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ON CARTAN SUBALGEBRAS OF ALTERNATIVE ALGEBRAS(1)

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Abstract. In 1966, Jacobson introduced the notion of a Cartan subalgebra for finite-dimensional Jordan algebras with unity over fields of characteristic not 2. Since finite-dimensional Jordan, alternative, and Lie algebras are known to be related through their structure theories, it would seem logical that such an analogue would also exist for finite-dimensional alternative algebras. In this paper, we show that this is the case. Moreover, the linear transformation we define that plays the role in alternative algebras that "ad ( )" plays in Lie algebras is identical with that used in the Jordan theory, and can be used in the Lie case as well. Hence we define Cartan subalgebras relative to this linear transformation for finite-dimensional alternative, Jordan, and Lie algebras, and observe that in the Lie case, they coincide with the classical definition of a Cartan subalgebra.

1. **Introduction.** In 1966, Jacobson [4] developed an analogue of Cartan subalgebras for finite-dimensional Jordan algebras with unity over fields of characteristic not 2. Since finite-dimensional Jordan, alternative, and Lie algebras are known to be related through their structure theories, it would seem logical that such an analogue would also exist for finite-dimensional alternative algebras independent of the fact that the symmetric algebra of an alternative algebra is a Jordan algebra.

In this paper, we show that this is indeed the case. Moreover, the linear transformation we define that plays the role in alternative algebras that "ad ( )" plays in the Lie case is identical with that used in the Jordan theory, and can be used for Lie algebras as well. Hence we define Cartan subalgebras relative to this linear transformation for finite-dimensional alternative, Jordan, and Lie algebras, and observe that in the Lie case they coincide with the classical definition of a Cartan subalgebra. However, in the Lie case as in the Jordan case, we are restricted to

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ground fields of characteristic not 2. Finally, following Barnes [1], we will see that we can characterize our Cartan subalgebras as "minimal Engel subalgebras".

Most of the notation in this paper is standard. If U is an algebra and I is an ideal of U, we write $I \lhd U$. If $u \in U$, by R_u and L_u we mean the linear transformations of U where $x \in U$, $xR_u = xu$ and $xL_u = ux$. We recall that if U is a Lie algebra, then $R_u = \mathrm{ad}\,(u)$. If $u_1, u_2, u_3 \in U$, we write $(u_1, u_2, u_3) = u_1u_2 \cdot u_3 - u_1 \cdot u_2u_3$ for the associator of u_1, u_2, and u_3, and $[u_1, u_2] = u_1u_2 - u_2u_1$ for the commutator of u_1 and u_2.

2. **Preliminaries.** An algebra A over a field F is *alternative* if $(a, a, b) = (b, a, a) = 0$ for all elements $a, b \in A$. From this definition, it is clear that a homomorphic image of an alternative algebra and a direct sum of alternative algebras are alternative. Since the associator is multilinear, we have that A_K is alternative for all extensions K of F, where $A_K = A \otimes_F K$ is the scalar extension of A to an algebra over K.

We will now give those properties of alternative algebras that we will require in our construction of Cartan subalgebras, and refer the reader to Schafer [8] for proofs.

Linearizing the identities that define A, we obtain

$$\begin{aligned} (a, b, c) &= -(a, c, b) = (c, a, b), \\ (a, b, a) &= 0 \end{aligned} \qquad \text{for all } a, b, c \in A. \tag{2.1}$$

In terms of R_a and L_a, $a \in A$, these become

$$\begin{aligned} R_aR_b - R_{ab} &= L_{ab} - L_bL_a = [L_b, R_a] \\ &= L_aL_b - L_{ba} = R_{ba} - R_bR_a = [R_b, L_a], \quad \text{for all } a, b \in A. \\ [R_a, L_a] &= 0 \end{aligned} \tag{2.2}$$

Since the subalgebra of A generated by any two elements of A is associative, we have

$$\begin{aligned} R_{a^i} &= (R_a)^i, \\ L_{a^i} &= (L_a)^i \end{aligned} \qquad \text{for all } a \in A \text{ and positive integers } i. \tag{2.3}$$

Now suppose e is an idempotent of A. Then A can be written as a direct sum

$$\begin{aligned} A &= A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}, \\ A_{ij} &= \{x_{ij} \in A \mid x_{ij}e = jx_{ij},\ ex_{ij} = ix_{ij}\}, \qquad i, j = 0, 1. \end{aligned} \tag{2.4}$$

If $\{e_1, \ldots, e_t\}$ is a set of pairwise orthogonal idempotents of A ($e_i^2 = e_i$, $e_ie_j = e_je_i = 0$ for $i \neq j$), we get a refined decomposition of A as

$$A = \sum_{i,j=1}^{t} \oplus A_{ij} \tag{2.5}$$

where if δ_{ij} is the Kronecker delta,

$$A_{ij} = \{x_{ij} \in A \mid e_kx_{ij} = \delta_{ki}x_{ij},\ x_{ij}e_k = \delta_{jk}x_{ij},\ k = 1, \ldots, t\}.$$

