MONOFUNCTORS AS REFLECTORS

BY

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Abstract. In a well-powered and co-well-powered complete category $\mathscr{K}$ with weak amalgamations, the class M of all reflective subcategories with a monofunctor as reflector forms a complete lattice; the limit-closure of the union of any class of elements of M belongs to M. If $\mathscr{K}$ has injective envelopes, then the set-theoretical intersection of any class of elements of M belongs to M.

1. **Introduction.** Let $\mathscr{K}$ be a well-powered and co-well-powered complete category. It is well known that for all limit-closed subcategories([2]) $\mathscr{X}$ of $\mathscr{K}$ the full subcategory $s\mathscr{X}$ of all subobjects of objects of $\mathscr{X}$ is a reflective subcategory of $\mathscr{K}$. We say that $\mathscr{K}$ has amalgamations([3]) iff for any two monomorphisms $m_i: A \to B_i$ $(i=1, 2)$ there is a commutative square of the form

$$\begin{array}{ccc} & B_1 & \\ m_1 \nearrow & & \searrow n_1 \\ A & & C \\ m_2 \searrow & & \nearrow n_2 \\ & B_2 & \end{array}$$

where n_1 and n_2 are monomorphisms. If we assume that $\mathscr{K}$ has amalgamations, then any limit-closed subcategory $\mathscr{X}$ with $s\mathscr{X} = \mathscr{K}$ is reflective; more generally, a limit-closed subcategory $\mathscr{X}$ for which the reflector of $\mathscr{K}$ onto $s\mathscr{X}$ is a monofunctor is reflective and has itself a monofunctor as reflector (Theorem 1). We denote by M the class of all reflective subcategories with a monofunctor as reflector. Thus a limit-closed subcategory $\mathscr{X}$ belongs to M if and only if $s\mathscr{X}$ belongs to M. M is partially ordered by inclusion and has a smallest element. We show that the limit-closure of the union of any class of elements of M belongs to M (Theorem 2); therefore M is a complete lattice. A category with injective envelopes always has amalgamations. In these categories the set-theoretical intersection of any class of members of M also belongs to M (Theorem 4).

([2]) In this paper "subcategory" always means *full* and *replete* subcategory.

([3]) In the terminology of [3], $\mathscr{K}$ has *weak* amalgamations.

In the category of all modules over a ring R a limit-closed subcategory $\mathscr{X}$ belongs to M iff there is a torsion-theory $(\mathscr{P}, \mathscr{Q})$ with $\mathrm{s}\mathscr{X} = \mathscr{Q}$. If $\mathscr{X}$ is the functorcategory $[\mathscr{C}, \mathfrak{S}]$, where $\mathscr{C}$ is small and $\mathfrak{S}$ is the category of sets, then $\mathscr{X}$ belongs to M iff $\mathrm{s}\mathscr{X}$ is the category of all separated presheaves with respect to some Grothendieck-topology.

2. **The reflectivity of limit-closed subcategories.** The following remark will be useful in the sequel. *In a category with amalgamations every bimorphism is an essential monomorphism.* For given a bimorphism b and a morphism x such that xb is a monomorphism, we find a commuting square

$$
\begin{array}{ccc}
 & \overset{b}{\nearrow} \cdot \overset{n_1}{\searrow} & \\
\cdot & & \cdot \\
 & \underset{xb}{\searrow} \cdot \underset{n_2}{\nearrow} &
\end{array}
$$

where n_1 and n_2 are monomorphisms. But b is an epimorphism, therefore the equality $n_1 b = n_2 x b$ implies $n_1 = n_2 x$. So x is a monomorphism.

LEMMA 1. *Let $\mathscr{K}$ be a well-powered and complete category with amalgamations. If $\mathscr{X}$ is a limit-closed subcategory and $\mathrm{s}\mathscr{X} = \mathscr{K}$, then $\mathscr{X}$ is reflective, and has a monofunctor as reflector.*

Proof. For any object K of $\mathscr{K}$ there is a monomorphism $m: K \to X$ with X in $\mathscr{X}$. Let m factorize as

$$
K \xrightarrow{m'} X' \xrightarrow{m''} X
$$

where m'' is the smallest subobject of X which lies in $\mathscr{X}$ and through which m factorizes.

(a) *m' is a bimorphism.* We only have to show that m' is an epimorphism. Given $x_i: X' \to K'$ with $x_1 m' = x_2 m'$ we find a monomorphism $u: K' \to X''$ with X'' in $\mathscr{X}$. The equalizer e of $u x_1$ and $u x_2$ belongs to $\mathscr{X}$, and m' can be factorized through e, therefore e must be an isomorphism. But $u x_1 = u x_2$ implies $x_1 = x_2$, because u is a monomorphism.

(b) *Every morphism $a: K \to Y$ with Y in $\mathscr{X}$ may be factorized through m'.* Let P be the product of X' and Y with the projections $p_1: P \to X'$ and $p_2: P \to Y$ and define $z: K \to P$ by $p_1 z = m'$ and $p_2 z = a$. The last equality implies that z is a monomorphism. P is an object of $\mathscr{X}$, therefore z factorizes as $K \xrightarrow{z'} P' \xrightarrow{z''} P$ where z'' is the smallest subobject of P which belongs to $\mathscr{X}$ and through which z can be factorized. In (a) we have seen that z' is a bimorphism and therefore an essential monomorphism. But

(*) $$(p_1 z'') z' = m'$$

is a monomorphism, therefore $p_1 z''$ is a monomorphism. The equality (*) also shows that $p_1 z''$ is a subobject through which m' factors. Because $p_1 z''$ belongs to $\mathscr{X}$,

p_1z'' must be an isomorphism. If we define $a'=p_2z''(p_1z'')^{-1}$ we get the equality

$$a'm' = p_2z''(p_1z'')^{-1}(p_1z''z') = p_2z''(p_1z'')^{-1}(p_1z'')z' = p_2z''z' = a.$$

(c) m' *is a reflection of* K *into* $\mathscr{X}$. In (b) we have seen that every morphism $a: K \to Y$ with Y in $\mathscr{X}$ may be factorized through m' and according to (a) this factorization is unique, because m' is an epimorphism.

