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NORMAL OPERATORS ON QUATERNIONIC HILBERT SPACES(1)

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Abstract. Simple modifications of standard complex methods are used to obtain a spectral theorem, a functional calculus and a multiplicity theory for normal operators on quaternionic Hilbert spaces. It is shown that the algebra of all operators on a quaternionic Hilbert space is a real C^*-algebra in which (a) every normal operator is unitarily equivalent to its adjoint and (b) every operator in the double commutant of a hermitian operator is hermitian. Unitary representations of locally compact abelian groups in quaternionic Hilbert spaces are studied and, finally, the complete structure theory of commutative von Neumann algebras on quaternionic Hilbert spaces is worked out.

Introduction. Recent years have witnessed renewed interest in linear problems involving quaternions, due mainly to the physicists who are studying the possibilities of a quaternionic quantum mechanics [2]. From the mathematical point of view this interest has resulted in spectral theorems for unitary and skew-hermitian operators on quaternionic Hilbert spaces (the theory of hermitian operators poses no difficulties and is entirely similar to the standard complex theory) [2], [3], the study of unitary representations of groups in quaternionic Hilbert spaces [1], [4], [8], [12] and some additional theorems closer to quantum mechanics [11]. So far as the present author is aware a systematic study of operators on quaternionic Hilbert spaces is not available in the literature.

In this article we study the central area of problems featuring normal operators and their structure. It is a little surprising that by slight modifications of standard complex methods one can obtain a theory nearly as complete as in the complex case. There are some algebraic complications because of the noncommutativity of the quaternions but the basic geometric nature of the arguments is unchanged and we can obtain a spectral theorem (§3), an analogue of the SNAG theorem expressing an arbitrary representation of a locally compact abelian group as an integral of irreducible ones in a unique fashion (§4: cf. [1]), a multiplicity function

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which determines a normal operator to within unitary equivalence (§5), a functional calculus (§6) and the complete structure theory of commutative W^*-algebras (§7). While it is true that much of this theory is but a reflection of the "complex" theory there are nevertheless differences which make the "quaternionic" theory interesting. For example, the set of all operators on a quaternionic Hilbert space is a Banach *-algebra only over the reals (since multiplication by a scalar is not a linear transformation in a vector space over a division ring unless the scalar belongs to the center of the division ring) and in this Banach *-algebra every normal operator is unitarily equivalent to its adjoint! There are other interesting algebras of operators on a quaternionic Hilbert space and it is likely that these might prove to be useful models for a general theory of real Banach algebras. (There is much similarity between our theorems in the last section and some of the theorems on real C^*-algebras in [6].)

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1. **Quaternionic Hilbert spaces.** Let $\boldsymbol{Q}=\{q=q_0+q_1i+q_2j+q_3k\}$ be the division ring of real quaternions. $q^*=q_0-q_1i-q_2j-q_3k$ will denote the conjugate of q and $|q|=\sqrt{(q_0^2+q_1^2+q_2^2+q_3^2)}$ the absolute value of q. $\boldsymbol{R}=\{q : q_1=q_2=q_3=0\}$ will be identified with the real field and $\boldsymbol{C}=\{q : q_2=q_3=0\}$ with the complex field. Then every quaternion is uniquely of the form $\alpha+k\beta=\alpha+\bar{\beta}k$. α is called the complex part of q.

Let $\mathscr{V}$ be a (left) vector space over $\boldsymbol{Q}$. An inner product on $\mathscr{V}$ is a map $(\cdot|\cdot): \mathscr{V}\times\mathscr{V}\to\boldsymbol{Q}$ with the properties,

(i) $(x|y)=(y|x)^*$,

(ii) $(px+qy|z)=p(x|z)+q(y|z)$,

(ii′) $(x|py+qz)=(x|y)p^*+(x|z)q^*$, and

(iii) $(x|x)\geqq 0$, $=0$ if and only if $x=0$,

for all $x, y, z\in\mathscr{V}$ and $p, q\in\boldsymbol{Q}$.

If $(\cdot|\cdot)$ is an inner product on $\mathscr{V}$, then $\|x\|=\sqrt{(x|x)}$ is a norm on $\mathscr{V}$. A left vector space over the quaternions together with an inner product on it which makes the resulting normed linear space complete is called a quaternionic Hilbert space.

The geometry of quaternionic Hilbert spaces is entirely similar to that of complex Hilbert spaces. In particular the projection theorem is valid and every bounded linear functional is of the form $x\to(x|y)$ for a unique y.

Throughout the rest of the article $\mathscr{H}$ will denote a quaternionic Hilbert space and $\mathscr{B}$ the set of all operators (i.e. bounded linear transformations) on $\mathscr{H}$. The existence of adjoints is proved in the usual way and the definitions of hermitian, unitary and normal operators follow. *$\mathscr{B}$ is then a Banach *-algebra only over the reals* since, for $q\in\boldsymbol{Q}$, the map $x\to qx$ is not linear unless q is in the center of $\boldsymbol{Q}$, i.e. is real.

1.1. EXAMPLE. Let $(X, \mathscr{S}, \mu)$ be a measure space with μ nonnegative. Then $L^2_Q(\mu)$, the space of all (equivalence classes of) quaternion-valued measurable functions with square integrable absolute values considered as a left vector space, together with the inner product $(f|g)=\int fg^* \, d\mu$, is a quaternionic Hilbert space. For any quaternion-valued, essentially bounded measurable function h on X, the operator of right multiplication by h, $R_h: f \to f \cdot h$, is normal. R_h is hermitian if and only if h is essentially real and R_h is unitary if and only if $|h|=1$ a.e. We shall prove in §5 that every normal operator on $\mathscr{H}$ is built up of operators of the form R_h.

