \geq 0$. The general case follows easily by using a Hahn decomposition for μ .

(b) Any regular Borel measure is determined by its values on closed compact sets. The uniqueness of μ thus follows from the fact that if K is any compact subset of X then $K \in \mathcal{B}(\beta X)$ so that $\mu(K) = \nu(K)$. We establish the existence of μ . By condition (1) there is a σ -compact set $J \subset X$ with $|\nu|(\beta X - J) = 0$. For the rest of the proof we consider first the case $\nu \geq 0$. Let $\hat{\nu}$ be the completion of ν . Then X , and hence every Borel set in X , is $\hat{\nu}$ -measurable. Let μ be the restriction of $\hat{\nu}$ to $\mathcal{B}(X)$. Clearly, $\hat{\nu}$ is regular and it follows easily from this that μ is regular. Lastly, for each A' in $\mathcal{B}(\beta X)$, $\nu(A') = \hat{\nu}(X \cap A') = \mu(X \cap A')$, which shows that $\mu' = \nu$.

For the general case ($\nu \geq 0$) it is sufficient to observe that if $\nu = \nu^+ - \nu^-$ is a Hahn decomposition of ν , so that $|\nu| = \nu^+ + \nu^-$, then $\nu^+(\beta X - J) = \nu^-(\beta X - J) = 0$. The above argument can then be applied to ν^+ and ν^- to obtain the positive and negative parts of μ .

COROLLARY 4.4. *The mapping $\mu \mapsto \mu'$ is a bijection between the finite regular signed Borel measures on X and those finite regular signed Borel measures on βX that satisfy the condition (1). Moreover, for every f in $C(X)$, $\int f d\mu = \int f' d\mu'$, where $f' \in C(\beta X)$ is the unique continuous extension of f .*

Proof. Let $J \subset X$ be a σ -compact set with $|\mu|(X - J) = 0$. Then $|\mu'|(\beta X - J) = 0$ too, while on J both the functions f and f' and the measures μ and μ' coincide.

We can now prove a converse to Lemma 4.2:

LEMMA 4.5. *Let X be a completely regular space (not necessarily Hausdorff) and let L be a β -continuous linear functional on $C(X)$. Then there is a unique regular signed Borel measure μ on X such that $L(f) = \int f d\mu$ for all f in $C(X)$.*

Proof. First assume that X is Hausdorff. Since L is β -continuous it is certainly σ -continuous. Now, the canonical isomorphism of $C(X)$ onto $C(\beta X)$, which assigns to each bounded continuous function f on X its unique continuous extension f' on βX , is an isometry for the sup norm. So, by the Riesz representation theorem, there is a unique regular signed Borel measure ν on βX such that $L(f) = \int_{\beta X} f' d\nu$ for every f in $C(X)$.

We claim that the measure ν satisfies condition (1) of Lemma 4.3. Indeed, suppose that this condition is not satisfied. Then there is an $\epsilon > 0$ such that $|\nu|(\beta X - K) > \epsilon$ for every compact set $K \subset X$. Now, since L is β -continuous there is a function ψ in $B_0(X)$ such that $\int_{\beta X} f' d\nu = L(f) < 1$ whenever $f \in C(X)$ and $\|f\psi\| \leq 1$. Let $K \subset X$ be a compact set such that $|\psi(x)| < \epsilon$ for $x \in X - K$. Then $M = \beta X - K$ is a locally compact space, and the restriction ν_M of ν to $\mathcal{B}(M)$ is a bounded signed regular Borel measure on M . However, by the Riesz representation theorem, the set of all such measures is the dual $C_0(M)^*$ of the space $C_0(M)$ of all continuous real-valued functions vanishing at infinity on M . Regarded as an element of $C_0(M)^*$, ν_M has the norm $|\nu_M|(M) = |\nu|(\beta X - K) > \epsilon$. Hence, by the Hahn-Banach theorem, there is a function f_M in $C_0(M)$ with $\|f_M\| < 1$ and $\int_M f_M d\nu_M > \epsilon$. Let f' in $C(\beta X)$ be the extension of f_M obtained by setting $f'(x) = 0$ for $x \in K$ and let $f = f'|_X$. Then $\int_{\beta X} f' d\nu > \epsilon$. On the other hand $\|f\psi\| < \epsilon$ whence, by the choice of ψ , $|\int_{\beta X} f' d\nu| < \epsilon$, which is a contradiction.

It now follows from Lemma 4.3 that there is a bounded signed regular Borel measure μ on X such that $\mu' = \nu$ and we then have $L(f) = \int_X f d\mu$ for every f in $C(X)$.

Now suppose X is completely regular but not Hausdorff. Introduce the quotient space $Y = X/\sim$, where $x \sim y$ means $f(x) = f(y)$ for every f in $C(X)$. Then [8, p. 155] Y is completely regular and Hausdorff and the canonical map of X onto Y establishes a bijection between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ under which open sets and closed compact sets are preserved. There is thus a natural one-to-one correspondence between the regular signed Borel measures on X and those on Y . Moreover, the natural isomorphism [3, p. 41] of $C(Y)$ onto $C(X)$ is a homeomorphism for the strict topology—this follows from the easily established fact that the closure of each compact set in X is compact. The validity of the lemma for the space X is thus an immediate consequence of its validity for Y .

From Lemmas 4.2 and 4.5 we obtain

THEOREM 4.6. *For any completely regular space X the dual of $C(X)$ under the strict topology is the space of all bounded signed regular Borel measures on X .*

REFERENCES

1. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104. MR **21** #4350.
2. R. C. Busby, *Double centralizers and extensions of C^* -algebras*; Trans. Amer. Math. Soc. **132** (1968), 79–99. MR **37** #770.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

4. I. Glicksberg, *Bishop's generalized Stone-Weierstrass theorem for the strict topology*, Proc. Amer. Math. Soc. **14** (1963), 329–333. MR **26** #4165.
5. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965. MR **32** #5826.
6. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.
7. J. L. Kelley, I. Namioka et al., *Linear topological spaces*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.
8. W. J. Pervin, *Foundations of general topology*, Academic Press Textbooks in Math., Academic Press, New York, 1964. MR **29** #2759.
9. F. D. Sentilles and D. C. Taylor, *Factorization in Banach algebras and the general strict topology*, Trans. Amer. Math. Soc. **142** (1969), 141–152. MR **40** #703.
10. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR **35** #970.
11. C. Todd, *Stone-Weierstrass theorems for the strict topology*, Proc. Amer. Math. Soc. **16** (1965), 654–659. MR **31** #3891.
12. E. Wattel, *The compactness operator in set theory and topology*, Mathematical Centre Tracts, 21, Mathematisch Centrum, Amsterdam, 1968. MR **39** #7551.
13. D. D. Weddington, *On k -spaces*, Proc. Amer. Math. Soc. **22** (1969), 635–638. MR **40** #2001.
14. D. Gulick, *The σ -compact-open topology and its relatives* (preprint).
15. F. D. Sentilles, *Bounded continuous functions on a completely regular space* (preprint).

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