(d) *The reflector of* $\mathscr{X}$ *onto* $\mathscr{X}$ *is a monofunctor*. Let $k: A \to B$ be a mono-morphism in $\mathscr{X}$. The reflections a of A and b of B define a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ k\ } & B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A' & \xrightarrow[\ k'\ ]{} & B'
\end{array}
$$

and k' is the image of k under the reflector $\mathscr{X} \to \mathscr{X}$. But k and b are monomorphisms and a is an essential monomorphism, so k' is a monomorphism.

REMARK 1. The proof discloses a special feature of categories with amalgama-tions. We have constructed the reflection by taking any monomorphism $m: K \to X$ with X in $\mathscr{X}$ and factorizing through a subobject of X. The subcategory $\mathfrak{K}$ of the compact spaces of the category $\mathfrak{B}$ of all completely regular Hausdorff spaces gives an example where the reflections are bimorphisms but not essential mono-morphisms.

REMARK 2. In the proof of Lemma 1 we did not need the existence of infinite products. Also the existence of finite products may be avoided if $\mathscr{X}$ has pushouts: the assertion of Lemma 1 remains true if $\mathscr{X}$ is a well-powered category with arbitrary intersections, pullbacks, pushouts and amalgamations. We have only to change the proof of (b). Instead of forming the product of X' and Y we take a pushout $\bar{a}: X' \to Q$ and $\bar{m}: Y \to Q$ of a and m and a monomorphism $u: Q \to X''$ with X'' in $\mathscr{X}$. If in the following diagram p_1 and p_2 are pullback-morphisms with respect to $u\bar{a}$ and $u\bar{m}$, we get a morphism z with $p_1z=m'$ and $p_2z=a$.

$$
\begin{array}{ccc}
K & \xrightarrow{\quad m' \quad} & X' \\
{\scriptstyle a}\downarrow \ {\scriptstyle z}\searrow & {\scriptstyle p_1} & \ \downarrow{\scriptstyle \bar{a}} \\
 & P & \\
{\scriptstyle p_2}\swarrow & \xrightarrow{\ \bar{m}\ } & Q \xrightarrow{u} \\
Y & \xrightarrow{\qquad\qquad} & X''
\end{array}
$$

The object P belongs to $\mathscr{X}$ (because X', Y and X'' are in $\mathscr{X}$), therefore P, p_1, p_2, z have all properties we need for continuing the proof of (b).

In a well-powered and co-well-powered complete category $\mathscr{X}$ every subcategory which is limit-closed and subobject-closed is reflective [1], [5], [8]. In particular,

if $\mathscr{X}$ is any limit-closed subcategory, s$\mathscr{X}$ is a reflective subcategory of $\mathscr{K}$, and $\mathscr{X}$ is reflective in $\mathscr{K}$ if and only if $\mathscr{X}$ is reflective in s$\mathscr{X}$. With $\mathscr{K}$ also s$\mathscr{X}$ is well-powered and complete. In order to apply Lemma 1 we have to impose a condition onto s$\mathscr{X}$ which implies that s$\mathscr{X}$ has amalgamations. If $\mathscr{K}$ has amalgamations and the reflector onto s$\mathscr{X}$ is a monofunctor, then s$\mathscr{X}$ has amalgamations: given two monomorphisms $m_i: A \to B_i$ ($i = 1, 2$) in s$\mathscr{X}$, these morphisms are also mono-morphisms in $\mathscr{K}$, therefore we find in $\mathscr{K}$ monomorphisms $n_i: B_i \to C$ with $n_1 m_1 = n_2 m_2$. If $c: C \to C'$ is the reflection of C into s$\mathscr{X}$, cn_i is the image of n_i under the reflector $\mathscr{K} \to$ s$\mathscr{X}$. We have assumed that this functor is a monofunctor, so cn_1 and cn_2 define an amalgamation of m_1 and m_2 in s$\mathscr{X}$. This proves the first part of the following theorem:

THEOREM 1. *Let $\mathscr{K}$ be a well-powered and co-well-powered complete category with amalgamations. Let $\mathscr{X}$ be a limit-closed subcategory of $\mathscr{K}$. If s$\mathscr{X}$ has a mono-functor as reflector, then $\mathscr{X}$ is reflective and has a monofunctor as reflector. Conversely, if $\mathscr{X}$ is reflective and has a monofunctor as reflector, s$\mathscr{X}$ has a monofunctor as reflector.*

It remains to show that a reflective subcategory $\mathscr{X}$ has a monofunctor as reflector if and only if s$\mathscr{X}$ has a monofunctor as reflector. Let $U_2: \mathscr{X} \to$ s$\mathscr{X}$ be the inclusion-functor and $S_1: \mathscr{K} \to$ s$\mathscr{X}$, $S_2:$ s$\mathscr{X} \to \mathscr{X}$ be the reflectors. $\mathscr{X}$ is monoreflective in s$\mathscr{X}$[4], therefore S_1 is a subfunctor of $U_2 S_2 S_1$. As a right adjoint functor U_2 is a monofunctor, so if $S_2 S_1$ is a monofunctor, also $U_2 S_2 S_1$ is a monofunctor—but every subfunctor of a monofunctor is a monofunctor, so S_1 is a monofunctor, s$\mathscr{X}$ satisfies the conditions of Lemma 1 (s$\mathscr{X}$ replacing $\mathscr{K}$) and therefore S_2 is also a monofunctor, so $S_2 S_1$ is a monofunctor.