What can we say about eigenvalues and eigenvectors for an operator A on $\mathscr{H}$? Suppose for $x \in \mathscr{H}$ and $q \in \boldsymbol{Q}$, $Ax=qx$. If $y=px$, $p \in \boldsymbol{Q}$, then $Ay=pqp^{-1}y \neq qy$ in general, so that while a multiple of an eigenvector is again an eigenvector it need not correspond to the same eigenvalue! The invariant object is therefore not the individual eigenvalue but the eigenclass: the conjugacy class of the eigenvalue [2]. However, we may retain the flavor of the complex theory and recover the individuality of the eigenvalues by considering a "cross section" of the conjugacy classes which contains exactly one element from each class and restricting ourselves to eigenvalues from this cross section. Since the conjugacy class of a quaternion is determined by its real part and norm [11] such a cross section is given by $\boldsymbol{C}^+=\{q : q_1 \geqq 0, q_2=q_3=0\}$, the set of complex numbers with nonnegative imaginary part.

This means that if $Ax=qx$ then there is a unique $\lambda \in \boldsymbol{C}^+$, and a $y=px$ such that $Ay=\lambda y$. (p is arbitrary if q is real and unique up to complex (left) multiples if q is nonreal.) If $z=(\alpha+\beta k)y$ is any vector in the ray S spanned by y (or x), then $Az=(\alpha+k\beta)\lambda y=(\lambda\alpha+\bar{\lambda}\beta)y$. If we define J to be the unique linear map on S for which $Jy=iy$, and let $\lambda=a+ib$ (a, b real), then $Az=(a+Jb)z$ for all $z \in S$. Thus in the quaternionic case the "imaginary" operator J (note that $J^2=-1$) replaces the multiplication by i in the complex case. We shall prove in §3, that every normal operator on $\mathscr{H}$ is an integral of operators of the form $a+Jb$.

2. **The symplectic image.** In this section we study the relation between $\mathscr{H}$ and its underlying complex structure and introduce the notion of an imaginary operator.

Since we have identified the complex field with a subfield of $\boldsymbol{Q}$, $\mathscr{H}$ may be considered to be a complex vector space as well. Let us call this $\mathscr{H}^s$. $\mathscr{H}^s$ is a complex Hilbert space with respect to the inner product $\langle x|y\rangle=$the complex part of $(x|y)$ and is called the symplectic image of $\mathscr{H}$ [3], [8]. The basic relations between $\mathscr{H}$ and $\mathscr{H}^s$ are summarized in the six statements below ($x, y \in \mathscr{H}^s$):

(1) The norms on $\mathscr{H}$ and $\mathscr{H}^s$ are identical. More generally $(x|y)=\langle x|y\rangle + \langle x|ky\rangle k=\langle x|y\rangle - k\langle kx|y\rangle$.

(2) Let K denote the map $x \to kx$ on $\mathscr{H}^s$. Then K is conjugate linear and $K^2 = -I$, where I is identity operator (on $\mathscr{H}$ or $\mathscr{H}^s$ equivalently). Further $\langle Kx|Ky\rangle =\langle y|x\rangle$. In particular $\langle x|Kx\rangle=0$ and $\|Kx\|=\|x\|$.

(3) If $\{e_r\}$ is a basis for $\mathscr{H}$, then $\{e_r, Ke_r\}$ is a basis for $\mathscr{H}^s$.

(4) Every operator A on $\mathscr{H}$ may be considered to be an operator A^s on $\mathscr{H}^s$. The map $A \to A^s$ is an injective norm and adjoint preserving real homomorphism. It is not surjective. In fact, noticing that if $A^s x = \lambda x$ ($\lambda \in \boldsymbol{C}$) then $A^s(kx) = \bar{\lambda}(kx)$, it is easy to see that the spectrum of A^s is symmetric about the real axis.

(5) If B is an operator on $\mathscr{H}^s$ then so is $KBK^{-1} = K^{-1}BK$ and $(KBK^{-1})^* = KB^*K^{-1}$. B is of the form A^s for some operator A on $\mathscr{H}$ if and only if B commutes with K.

(6) Let P be the projection in $\mathscr{H}^s$ on the subspace S of $\mathscr{H}^s$. Then KPK^{-1} is the projection in $\mathscr{H}^s$ on the subspace $K[S] = \{kx : x \in S\}$ of $\mathscr{H}^s$. If S and $K[S]$ are orthogonal in $\mathscr{H}^s$ and T is their direct sum then T is in fact a subspace of $\mathscr{H}$ and a basis for S in $\mathscr{H}^s$ is also a basis for T in $\mathscr{H}$. (For if $x, y \in S$ and $\langle x|y\rangle = 0$, then $ky \in K[S]$ and $\langle x|ky\rangle = 0$ so that $(x|y) = 0$ because of (1) above.) Further if Q is the projection in $\mathscr{H}$ on T then $Q^s = P + KPK^{-1}$.

2.1. Definition. An operator J on $\mathscr{H}$ is said to be imaginary if there exists a basis $\{e_r\}$ of $\mathscr{H}$ such that $Je_r = ie_r$ or 0 for all r.

The proof of the proposition below illustrates in embryo the way in which the symplectic image is used to study operators on $\mathscr{H}$.

2.2. Proposition. *Let J be an operator on $\mathscr{H}$. Then the following statements are equivalent.*

(i) *J is imaginary.*

(ii) *$J^* = -J$ and $I - JJ^*$ is the projection on the null space of J.*

(iii) *J is normal and $-J^2$ is a projection.*

(iv) *J is normal and $J^2 + J^4 = 0$.*

(v) *$J^s = iP - iKPK^{-1}$ for a unique projection P in $\mathscr{H}^s$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Trivial.

(iv) $\Rightarrow$ (v). It is easy to see that the spectral measure of J^s is concentrated at $\{\pm i\}$. Since $Jx = ix$ if and only if $J(kx) = -i(kx)$, it follows from (6) above that J has the required form.