REMARK 3. It should be noted that there are limit-closed subcategories $\mathscr{X}$ where the reflector $\mathscr{K} \to$ s$\mathscr{X}$ is *not* a monofunctor, but nevertheless s$\mathscr{X}$ has weak amal-gamations (and therefore Lemma 1 can be applied): the subcategory $\mathscr{X} =$ s$\mathscr{X} = \mathfrak{F}$ of the torsionfree groups in the category $\mathfrak{G}$ of all groups has amalgamations (the free product of two torsionfree groups with amalgamations of a common subgroup is a torsionfree group), but the reflector $\mathfrak{G} \to \mathfrak{F}$ is not a monofunctor (there are groups generated by torsion-elements which contain torsionfree subgroups). We have restricted our attention to subcategories with a monofunctor as reflector (and not to the wider class of reflective subcategories having amalgamations) because of Theorem 2 and Theorem 4. We do not know if the reflective subcategories with amalgamations form a lattice.

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(4) $\mathscr{X}$ is a monoreflective (epireflective, bireflective) subcategory of $\mathscr{L}$ means that $\mathscr{X}$ is a reflective subcategory and moreover for every object L in $\mathscr{L}$ the reflection morphism is a monomorphism (epimorphism, bimorphism). The notion of a monoreflective subcategory should not be confused with the concept of a reflective subcategory with a monofunctor as reflector. In fact, in a category with amalgamations every monoreflective subcategory has a monofunctor as reflector, but the converse need not be true.

3. **The lattice M.**   We assume throughout this section that $\mathcal{K}$ is a well-powered and co-well-powered complete category with amalgamations.

Let M denote the class of all reflective subcategories with a monofunctor as reflector. Theorem 1 shows that a limit-closed subcategory $\mathcal{X}$ belongs to M if and only if $s\mathcal{X}$ belongs to M. M is partially ordered by inclusion and has a unique minimal element: the subcategory of $\mathcal{K}$ which contains just the terminal objects of $\mathcal{K}$.

If $\mathcal{Y}$ is any subcategory of $\mathcal{K}$, the limit-closure $L\mathcal{Y}$ is defined as the smallest limit-closed subcategory of $\mathcal{K}$ which contains $\mathcal{Y}$.

THEOREM 2. *If $\{\mathcal{X}_i; i \in I\}$ is any class of elements of M, and $\mathcal{X}$ is the limit-closure of the union of the $\mathcal{X}_i$, then $\mathcal{X}$ is an element of M.*

REMARK. In general, the limit-closure of a class of reflective subcategories need not be reflective. In a well-powered complete category every limit-closed subcategory $\mathcal{X}$ is the limit-closure of the reflective subcategories contained in $\mathcal{X}$: for every object X in $\mathcal{X}$ the limit-closure $L(X)$ (the smallest limit-closed subcategory which contains X) is a reflective subcategory—this is a consequence of the special adjoint functor theorem of Freyd [10]. But in [6] Herrlich has constructed a limit-closed subcategory of a well-powered and co-well-powered complete category which is not reflective.

Proof of Theorem 2.   We want to show that $s\mathcal{X}$ has a monofunctor as reflector; then the assertion is a consequence of Theorem 1.

Let K be an object of $\mathcal{K}$, and let $\mathscr{E} = \{e_i : K \to E_i; i \in I\}$ be the class of reflections of K into the subcategories $s\mathcal{X}_i$.

(a) If J is a sub*set* of the index-class I such that every quotient e_i in $\mathscr{E}$ is equivalent to a quotient $e_j\mathscr{E}$ with j in J, then the reflection of $\mathcal{K}$ into $s\mathcal{X}$ may be constructed in the following way: we form the product $P = \prod_{j \in J} E_j$ with the projections $p_j: P \to E_j$ and define f by $p_j f = e_j$. If we factorize f as

$$K \xrightarrow{\ e\ } E \xrightarrow{\ m\ } P$$

where m is the smallest subobject of P through which f can be factorized, e is a reflection of $\mathcal{K}$ into $s\mathcal{X}$.

(b) If there is given a monomorphism $k: K \to K'$, then the reflection classes $\mathscr{E} = \{e_i : K \to E_i; i \in I\}$ and $\mathscr{E}' = \{e_i' : K' \to E_i'; i \in I\}$ both contain only a set of nonequivalent quotients, because $\mathcal{K}$ is co-well-powered. Therefore we find a subset J of I such that every quotient e_i in $\mathscr{E}$ is equivalent to a quotient e_j with $j \in J$ *and* every quotient e_i' in $\mathscr{E}'$ is equivalent to a quotient e_j' with $j \in J$. For every $j \in J$ we define k_j by $e_j'k = k_j e_j$. k_j is the image of k under the reflector $\mathcal{K} \to s\mathcal{X}_i$. This is a monofunctor (Theorem 1), therefore the morphisms k_j are monomorphisms. The product $\prod_{j \in J} k_j: P \to P'$, where $P = \prod_{j \in J} E_j$ and $P' = \prod_{j \in J} E_j'$ is a morphism which satisfies the equality $(\prod k_j) f = f'k$ where f and f' are defined as in (a).

Moreover $\prod k_j$ is a monomorphism. Factorizing $f = me$ and $f' = m'e'$ as in (a) gives us the following commutative diagram:

$$
\begin{array}{ccccc}
K & \xrightarrow{\;e\;} & E & \xrightarrow{\;m\;} & P \\
{\scriptstyle k}\downarrow & & {\scriptstyle k^*}\downarrow & & \downarrow{\scriptstyle \prod k_j} \\
K' & \xrightarrow{\;e'\;} & E' & \xrightarrow{\;m'\;} & P'
\end{array}
$$

e is an extreme epimorphism, m' is a monomorphism, therefore we find k^* with $k^*e = e'k$ and $m'k^* = (\prod k_j)m$. The first equality shows that k^* is the image of k under the reflector $\mathscr{X} \to s\mathscr{X}$, the second equality implies that k^* is a monomorphism (because $\prod k_j$ and m are monomorphisms).

As a consequence we have the following:

COROLLARY. *M is a complete lattice.*

Proof. M is partially ordered and has a unique smallest element. In this situation the existence of arbitrary unions in M implies the existence of arbitrary intersections in M.

In order to show that the intersections in M are just the set-theoretical intersections of the elements of M, we need a new assumption: the existence of injective envelopes. This will be done in the next section.

4. Categories with injective envelopes. Let us remind ourselves of some definitions. An object I is called injective, iff for every monomorphism $m: A \to B$ and every morphism $a: A \to I$ there is a morphism $b: B \to I$ with $bm = a$. A category $\mathscr{X}$ has (enough) injectives, iff for every object K in $\mathscr{X}$ there is a monomorphism $m: K \to I$ with I injective; $\mathscr{X}$ has injective envelopes iff always such a morphism exists which is an essential monomorphism. It is well known (see for example [3]), that a category with finite products and injectives has amalgamations, so in the sequel we need not mention this assumption.

LEMMA 2. *Let $\mathscr{X}$ be a well-powered and co-well-powered complete category. If $\mathscr{I}$ is any class of injective objects of $\mathscr{X}$, then the limit-closure $\mathrm{L}\mathscr{I}$ of $\mathscr{I}$ is a reflective subcategory with a monofunctor as reflector.*

Proof. We set $\mathscr{X} = \mathrm{L}\mathscr{I}$.

(a) For every object Y in $s\mathscr{X}$ there is a monomorphism $u: Y \to I$ where I belongs to $\mathscr{X}$ and is injective in $\mathscr{X}$. Y is an object of $\mathrm{SL}\mathscr{I}$, therefore a subobject of a product of elements of $\mathscr{I}$. Let $u: Y \to I$ be a monomorphism where I is a product of elements of $\mathscr{I}$. $\mathscr{X}$ is the limit-closure of $\mathscr{I}$, so I belongs to $\mathscr{X}$; the elements of $\mathscr{I}$ are injective in $\mathscr{X}$, a product of injective objects is injective, so I is injective in $\mathscr{X}$.

(b) s$\mathscr{X}$ has a monofunctor as reflector. Given a monomorphism $m: A \to B$ in $\mathscr{X}$, we look at the reflections a of A and b of B into s$\mathscr{X}$. There is a morphism m' such that the following diagram is commutative

$$
\begin{array}{ccc}
A & \xrightarrow{\;m\;} & B \\
{\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} \\
A' & \xrightarrow[\;m'\;]{} & B'
\end{array}
$$

and m' is the image under the reflector $\mathscr{X} \to$ s$\mathscr{X}$. A' is an object of s$\mathscr{X}$; according to (a) we find a monomorphism $u: A' \to I$ where I belongs to $\mathscr{X}$ and is injective in $\mathscr{X}$. m is a monomorphism, I is injective, so there exists a morphism x with $xm = ua$. $x: B \to I$ can be factorized through b, because I belongs to s$\mathscr{X}$: $x = x'b$. But the equality $ua = xm = x'bm = x'm'a$ implies $u = x'm'$ (because a is a reflection morphism). u is a monomorphism, so m' is a monomorphism.

(c) We want to apply Lemma 1 with s$\mathscr{X}$ instead of $\mathscr{X}$. s$\mathscr{X}$ is well-powered and complete, (a) implies that s$\mathscr{X}$ has enough injectives, so s$\mathscr{X}$ has amalgamations. Therefore $\mathscr{X}$ is a reflective subcategory of s$\mathscr{X}$ and the reflector of s$\mathscr{X}$ onto $\mathscr{X}$ is a monofunctor. The reflector of $\mathscr{X}$ onto $\mathscr{X}$ is the composition of the reflector $\mathscr{X} \to$ s$\mathscr{X}$ and s$\mathscr{X} \to \mathscr{X}$. Both functors are monofunctors, so $\mathscr{X}$ has a monofunctor as reflector.