(v) $\Rightarrow$ (i). If $\{e_r\}$ is a basis for the range of P in $\mathscr{H}^s$, then by (6) above $\{e_r\}$ is a basis for the range of $P + KPK^{-1}$ in $\mathscr{H}$. If now $\{f_s\}$ is a basis for the null space of J in $\mathscr{H}$ then $\{e_r, f_s\}$ is a basis for $\mathscr{H}$ such that $Je_r = ie_r$ and $Jf_s = 0$ for all r, s.

3. **The spectral theorem.** Let $(X, \mathscr{S})$ be a measure space and E a spectral measure defined on $(X, \mathscr{S})$ with values in $\mathscr{H}$, i.e. E is a countably additive set function defined on $\mathscr{S}$ whose values are projections on $\mathscr{H}$ and $E(X) = I$. Let f be a real valued E-essentially bounded measurable function on X. Then the map

$$(x, y) \to \int f\, d(E(\cdot)x|y)$$

is a hermitian-symmetric bounded sesquilinear form on $\mathscr{H}$ and hence there exists a unique hermitian operator A such that $(Ax|y) = \int f\, d(E(\cdot)x|y)$. As usual we

write $A=\int f\,dE$. The theory of hermitian operators follows as in the complex case.

The nonhermitian case cannot be handled in a similar fashion because the spectral integral $\int f\,dE$ is not meaningful if f is not E-essentially real. For in such a case we can find a quaternion q for which $f\cdot q\neq q\cdot f$ on a set of nonzero E-measure, so that the integral above as a function of x and y is no more linear in x. We circumvent this difficulty by introducing integration with respect to "spectral systems."

3.1. Definition. An imaginary operator J is said to be admissible with respect to the spectral measure E if (i) J commutes with $E(M)$ for all $M\in\mathscr{S}$ and (ii) there is an $X_0\in\mathscr{S}$ such that $I-JJ^*=E(X_0)$ (i.e. $E(X_0)$ is the projection on the null space of J). The pair (E,J) where E is a spectral measure and J is an admissible imaginary operator is called a spectral system.

It is evident that X_0 is unique up to E-null sets. When we associate canonical spectral systems with normal operators we shall take $X=\boldsymbol{C}^+$ and $X_0=\boldsymbol{R}$. For the present we assume that we have chosen and fixed an X_0.

Let $\mathscr{M}$ denote the *real* Banach *-algebra (with the E-essential supremum norm) of all equivalence classes of *complex-valued* E-essentially bounded measurable functions on X, whose restrictions to X_0 are real.

3.2. Definition. If $f=f_1+if_2\in\mathscr{M}$ (f_1,f_2 real), then $\int f\,dE$ (given J) is the operator on $\mathscr{H}$ defined by

$$\int f\,dE=\left(\int f_1\,dE\right)+J\left(\int f_2\,dE\right).$$

The map $f\to\int f\,dE$ has all the usual properties (see §§37, 38 in [5]) and they may be proved in much the same way. We illustrate with one proof.

3.3. Proposition. *If $A=\int f\,dE$, then $\|Ax\|^2=\int|f|^2\,d(E(\cdot)x|x)$ for all $x\in\mathscr{H}$.*

Proof. Let $f=f_1+if_2$, $B=\int f_1\,dE$ and $C=\int f_2\,dE$. Then noticing that B, C, J are mutually commuting, that $JJ^*C=C$ and recalling that $J^*=-J$, it is easy to prove that $\|Ax\|^2=\|Bx\|^2+\|Cx\|^2$ for all $x\in\mathscr{H}$. Since the proposition is true when f is real, the proof is complete.

The spectral theorem for normal operators on a quaternionic Hilbert space can now be stated.

3.4. Theorem (Spectral Theorem). *Let A be a normal operator on a quaternionic Hilbert space $\mathscr{H}$. Then there exists a unique spectral system (E,J), where E is a spectral measure on the Borel sets of $\boldsymbol{C}^+$ and J satisfies $I-JJ^*=E(\boldsymbol{R})$, such that $A=\int\lambda\,dE$ (given J), λ being the identity function on $\boldsymbol{C}^+$.*

Proof. Let E_s be the spectral measure of A^s on $\mathscr{H}^s$. We observed earlier that if S is the eigen subspace of A^s corresponding to the eigenvalue λ, then $K[S]$ is the eigen subspace of A^s corresponding to the eigenvalue $\bar{\lambda}$. More generally we may prove, using the Stone-Lengyel characterization of spectral subspaces (§§41, 42 in

[5]), that for all compact sets M, $E_s(\overline{M})=KE_s(M)K^{-1}$. Since E_s is regular this relation must be true for all Borel sets M. It follows (see §2) that if $M=\overline{M}$, then $E_s(M)$ must in fact be the symplectic image of a projection on $\mathscr{H}$ and consequently that if $E(M)$ is defined for all Borel M contained in $\boldsymbol{C}^+$ by $E(M)^s=E_s(M)+KE_s(M)K^{-1}$, then E is a spectral measure in $\mathscr{H}$ based on $\boldsymbol{C}^+$. Note that we then have $E(M)^s=E(M\cup\overline{M})$ and in particular $E(\boldsymbol{R})^s=E_s(\boldsymbol{R})$.

Define J on $\mathscr{H}$ by $J^s=iE_s(\boldsymbol{C}^+-\boldsymbol{R})-iE_s(\boldsymbol{C}-\boldsymbol{C}^+)$. J is then imaginary and commutes with E since J^s commutes with E_s. Further $JJ^*=I-E(\boldsymbol{R})$ so that the null space of J is the range of $E(\boldsymbol{R})$ and J is admissible with respect to E. Consequently (E,J) is a spectral system with the required properties.