In the presence of injective envelopes we can characterize the elements of M in the following way:

THEOREM 3. *Let $\mathscr{X}$ be a well-powered and co-well-powered complete category with injective envelopes. Then the following assertions are equivalent for a limit-closed subcategory $\mathscr{X}$:*

(i) *$\mathscr{X}$ is an element of M,*

(ii) *$\mathscr{X}$ is closed under injective envelopes,*

(iii) *for every object X in $\mathscr{X}$ there is a monomorphism $X \to I$ in $\mathscr{X}$, where I is injective in $\mathscr{X}$.*

Proof. (i) $\to$ (ii). Let $m: X \to I$ be an injective envelope with X in $\mathscr{X}$. If $c: I \to I'$ is the reflection of I in $\mathscr{X}$, cm is the image of m under the reflector $S: \mathscr{X} \to \mathscr{X}$. But m is a monomorphism and S is a monofunctor, so cm is a monomorphism. m is an essential monomorphism implies that c is a monomorphism. The injectivity of I gives us a morphism r with $rc = 1_I$. c is a reflection, so the equality $crc = c$ implies $cr = 1_{I'}$. Therefore c is an isomorphism and I belongs to $\mathscr{X}$.

(ii) $\to$ (iii). Trivial.

(iii) $\to$ (i). Let $\mathscr{I}$ be the class of injective objects of $\mathscr{X}$ which belong to $\mathscr{X}$. Lemma 2 implies that L$\mathscr{I}$ is a reflective subcategory with a monofunctor as reflector. But sL$\mathscr{I} =$ s$\mathscr{X}$, so the assertion follows from Theorem 1.

Property (ii) is hereditary with respect to intersections. Therefore we get the following theorem as an immediate consequence of the equivalence of (i) and (ii) in Theorem 3:

THEOREM 4. *Let $\mathscr{X}$ be a well-powered and co-well-powered complete category with injective envelopes. If $\{\mathscr{X}_i; i \in I\}$ is any class of elements of M, and $\mathscr{X}$ is the (set-theoretical) intersection of the $\mathscr{X}_i$, then $\mathscr{X}$ is an element of M.*

5. **Locally minimal and locally maximal elements of M.** Let $\mathscr{X}$ be a well-powered and co-well-powered complete category with injective envelopes. If we look at all reflective subcategories $\mathscr{X}$ with fixed $\mathscr{L} = s\mathscr{X}$, there is a biggest subcategory, namely $\mathscr{L}$, but there is also a smallest subcategory: the limit-closure of the injective objects of $\mathscr{X}$ which belong to $\mathscr{L}$. The following theorem characterizes these subcategories. We start with the category $\mathscr{L}$ and denote by M_0 the class of all reflective subcategories $\mathscr{X}$ of $\mathscr{L}$ with $s\mathscr{X} = \mathscr{L}$.

THEOREM 5. *Let $\mathscr{L}$ be a well-powered complete category with injectives. If $\mathscr{X}$ is a limit-closed subcategory, the following assertions are equivalent:*

(i) *$\mathscr{X}$ is the limit-closure of the injectives of $\mathscr{L}$,*
(ii) *$\mathscr{X}$ is the smallest element of M_0,*
(iii) *$\mathscr{X}$ belongs to M_0 and has no proper bimorphisms.*

Proof. (i) $\leftrightarrow$ (ii). Let $\mathscr{I}$ be the class of all injectives of $\mathscr{L}$. $\mathscr{L}$ has enough injectives, therefore we have $s\mathscr{I} = \mathscr{L}$, but then a fortiori $sL(\mathscr{I}) = \mathscr{L}$. Lemma 1 implies that $L(\mathscr{I})$ belongs to M_0. If $\mathscr{X}$ is any element of M_0, then $\mathscr{X}$ contains all injective objects of $\mathscr{L}$; given an injective object I, we look at the reflection $c: I \to I'$ of I into $\mathscr{X}$. $\mathscr{X}$ is monoreflective, therefore c is a monomorphism and the injectivity of I implies that c is a coretraction. But monoreflective subcategories are epireflective, so c is also an epimorphism and therefore an isomorphism. So we have shown that $\mathscr{I}$, and therefore $L\mathscr{I}$, is contained in $\mathscr{X}$.

(i) $\to$ (iii). We first observe that the limit-closure $L\mathscr{I}$ of the injectives of $\mathscr{L}$ is just the full subcategory $\mathscr{X}$ of all objects X for which there exists an equalizer-diagram

(*)
$$X \xrightarrow{e} I \; \underset{v}{\overset{u}{\rightrightarrows}} \; J$$

where I and J are injective. Clearly, $\mathscr{X}$ is contained in $L\mathscr{I}$. So we have to show that $\mathscr{X}$ is limit-closed. But $\mathscr{I}$ is product-closed, so it is easy to see that $\mathscr{X}$ is product-closed. It remains to verify that $\mathscr{X}$ is closed under the formation of equalizer-diagrams

$$X \xrightarrow{z} X_1 \; \underset{b}{\overset{a}{\rightrightarrows}} \; X_2.$$

For every X_i we have equalizer-diagrams of the form (*), say

$$X_i \xrightarrow{e_i} I_i \underset{v_i}{\overset{u_i}{\rightrightarrows}} J_i$$

where I_i and J_i are injective. In particular, I_2 is injective and e_1 is a monomorphism, so we can find morphisms $a': I_1 \to I_2$ and $b': I_1 \to I_2$ with $a'e_1 = e_2 a$ and $b'e_1 = e_2 b$. One verifies that the morphism $e_1 z$ is the limit of the following diagram

and this implies that there exists a diagram of the form (*) for X. (Look at the usual construction of limits by means of products and equalizers.)

Now given a bimorphism $b: X \to Y$ in $\mathscr{X} = \mathrm{L}\mathscr{I}$, we take an equalizer-diagram of the form (*), where I and J are injective. b is a monomorphism, I is injective, so there exists a morphism d with $db = e$. But equalizers are extreme monomorphisms, so the epimorphism b must be an isomorphism. So all bimorphisms in $\mathrm{L}\mathscr{I}$ are isomorphisms.

(iii) $\to$ (ii). We show that a subcategory $\mathscr{X}$ with the properties of (iii) is a minimal element of M_0. Given an element Y of M_0 which is contained in $\mathscr{X}$. The equality $\mathrm{s}\mathscr{Y} = \mathscr{L}$ implies that $\mathscr{Y}$ is a monoreflective and therefore bireflective subcategory of $\mathscr{L}$. But then $\mathscr{Y}$ is also a bireflective subcategory of $\mathscr{X}$. All bimorphisms of $\mathscr{X}$ are isomorphisms, so the reflections of $\mathscr{X}$ into $\mathscr{Y}$ are in fact isomorphisms and $\mathscr{X} = \mathscr{Y}$.

We call a reflective subcategory $\mathscr{X}$ *locally minimal* (*maximal*) iff $\mathscr{X}$ belongs to M and $\mathscr{X}$ is minimal (maximal) in the class of reflective subcategories with fixed $\mathrm{s}\mathscr{X}$. Thus the locally minimal subcategories of $\mathscr{K}$ are those of the form $\mathrm{L}\mathscr{I}$, the locally maximal subcategories are those of the form $\mathrm{SL}\mathscr{I}$, where $\mathscr{I}$ is a class of injective objects. In a category without proper bimorphisms, another characterization can be given using the following theorem:

THEOREM 6. *Let $\mathscr{K}$ be a well-powered and co-well-powered complete category with injectives. A reflective subcategory is locally minimal if and only if the reflector maps monomorphisms onto regular monomorphisms*[5].

Proof. We need the following result which is essentially due to Kelly [7]:

([5]) A monomorphism m is regular, iff m is the equalizer of two morphisms.

LEMMA 3. *Let $\mathscr{L}$ be a well-powered complete category with amalgamations. Then every morphism x may be factorized as $x=me$ where e is an epimorphism and m is a regular monomorphism.*

Proof of the lemma. (a) Every regular monomorphism is the equalizer of two monomorphisms. If m is the equalizer of $a_i\colon X \to Y$ ($i=1, 2$), we take the product Z of X and Y with the projections $x\colon Z \to X$ and $y\colon Z \to Y$, and define a_i' by $xa_i'=1_X$ and $ya_i'=a_i$. The last equation shows that a_1' is a monomorphism, and it is easy to see that m is the equalizer of a_1' and a_2'.

(b) The composition of two regular monomorphisms is a regular monomorphism. Let $m\colon X \to Y$ be the equalizer of $a_i\colon Y \to A$, and $n\colon Y \to Z$ be the equalizer of $b_i\colon Z \to B$ ($i=1, 2$). We can assume that the a_i are monomorphisms. The existence of amalgamations gives us a commutative diagram, where all morphisms are monomorphisms:
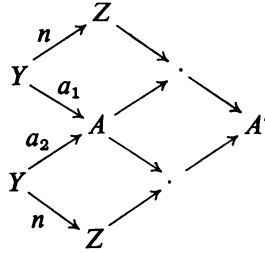
Therefore we have monomorphisms $n'\colon A \to A'$ and $a_i'\colon Z \to A'$ with $a_i'n=n'a_i$ ($i=1, 2$). We define C as the product of A' and B with the projections $p_{A'}\colon C \to A'$ and $p_B\colon C \to B$, and for $i=1, 2$ we define $c_i\colon Z \to C$ by $p_{A'}c_i=a_i'$ and $p_Bc_i=b_i$. nm is the equalizer of c_1 and c_2: we have the equality $c_1nm=c_2nm$, and given u with $c_1u=c_2u$, we have

$$b_1u = p_Bc_1u = p_Bc_2u = b_2u,$$

and therefore we can factorize u as $u=nu'$. Now the equality

$$n'a_1u' = a_1'nu' = a_1'u = p_{A'}c_1u = p_{A'}c_2u = a_2'nu' = n'a_2u'$$

implies $a_1u'=a_2u'$, because n' is a monomorphism. Therefore we can factorize u' as $u'=mu''$, and $u=(nm) u''$ gives us the factorization we have looked for.

(c) Given an arbitrary morphism $x\colon A \to B$ in $\mathscr{L}$, take the smallest subobject m of B which is a regular monomorphism and through which x factors, say $x=me$. m exists, because our assumptions imply that the regular monomorphisms are closed under intersections. From (b) it follows that e is an epimorphism. This proves the lemma.

If $\mathscr{X}$ is locally minimal, then, according to Theorem 5, $\mathscr{X}$ has no proper bimorphisms. $\mathscr{X}$ satisfies the conditions of Lemma 3, so every monomorphism of $\mathscr{X}$ is a regular monomorphism, and the reflector $S\colon \mathscr{X} \to \mathscr{X}$ maps monomorphisms to

regular monomorphisms. Conversely, if $\mathscr{X}$ is reflective and the reflector $S\colon \mathscr{K} \to \mathscr{X}$ maps monomorphisms onto regular monomorphisms, S is, in particular, a monofunctor and so $\mathscr{X}$ belongs to M. If b is a bimorphism of $\mathscr{X}$, b is a monomorphism of $\mathscr{K}$, so $b = S(b)$ is a regular monomorphism of $\mathscr{X}$. But a morphism which is an epimorphism and a regular monomorphism must be an isomorphism. So $\mathscr{X}$ has no proper bimorphisms and $\mathscr{X}$ is locally minimal.