We have now to prove that $A=B+JC$ where $B=\int \operatorname{Re}(\lambda)\,dE$ and $C=\int \operatorname{Im}(\lambda)\,dE$. It is easy to prove that

$$\int \operatorname{Re}(\lambda)\,d\langle E_s(\cdot)x|y\rangle = \int_{\boldsymbol{C}^+-\boldsymbol{R}} \operatorname{Re}(\lambda)\,d\langle E^s(\cdot)x|y\rangle = \langle B^s x|y\rangle$$

and that

$$i\int \operatorname{Im}(\lambda)\,d\langle E_s(\cdot)x|y\rangle = \int_{\boldsymbol{C}^+-\boldsymbol{R}} \operatorname{Im}(\lambda)\,d\langle E^s(\cdot)x|J^{*s}y\rangle = \langle C^s x|J^{*s}y\rangle$$

by splitting the integrals on the left-hand side into three integrals on $\boldsymbol{R}$, $\boldsymbol{C}^+-\boldsymbol{R}$ and $\boldsymbol{C}-\boldsymbol{C}^+$ respectively and reassembling them suitably, remembering the definitions of E and J. These equalities imply that $A^s=B^s+J^sC^s$ and hence that $A=B+JC$. Uniqueness of the spectral system may be deduced from the uniqueness of the spectral measure in the complex case using the symplectic image. The details are omitted.

3.5. Corollary. *Every normal operator A on $\mathscr{H}$ can be written uniquely in the form $B+JC$ where B, J, C are mutually commuting, B is hermitian, C is positive, J is imaginary and $Jx=0$ if and only if $Cx=0$.*

3.6. Corollary. *An operator B commutes with A if and only if B commutes with E and J.*

4. **Unitary representations of** LCA **groups.** Let G be a second-countable locally compact abelian group. We wish to prove that every (unitary) representation of G in $\mathscr{H}$ is an integral of irreducible representations in an essentially unique way.

Using the spectral theory in the previous section we may prove as in the complex case that the irreducible representations of G can be identified with the continuous homomorphisms of G into $|\boldsymbol{Q}|$, the unit quaternions. However, unlike in the complex case two distinct homomorphisms h_1 and h_2 can still be conjugate to each other ($h_1=qh_2q^{-1}$ for some $q\in|\boldsymbol{Q}|$) and thus give rise to *equivalent* representations. Consequently in decomposing an arbitrary representation of G in terms of irreducible ones, the "correct" support of the spectral measure is not the space of homomorphisms but the set of conjugacy classes of such homomorphisms or (as in

the case of the eigenvalue problem) a "cross section" of such conjugacy classes. If this point of view is not adopted, the spectral measure will not be unique [1].

How can we choose a nice cross section? Observe first that since every maximal abelian subgroup of $|\boldsymbol{Q}|$ is conjugate to $|\boldsymbol{C}|$, the unit complex numbers, given any homomorphism h of G into $|\boldsymbol{Q}|$ we can find a $q \in |\boldsymbol{Q}|$ such that $\lambda(g)=qh(g)q^{-1}$ is a (complex) character of G. λ need not be unique since on replacing q by kq we see that $\bar{\lambda}$ is also conjugate to h. But this is the extent of nonuniqueness. For if λ_1 and λ_2 are characters such that $\lambda_1=q\lambda_2q^{-1}$ then $\lambda_2=\lambda_1$ or $\bar{\lambda}_1$. To see this let $q=\alpha+k\beta$, α, β complex. Then

$$\lambda_1 = |\alpha|^2\lambda_2+|\beta|^2\bar{\lambda}_2+k\bar{\alpha}\beta(\lambda_2-\bar{\lambda}_2).$$

Since the left-hand side of the equation is complex the coefficient of k on the right-hand side must be 0. Therefore either λ_2 is real or one of α or β must be 0. In any case $\lambda_2=\lambda_1$ or $\bar{\lambda}_1$.

Let X denote the (complex) character group of G and X_0 the subgroup of all real characters of G. By a *Borel section* for X we mean a Borel subset A of X containing X_0 which has the property that if $\lambda \notin X_0$ then exactly one of λ and $\bar{\lambda}$ belongs to A. Since we have assumed G to be second countable, an application of the Federer-Morse lemma [11] shows that Borel sections exist. Further it is easy to see, using the methods of [7], that if A and B are two Borel sections endowed with the relativised Borel structures then the natural one-one correspondence between their points is a Borel isomorphism.

Any Borel section A for X is a "cross section" for the equivalence classes of the irreducible quaternionic representations of G.

4.1. THEOREM (THE SNAG THEOREM). *Let U be a unitary representation of the locally compact, second countable abelian group G in a quaternionic Hilbert space $\mathscr{H}$. Then given any Borel section A of X (the character group of G), there exists a unique spectral system (E, J) based on A and acting in $\mathscr{H}$ with $I-JJ^*=E(X_0)$ such that $U_g=\int \lambda(g)\, dE(\lambda)$ (given J). Moreover, the spectral measure E is defined independently of the section A in the sense that given any two sections, the natural Borel isomorphism between them preserves E.*

Proof. Consider U^s on $\mathscr{H}^s$. Let E_s be the unique spectral measure on X associated to U^s on $\mathscr{H}^s$ by the standard complex SNAG theorem. Then one can check that

$$\int \lambda(g)\, d\langle KE_s(\lambda)K^{-1}x|y\rangle = \int \lambda(g)\, d\langle E_s(\bar{\lambda})x|y\rangle$$

so that $KE_s(M)K^{-1}=E_s(\overline{M})$ for all Borel M. Consequently if for all Borel N contained in A we define $E(N)$ by $E(N)^s=E_s(N\cup\overline{N})$, then E becomes a spectral measure based on A acting in $\mathscr{H}$. And if J is defined by $J^s=iE_s(A-X_0)-iE_s(X-A)$, then (E, J) is the required spectral system. The details are omitted.