COROLLARY. *Let $\mathscr{K}$ be a well-powered and co-well-powered complete category with injectives and without proper bimorphisms. Then*

(a) *A reflective subcategory is locally minimal if and only if the reflector preserves regular monomorphisms.*

(b) *A reflective subcategory is locally maximal if and only if it is epireflective and the reflector preserves monomorphisms.*

Proof. (a) Lemma 3 implies that every monomorphism of $\mathscr{K}$ is a regular monomorphism, so (a) is a consequence of Theorem 6. (b) In a well-powered and co-well-powered complete category without bimorphisms the epireflective subcategories are just those limit-closed subcategories $\mathscr{X}$ which satisfy the equality $\mathscr{X} = \mathrm{s}\mathscr{X}$ [11].

REMARK. If $\mathscr{X}$ is a reflective subcategory of $\mathscr{K}$ and the reflector $S\colon \mathscr{K} \to \mathscr{X}$ preserves regular monomorphisms, then S preserves equalizers of cokernel pairs. For given a cokernel pair q_1, q_2 (of a morphism x), and the equalizer m of q_1 and q_2, then q_1, q_2 is also the cokernel pair of m, that is

$$\begin{array}{ccc} & m \nearrow{}^{\bullet} \searrow q_1 & \\ {}^{\bullet} & & {}^{\bullet} \\ & m \searrow{}_{\bullet} \nearrow q_2 & \end{array}$$

is a pushout diagram. But as a left adjoint, S preserves pushouts, so

$$\begin{array}{ccc} & S(m) \nearrow{}^{\bullet} \searrow S(q_1) & \\ {}^{\bullet} & & {}^{\bullet} \\ & S(m) \searrow{}_{\bullet} \nearrow S(q_2) & \end{array}$$

is a pushout diagram in $\mathscr{X}$, and $S(q_1)$, $S(q_2)$ is the cokernel pair of $S(m)$. But $S(m)$ is a regular monomorphism, so $S(m)$ is the equalizer of this cokernel pair: $S(m)$ is the equalizer of $S(q_1)$ and $S(q_2)$. If $\mathscr{K}$ has cokernel pairs, S preserves regular monomorphisms if and only if S preserves equalizers of kernel pairs.

6. **Applications.** (a) *Module categories.* Let R be an associative ring with 1 and $\mathfrak{M}_R$ be the category of all R-modules. In studying the localization of rings the concept of a (hereditary) torsion-theory ($\mathscr{P}$, $\mathscr{Q}$) [2], [9] has been useful. As Gabriel [4] has shown, every torsion-theory ($\mathscr{P}$, $\mathscr{Q}$) determines a reflective subcategory $\mathscr{E}$ with $\mathscr{Q} = \mathrm{s}\mathscr{E}$ and with an exact functor as reflector. Both $\mathscr{Q}$ and $\mathscr{E}$ are reflective

subcategories with a monofunctor as reflector and the following proposition shows that the elements of M are just those limit-closed subcategories $\mathscr{X}$ with $\mathscr{E} \subseteq \mathscr{X} \subseteq \mathscr{D}$ for a suitable torsion-theory (see also [12]).

PROPOSITION 1. *For a limit-closed subcategory $\mathscr{X}$ of $\mathfrak{M}_R$ the following properties are equivalent*:

   (i) *$\mathscr{X}$ is reflective and has a monofunctor as reflector*,

   (ii) *there is a torsion-theory $(\mathscr{P}, \mathscr{D})$ with $s\mathscr{X} = \mathscr{D}$*,

   (iii) *there is a reflective subcategory $\mathscr{E}$ with an exact reflector and $\mathscr{E} \subseteq \mathscr{X} \subseteq s\mathscr{E}$.*

Proof. (i) → (ii). According to Theorem 3, $\mathscr{X}$ is closed under essential extensions, therefore $s\mathscr{X}$ is closed under products, subobjects and essential extensions, and it is well known (see for example [13]) that these properties characterize the class $\mathscr{D}$ of a torsion-theory $(\mathscr{P}, \mathscr{D})$.

(ii) → (iii). Take for $\mathscr{E}$ the reflective subcategory $\mathscr{E}$ with $\mathscr{D} = s\mathscr{E}$ and with an exact reflector constructed by Gabriel. Then $\mathscr{X} \subseteq s\mathscr{E}$ and we have to show that $\mathscr{E}$ is contained in $\mathscr{X}$. But $\mathscr{E}$ is an abelian category and contains therefore no proper bimorphism. Applying Theorem 5 for $\mathscr{L} = \mathscr{D}$ shows that $\mathscr{E}$ is contained in every reflective subcategory $\mathscr{Y}$ with $s\mathscr{Y} = \mathscr{D}$. But $\mathscr{D}$ has a monofunctor as reflector, so $\mathscr{X}$ is in fact reflective (Theorem 1), and $\mathscr{E}$ is contained in $\mathscr{X}$.