The very definition of E shows that it is defined independently of the section.

5. **Multiplicity theory.** In this section we show that every normal operator is determined up to unitary equivalence by a multiplicity function based on $\boldsymbol{C}^+$ and deduce two corollaries which are not true in the complex case.

Let $(X, \mathscr{S})$ be a measurable space and X_0 an arbitrary but fixed set in $\mathscr{S}$. All spectral systems (E, J) we consider will satisfy $I-JJ^*=E(X_0)$.

Let (E, J) be a spectral system. Let $\mathscr{P}$ denote the class of all projections on $\mathscr{H}$ which commute with $\mathscr{E}$, the range of E. Let $\mathscr{Q}$ denote the set of projections in $\mathscr{P}$ which commute with J. $\mathscr{P}$ and $\mathscr{Q}$ are then complete lattices and $\mathscr{P}\neq\mathscr{Q}$ in general. (It is a consequence of our results that $\mathscr{P}=\mathscr{Q}$ if and only if $E(X-X_0)$ is a row.) In the standard theory of spectral measures one decomposes $\mathscr{E}$ in terms of $\mathscr{P}$. This is not sufficient for our purposes because if (E, J) is the canonical spectral system associated to a normal operator A, then the family of subspaces reducing A is not $\mathscr{P}$, but $\mathscr{Q}$. We have therefore to decompose $\mathscr{E}$ in terms of $\mathscr{Q}$.

Two spectral systems (E, J) and (F, L) acting in quaternionic Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively are said to be isomorphic if there exists a unitary operator φ from $\mathscr{H}$ onto $\mathscr{K}$ such that $\varphi E(M)=F(M)\varphi$ for all $M\in\mathscr{S}$ and $\varphi J=L\varphi$.

5.1. Definitions. (a) If μ is a finite nonnegative measure on $(X, \mathscr{S})$ the canonical spectral system (E_μ, J_μ) associated to μ is defined by the equations

(i) $E_\mu(M)f=f\cdot\chi_M$ and

(ii) $J_\mu f=f\cdot(1-\chi_{X_0})\cdot i$

for all $f\in L^2_{\mathbb{Q}}(\mu)$ and $M\in\mathscr{S}$, where χ_M denotes the characteristic function of the set M.

(b) A subspace S of $\mathscr{H}$ is said to be of type μ for the spectral system (E, J) if $S\in\mathscr{Q}$ and the restriction of (E, J) to S is isomorphic to the canonical spectral system (E_μ, J_μ).

For the rest of this section we identify a projection with its range. $S\in\mathscr{P}$ is called a cycle if there exists an $x\in S$ such that $S=Z(x)$, the subspace spanned by $\{E(M)x : M\in\mathscr{S}\}$. For $x\in\mathscr{H}$, μ_x will denote the measure on $(X, \mathscr{S})$ defined by $\mu_x(M)=(E(M)x|x)$ for all $M\in\mathscr{S}$.

5.2. Proposition. (a) *The following are equivalent*:

(i) *$S\in\mathscr{Q}$ and is a cycle.*

(ii) *There is a vector $x\in S$ such that S is of type μ_x for (E, J).*

(iii) *There is an $x\in S$ such that $Jx\in S$ and $S=Z(x)$.*

(b) *Every $S\in\mathscr{Q}$, in particular $\mathscr{H}$, is a direct sum of mutually orthogonal cycles belonging to $\mathscr{Q}$.*

Proof. (a) (i) $\Rightarrow$ (ii). We are told that $S=Z(v)$ for some $v\in S$ and $Z(v)\in\mathscr{Q}$. Let $y=E(X-X_0)v$ and $z=E(X_0)v$. Then $v=y+z$ and y and z are very orthogonal so that $Z(v)=Z(y)\oplus Z(z)$. Since $Z(y)=Z(v)E(X-X_0)$ and $Z(z)=Z(v)E(X_0)$, $Z(y)$ and $Z(z)$ are in $\mathscr{Q}$.

Let $J^s=iP^+-iP^-$ be the canonical representation of J^s on $\mathscr{H}^s$. (See 2.2.) We first prove that there is a $y_0\in P^+$ such that $Z(y_0)=Z(y)$. Suppose $y\notin P^+$. Let

$y=y^+ + ky^-$ with $y^+, y^- \in P^+$. Since $Z(y) \in \mathscr{Q}$, $Jy = iy^+ - iky^- \in Z(y)$. Also, $iy = iy^+ + iky^- \in Z(y)$. Consequently, $y^+, y^- \in Z(y)$ and $Z(y^+) \subset Z(y)$. Let now $y_1 = y^- - Z(y^+)y^-$. Then $y_1 \in P^+$ and y_1 is orthogonal to $Z(y^+)$. Hence $Z(y_1)$ is orthogonal to $Z(y^+)$. On the other hand $Z(y_1)$ and $Z(y^+)$ are both contained in $Z(y)$ and by the well-known properties of cycles, if $y_0 = y^+ + y_1$, then $Z(y_0) = Z(y^+) + Z(y_1)$. It is now easy to check that $y_0 \in P^+$ and $Z(y_0) = Z(y)$.

Let $x = y_0 + z$. Then $Z(x) = Z(y_0) + Z(z) = Z(y) + Z(z) = Z(v) = S$ and $Jx = iy_0 = iE(X - X_0)x$.

For convenience, let us write $\mu = \mu_x$ and $X_0 = Y$. By standard methods [5] we can find an isomorphism φ from $L^2_Q(\mu)$ to $Z(x)$ such that $\varphi(1) = x$ and $\varphi E_\mu = E \cdot \varphi$. For this same φ,

$$\begin{aligned} \varphi J_\mu(\chi_M) &= \varphi(\chi_M \cdot \chi_{X-X_0} \cdot i) = i\varphi E_\mu(M \cap (X - Y)) \cdot 1 \\ &= iE(M)E(X-Y)\varphi(1) = iE(M)E(X-Y)x = JE(M)x \\ &= J\varphi(\chi_M), \end{aligned}$$

for all $M \in \mathscr{S}$, and it follows that $\varphi J_\mu(f) = J\varphi(f)$ for all $f \in L^2_Q(\mu)$. Consequently $S = Z(x)$ is of type μ_x for (E, J).

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). J leaves S invariant because $JE(M)x = E(M)Jx \in Z(x)$ for all M. Since $J^* = -J$, $S \in \mathscr{Q}$.

(b) It is sufficient to prove that if $0 \neq S \in \mathscr{Q}$, then there exists a cycle $S_0 \subset S$ and belonging to $\mathscr{Q}$. If $SE(X_0) \neq 0$ then for any nonzero $x \in SE(X_0)$, $Jx = 0$ and $Z(x) = S_0$ will do, by (a)(iii) above. If $SE(X - X_0) \neq 0$, then (with notation as in (a)) $SP^+ \neq 0$ and for any $x \in SP^+$, $Jx = ix$ so that $Z(x) = S_0$ will do.

The proposition above shows that there exist sufficiently many cycles in $\mathscr{Q}$. Knowing this it is an easy matter to proceed with the decomposition of $\mathscr{E}$ in terms of projections in $\mathscr{Q}$, and show that, if for any finite nonnegative measure μ on $(X, \mathscr{S})$ we define $u(\mu)$, the multiplicity of μ, to be the power of any maximal family of mutually orthogonal subspaces of type μ for (E, J), then $u(\mu)$ is a well-defined multiplicity function, and prove the theorem below. We omit the details.

5.3. THEOREM. *Let (E, J) and (F, L) be two spectral systems based on $(X, \mathscr{S})$ and acting in quaternionic Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively with $JJ^* = E(X - X_0)$ and $LL^* = F(X - X_0)$ for some $X_0 \in S$. Then (E, J) and (F, L) are isomorphic if and only if they have the same multiplicity function.*

It is a little surprising that even though the multiplicity function u is defined in terms of both E and J it really depends only on E. This is noticed in the proof of the corollary below.

5.4. COROLLARY. *Let E be a spectral measure and J and L two imaginary operators admissible with respect to E such that $JJ^* = LL^*$. Then (E, J) and (E, L) are isomorphic. In particular (E, J) and (E, J^*) are isomorphic.*

Proof. It is clearly sufficient to prove that the multiplicity function u of (E, J) is the same as the multiplicity function u_0 of E (or $(E, 0)$). If $\mu(X-X_0)=0$, then, trivially, a subspace is of type μ for (E, J) if and only if it is of type μ for E, so that $u(\mu)=u_0(\mu)$. It is therefore sufficient to consider the case when $JJ^*=I$. Let (μ_r) be a basis for u and let (S_{rs}) be a maximal family of mutually orthogonal subspaces of type μ_r for (E, J) such that $\bigoplus_r \bigoplus_s S_{rs}=\mathscr{H}$. Each $\bigoplus_s S_{rs}$ is then a primitive column of uniform multiplicity [10]. But then $u(\mu_r)=u_0(\mu_r)$ and (μ_r) is a basis for u_0 too. Hence $u=u_0$.

Let now A be a normal operator on H and (E, J) the associated spectral system.

5.5. Definition. Let μ be any finite nonnegative measure with compact support in $\boldsymbol{C}^+$. The canonical operator A_μ associated to μ is the operator on $L^2_{\boldsymbol{Q}}(\mu)$ defined by $(A_\mu f)(\lambda)=f(\lambda)\cdot\lambda$, $\lambda\in\boldsymbol{C}^+$.

It is easy to check that the spectral system of A_μ is (E_μ, J_μ). Further if we define a subspace S of $\mathscr{H}$ to be of type μ for A if S reduces A and A restricted to S is unitarily equivalent to A_μ, then S is of type μ for A if and only if S is of type μ for (E, J). We then have the following theorem.

5.6. Theorem (Multiplicity Theorem). *Let A be a normal operator on a quaternionic Hilbert space $\mathscr{H}$. For every finite nonnegative measure μ with compact support on the Borel subsets of $\boldsymbol{C}^+$ define $u(\mu)$ to be the cardinality of any maximal family of mutually orthogonal subspaces of type μ for A. Then u is a well-defined multiplicity function. Two normal operators are unitarily equivalent if and only if their multiplicity functions are the same.*

We now deduce two corollaries of this theorem which have no analogues in the complex case.

In the heuristic explanation of the properties of normal operators on complex Hilbert spaces it is sometimes said that normal operators behave like complex numbers. In a similar spirit we may say that normal operators on quaternionic Hilbert spaces behave like quaternions! The two theorems below reflect the two properties of quaternions that (i) every quaternion q is conjugate to q^* and (ii) if q is real and p commutes with every quaternion commuting with q then p is real.

5.7. Corollary. *Every normal operator A on a quaternionic Hilbert space is unitarily equivalent to its adjoint A^*.*

Proof. Observe that if the spectral system of A is (E, J), then the spectral system of A^* is (E, J^*) and use Corollary 5.4.

5.8. Corollary. *If A is a hermitian operator on a quaternionic Hilbert space and B is an operator which commutes with every operator commuting with A, then B is hermitian.*

Proof. In view of our structure theory it is sufficient to prove this when $A=A_\mu$ where μ is a finite nonnegative measure with compact support contained in $\boldsymbol{R}$.