(iii) → (i). We have the equality $s\mathscr{E} = s\mathscr{X}$ and $\mathscr{E}$ has a monofunctor as reflector, so the assertion is a consequence of Theorem 1. The locally maximal elements of M are just the subcategories $\mathscr{D}$, where $(\mathscr{P}, \mathscr{D})$ is a torsion-theory, the locally minimal elements are the reflective subcategories with an exact reflector (= the localizations of Gabriel). It is easy to see that in general these are not the only elements of M— for example in the category $\mathfrak{M}_Z$ of all abelian groups the full subcategory of the p-divisible torsionfree groups (where p is a prime) is a reflective subcategory with a monofunctor as reflector, but is neither locally minimal nor locally maximal. This is a special case of the following construction:

PROPOSITION 2. *Let $(\mathscr{P}, \mathscr{D})$ be a torsion-theory in $\mathfrak{M}_R$ and $\mathscr{P}'$ be a subcategory of $\mathscr{P}$. Then the full subcategory $\mathscr{X}$ of all objects X of Q with $\mathrm{Ext}\,(P, X) = 0$ for all $P \in \mathscr{P}'$ is a reflective subcategory with a monofunctor as reflector.*

Proof. We only have to show that $\mathscr{X}$ is limit-closed. $\mathscr{X}$ is product-closed, so we consider the case where $k \colon K \to X_1$ is the kernel of $x \colon X_1 \to X_2$, where x is a morphism in $\mathscr{X}$. If $x = me$ is the factorization of x in an epimorphism e and a monomorphism m, the exact sequence

$$0 \longrightarrow K \overset{k}{\longrightarrow} X_1 \overset{e}{\longrightarrow} Q \longrightarrow 0$$

determines an exact sequence

$$0 \to \mathrm{Hom}\,(P, K) \to \mathrm{Hom}\,(P, X_1) \to \mathrm{Hom}\,(P, Q) \to \mathrm{Ext}\,(P, K) \to \mathrm{Ext}\,(P, X_1).$$

Q is a subobject of X_2 in $\mathcal{X}$, so belongs to $\mathcal{Q}$, P is an object of $\mathcal{P}$, therefore we have Hom $(P, Q) = 0$. Ext $(P, X_1) = 0$ now implies Ext $(P, K) = 0$.

(b) *Functor-categories.* Let $\mathcal{C}$ be a small category and $[\mathcal{C}, \mathcal{S}]$ the category of all functors from $\mathcal{C}$ into the category $\mathcal{S}$ of sets. If t is a Grothendieck-topology on $\mathcal{C}$, we denote by $\mathcal{Q}_t$ the full subcategory of the separated presheaves, by $\mathcal{E}_t$ the full subcategories of all sheaves with respect to t [14]. Both $\mathcal{Q}_t$ and $\mathcal{E}_t$ are reflective subcategories with monofunctors as reflectors and the elements of M are just those limit-closed subcategories $\mathcal{X}$ with $\mathcal{E}_t \subseteq \mathcal{X} \subseteq \mathcal{Q}_t$ for a suitable topology:

PROPOSITION 3. *For a limit-closed subcategory $\mathcal{X}$ of $[\mathcal{C}, \mathcal{S}]$ the following properties are equivalent*:

   (i) *$\mathcal{X}$ is reflective and has a monofunctor as reflector,*

   (ii) *s$\mathcal{X} = \mathcal{Q}_t$ for a Grothendieck-topology t,*

   (iii) *there is a reflective subcategory $\mathcal{E}$ with a reflector respecting finite limits and $\mathcal{E} \subseteq \mathcal{X} \subseteq s\mathcal{E}$.*

The proof is quite similar to (a) and will be omitted.

The locally maximal elements of M are just the categories of the separated presheaves, the locally minimal elements are the categories of all sheaves with respect to a topology.

(c) *Categories of topological spaces.* We call a full subcategory of the category $\mathfrak{T}$ of all topological spaces a category of topological spaces. There are some categories $\mathcal{X}$ of topological spaces which satisfy the assumptions of Theorem 1: The category of all T_1-spaces and the category $\mathfrak{T}$ itself both have amalgamations. However there are only trivial subcategories of the form s$\mathcal{X}$ which have a monofunctor as reflector:

PROPOSITION 4. *Let $\mathcal{X}$ be a category of topological spaces which contains all discrete spaces. A reflective subcategory with a monofunctor as reflector is either monoreflective or contains only spaces with at most one point.*

Proof. We may assume that the subcategory $\mathcal{X}$ has a monofunctor as reflector and contains a space X with more than one point. First we notice that for a discrete space D the reflection morphism $r_D: D \to D'$ into $\mathcal{X}$ is a monomorphism: for given two distinct points d_1 and d_2 in D we find a (set-theoretical) map $D \xrightarrow{x} X$ which does not identify d_1 and d_2. But D is discrete, so x is continuous. Therefore we can factorize x through r_D, and r_D does not identify d_1 and d_2. Now for an arbitrary space K in $\mathcal{X}$ we find a bijective map $b: D \to K$ where D is discrete. We have the equality $r_K b = R(b) r_D$ where r_K and r_D are the reflection morphisms and $R: \mathcal{X} \to \mathcal{X}$ is the reflector. R is a monofunctor, so $R(b)$ is a monomorphism. This implies that $r_K b$ is a monomorphism. b is bijective, so r_K is a monomorphism.

(d) *Categories with only coretractions as monomorphisms.* If every monomorphism of $\mathcal{X}$ is a coretraction, then every functor $\mathcal{X} \to \mathcal{L}$ (where $\mathcal{L}$ is an arbitrary category) is a monofunctor. In this case M is the class of all reflective subcategories.

The assumption is verified in every abelian category of global dimension 0, for example in the module-categories $\mathfrak{M}_R$ where R is (artinian) semisimple. However, if in $\mathscr{X}$ every monomorphism is a coretraction, the same assertion is true in every full subcategory, and so every reflective subcategory is even epireflective.

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