But then B commutes with E_μ and hence there exists a bounded measurable function h_0 such that $Bf=f\cdot h_0$ for all $f\in L^2_Q(\mu)$. (This may be proved as in the complex case [10].) Since B has to commute with every operator of the form $f\to f\cdot q$ where q is any fixed quaternion, we must have $q\cdot h_0=h_0\cdot q$ a.e. $[\mu]$. Therefore h_0 is essentially real and B is hermitian.

6. **Functional calculus.** We propose now to define functions of a normal operator A on $\mathscr{H}$ and to show that the set of all functions of A coincides with the smallest commutative (real) W^*-algebra containing A (and I).

Let (E, J) be the spectral system associated to A and let $\mathscr{M}$ denote the *real* Banach $*$-algebra (with the E-ess sup norm) of all *complex-valued* E-essentially bounded measurable functions on $\boldsymbol{C}^+$ whose restrictions to $\boldsymbol{R}$ are real. For $f\in\mathscr{M}$, let $f(A)$ denote the operator $\int f\,dE$ (given J). (This implies that if A is hermitian then so is every $f(A)$!)

Since the range of $f\in\mathscr{M}$ need not be contained in $\boldsymbol{C}^+$, it is not obvious what the spectral system of $f(A)$ should be and whether a function of a function of A is again a function of A. These questions are answered by the proposition below, whose proof is omitted.

6.1. Proposition. *Let $f\in M$. Define $f^+\in\mathscr{M}$ by $f^+(\lambda)=f(\lambda)$ or $(f(\lambda))^-$ according as $f(\lambda)\in\boldsymbol{C}^+$ or not.*

(a) *The spectral system (F, L) of $f(A)$ is given by $F(M)=E((f^+)^{-1}(M))$ and $L=JE(f^{-1}(\boldsymbol{C}^+-\boldsymbol{R}))-JE(f^{-1}(\boldsymbol{C}-\boldsymbol{C}^+))$.*

(b) *For $f, g\in\mathscr{M}$, let*

$$\begin{aligned} N &= \{\lambda : g(\lambda)\in\boldsymbol{C}^+-\boldsymbol{R} \text{ and } f(g(\lambda))\in\boldsymbol{C}-\boldsymbol{C}^+\} \\ &\quad\cup\{\lambda : g(\lambda)\in\boldsymbol{C}-\boldsymbol{C}^+ \text{ and } f(\bar{g}(\lambda))\in\boldsymbol{C}^+-\boldsymbol{R}\}. \end{aligned}$$

Define $h\in\mathscr{M}$ by

$$\begin{aligned} h(\lambda) &= [f^+(g^+(\lambda))]^- && \text{if } \lambda\in N, \\ &= f^+(g^+(\lambda)) && \text{if } \lambda\notin N. \end{aligned}$$

Then $f(g(A))=h(A)$.

6.2. Theorem. *Let $\mathscr{H}$ be separable. Let $\Phi(A)$ denote the class of all functions of A, $[A]$, the smallest (real) W^*-algebra of operators containing A (and I) and $[A]''$, the double commutant of A. Then $\Phi(A)=[A]=[A]''$.*

Proof. Let (E, J) be the spectral system of A and $\mathscr{E}$, the range of E.

(i) $\Phi(A)\subset[A]$. If f is real, then it is easy to see that $f(A)\in[A]$. To prove that $J\in[A]$, note that $(A-A^*)/2=\int i\,\mathrm{Im}\,(\lambda)\,dE$ (given J), and define, for every natural number n, a function $f_n\in\mathscr{M}$ by

$$\begin{aligned} f_n(\lambda) &= 1/\mathrm{Im}\,(\lambda) && \text{if } \mathrm{Im}\,(\lambda)\geqq 1/n, \\ &= 0 && \text{if } \mathrm{Im}\,(\lambda) < 1/n. \end{aligned}$$

Then $f_n(A) \in [A]$ for all n and

$$JE(\{\lambda : \operatorname{Im}(\lambda) \geqq 1/n\}) = (A-A^*)f_n(A)/2 \in [A].$$

Taking the limit as $n \to \infty$, we see that

$$J = JE(\{\lambda : \operatorname{Im}(\lambda) > 0\} \in [A].$$

It follows that $\Phi(A) \subset [A]$.

(ii) $[A] \subset [A]''$. Trivial.

(iii) $[A]'' \subset \Phi(A)$. If $B \in [A]''$, then B is normal. Let (F, L) be the spectral system of B and $\mathscr{F}$, the range of F. As in the complex case [10] we show that $\mathscr{F} \subset \mathscr{E}$ and hence that there exists a Borel map $g: \boldsymbol{C}^+ \to \boldsymbol{C}^+$ such that $F(M) = E(g^{-1}(M))$. By Corollary 5.8, the restriction of B to $E(\boldsymbol{R})$ is hermitian and hence $E(\boldsymbol{R}) \subset F(\boldsymbol{R}) = E(g^{-1}(\boldsymbol{R}))$. This means that g is E-essentially real on $\boldsymbol{R}$ and we can assume without loss of generality that $g \in \mathscr{M}$.

Now, L commutes with J. Further $E(\boldsymbol{R}) \subset F(\boldsymbol{R})$ implies that $Lx = 0$ whenever $Jx = 0$. It is then not difficult to show, by studying J^s and L^s on $\mathscr{H}^s$, that $L = JP - JQ$ where $P = \frac{1}{2}L(L^* + J^*)$ and $Q = \frac{1}{2}L(L^* - J^*)$ are mutually orthogonal projections which commute with every projection commuting with A and consequently belong to $\mathscr{E}$. Suppose $Q = E(N)$ and define $f \in \mathscr{M}$ by $f(\lambda) = \bar{g}(\lambda)$ or $g(\lambda)$ according as whether $\lambda \in N$ or not. One may then verify that $B = f(A)$. The details are omitted.

The proof of the theorem is complete.

7. **Commutative W^*-algebras.** In this section we obtain the complete structure theory of (real) commutative W^*-algebras of operators (with identity) (CW^*A's) on a quaternionic Hilbert space. All results are stated without proof since they can be obtained by combining the techniques of the preceding sections with the standard complex methods of [9].

7.1. Definition. (a) A CW^*A $\mathscr{A}$ is called an R-algebra if every $A \in \mathscr{A}$ is hermitian.

(b) A CW^*A $\mathscr{A}$ is called a C-algebra if there is an (imaginary) operator $J \in \mathscr{A}$ such that $J^2 = -I$.

An example of an R-algebra (a C-algebra) is the set of all functions of a hermitian operator (a skew-hermitian operator with 0 null space).

7.2. Theorem. *If $\mathscr{A}$ is any CW^*A, then there exists a unique projection $P \in \mathscr{A}$ such that $\mathscr{A}P$ is an R-algebra (the "real part" of $\mathscr{A}$) and $\mathscr{A}(I-P)$ is a C-algebra (the "complex part" of $\mathscr{A}$). Further two CW^*A's are algebraically isomorphic (unitarily equivalent) if and only if their real and complex parts are separately algebraically isomorphic (unitarily equivalent).*

7.3. Definition. An R-algebra (a C-algebra) is maximal if it is not strictly contained in any R-algebra (C-algebra).

For any $W^*A\ \mathscr{A}$, let $\mathscr{S}(\mathscr{A})$ denote the W^*A of all hermitian operators in $\mathscr{A}$, $\mathscr{A}'$, the commutant of $\mathscr{A}$ and let $\mathscr{C}(\mathscr{A})=\mathscr{A}\cap\mathscr{A}'$ be the center of $\mathscr{A}$.

7.4. THEOREM. (i) *An R-algebra $\mathscr{A}$ is maximal if and only if $\mathscr{A}=\mathscr{S}(\mathscr{A}')$. In this case we also have $\mathscr{A}=\mathscr{C}(\mathscr{A}')$.*

(ii) *A C-algebra $\mathscr{A}$ is maximal if and only if $\mathscr{A}=\mathscr{A}'$.*

7.5. COROLLARY. *A CW^*A is a maximal abelian selfadjoint algebra if and only if it is a maximal C-algebra.*

7.6. DEFINITION. Let $(X, \mathscr{S}, \mu)$ be a nonnegative measure space. Let $\mathscr{A}_\mu$ be the set of all operators on $L^2_Q(\mu)$ of the form $f\to f\cdot g$, for some quaternion-valued, essentially bounded function g on X. The subcollection $\mathscr{A}^R_\mu$ ($\mathscr{A}^C_\mu$) of all operators in $\mathscr{A}_\mu$ with g essentially real (complex) is called the real (complex) multiplication algebra of the measure space $(X, \mathscr{S}, \mu)$.

7.7. THEOREM. *Let $\mathscr{A}$ be an R-algebra (a C-algebra). The following conditions are equivalent.*

(i) *There exists a cyclic vector for $\mathscr{A}$.*

(ii) *$\mathscr{A}$ is maximal and every projection P in $\mathscr{A}$ has the property that every orthogonal family $\{P_j\}$ of nonzero projections in $\mathscr{A}$, such that $P_j\leqq P$ for all j, is necessarily countable.*

(iii) *$\mathscr{A}$ is unitarily equivalent to the real (complex) multiplication algebra of a finite measure space.*

7.8. THEOREM. *Let $\mathscr{A}$ be an R-algebra (a C-algebra) on $\mathscr{H}$. For each cardinal $n\leqq\dim(\mathscr{H})$, there exists a projection P_n in $\mathscr{A}$ such that* (i) *P_n is either* 0 *or an n-fold copy of a maximal R-algebra (C-algebra),* (ii) *the P_n are mutually orthogonal and $\sum_n P_n=I$ and* (iii) *the map $n\to P_n$ of cardinals $\leqq\dim\mathscr{H}$, to projections in $\mathscr{A}$ with properties* (i) *and* (ii) *is unique. Further, if the Boolean algebra $B(n)$ of all projections in $\mathscr{A}P_n$ is called the measure algebra of $\mathscr{A}$ of multiplicity n, then two R-algebras (C-algebras) are unitarily equivalent if and only if their measure algebras for the same multiplicities are isomorphic.*

7.9. THEOREM. *Let $\mathscr{H}$ be separable. If $\mathscr{A}$ is any CW^*A on $\mathscr{H}$, then there exists a normal operator $A\in\mathscr{A}$ such that $\mathscr{A}$ consists precisely of all functions of A. If $\mathscr{A}$ is an R-algebra (a C-algebra) then A is hermitian (skew hermitian with trivial null space). In particular any commuting set of normal (hermitian) operators on $\mathscr{H}$ can be expressed as functions of a single normal (hermitian) operator.*

7.10. THEOREM. *If $\mathscr{A}$ is any CW^*A, then $\mathscr{A}=\mathscr{A}''$.*

We conclude with a few remarks on nonabelian algebras. In the complex case an algebra of operators is abelian if and only if it consists only of normal operators. (If A and B are hermitian, then the normality of $A+iB$ implies that $AB=BA$.) But not so in the quaternionic case. E.g., the "quaternionic" multiplication algebra

$\mathscr{A}_\mu$ (Definition 7.6) is a noncommutative algebra of normal operators. (It is easy to conjecture that an arbitrary "normal" algebra of operators on a quaternionic Hilbert space is decomposable in terms of such $\mathscr{A}_\mu$.) It thus seems necessary to distinguish the "diagonalizable" nonabelian algebras (owing their existence to the noncommutativity of the quaternions) from the general "nondiagonalizable" ones. The study of the latter seems to be difficult. Even the Double Commutant Theorem, whose proof is quite elementary in the complex case, does not seem to be available.

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