This decomposition is called the *Pierce decomposition* of A relative to the set $\{e_1, \ldots, e_t\}$. We have the following multiplicative relationships between the subspaces A_{ij}:

$$
\begin{aligned}
&A_{ij}A_{jk} \subseteq A_{ik}, && i, j, k = 1, \ldots, t,\\
(2.6)\qquad &A_{ij}A_{ij} \subseteq A_{ji}, && i, j = 1, \ldots, t,\\
&A_{ij}A_{pq} = 0, && j \neq p, \quad (i,j) \neq (p,q), \quad i, j, p, q = 1, \ldots, t.
\end{aligned}
$$

If A is finite dimensional, then the following are equivalent:

(i) A is nilpotent,

(ii) A is solvable,

(iii) A is nil.

In this case, A has a unique maximal solvable ideal $S'(A)$ and $S'(A/S'(A))=0$. $S'(A)$ is called the *radical* of A. If $S'(A)=0$, A is *semisimple*.

Finally, an idempotent e of A is *primitive* if e cannot be written as a sum of nonzero pairwise orthogonal idempotents.

3. **The universal enveloping algebra of an alternative algebra.** Suppose A is an alternative algebra over a field F. We will denote its anti-isomorphic image by A', and set $B=A \oplus A'$. If $a \in A$, we will write a' for the anti-isomorphic image of a.

(3.1) Suppose ρ is a linear map from B into an associative algebra V such that

$$
\text{(i)}\qquad
\begin{aligned}
[a_1'\rho, a_2\rho] &= (a_1a_2)\rho-(a_1\rho)(a_2\rho)\\
&= (a_2a_1)'\rho-(a_1'\rho)(a_2'\rho) = [a_1\rho, a_2'\rho],
\end{aligned}
$$

$$
\text{(ii)}\qquad (a_1a_1)\rho-(a_1\rho)(a_1\rho) = (a_1a_1)'\rho-(a_1'\rho)(a_1'\rho) = 0
$$

for all $a_1, a_2 \in A$. Then ρ is called a *representation* of A. We note that if the characteristic of F is different from 2, then the conditions given in (ii) are a consequence of those given in (i).

Every alternative algebra A has a representation. Indeed, the linear map ρ_R where, if $a \in A$, $a\rho_R=R_a$ and $a'\rho_R=L_a$ is a representation of A, and is called the *regular representation* of A. Here V is the ring of endomorphisms of A.

(3.2) Suppose U is an associative algebra for which there is a linear map $i: B \to U$ such that for each representation ρ of A into an associative algebra V, there is a unique homomorphism $\rho^*: U \to V$ such that $b\rho=(bi)\rho^*$ for all $b \in B$. Then the pair (U, i) is called a *universal enveloping algebra* for A.

To see that such an algebra exists, let $T(B)$ be the tensor algebra based on the vector space B, i.e. $T(B)=B_1 \oplus B_2 \oplus \cdots$ where $B_1=B$ and inductively for $n>1$, $B_n=B_{n-1} \otimes B_1$. The vector space operations in $T(B)$ are the usual ones, and $T(B)$ has an associative product $\otimes$ consistent with this notation. Now any representation

ρ of A into an associative algebra V can be uniquely extended to a homomorphism ρ' of $T(B)$ into V. From (3.1), the kernel of ρ' contains all elements of the form

$$
(3.3)\quad
\begin{aligned}
&\text{(i)} && a_1' \otimes a_2 - a_2 \otimes a_1' - a_1a_2 + a_1 \otimes a_2,\\
&\text{(ii)} && a_1a_2 - a_1 \otimes a_2 - (a_2a_1)' + a_1' \otimes a_2',\\
&\text{(iii)} && (a_2a_1)' - a_1' \otimes a_2' - a_1 \otimes a_2' + a_2' \otimes a_1,\\
&\text{(iv)} && a_1a_1 - a_1 \otimes a_1,\\
&\text{(v)} && (a_1a_1)' - a_1' \otimes a_1' \quad \text{where } a_1, a_2 \in A.
\end{aligned}
$$

Suppose K is the ideal of $T(B)$ generated by elements of this form. Then ρ' will induce a unique homomorphism $\rho^*: T(B)/K \to V$. Hence if i is the restriction to B of the natural homomorphism from $T(B)$ onto $T(B)/K$, we have $b\rho = (bi)\rho^*$ for all $b \in B$, and it follows that $(T(B)/K, i)$ is a universal enveloping algebra of A. We will write $U(A) = T(B)/K$, and for each $a \in A$, $ai = \bar{a}$ and $a'i = \bar{a}'$.

(3.4) We have the following properties for the universal enveloping algebra [5, Chapter 5]:

(i) universal enveloping algebras are determined within isomorphism by A, consequently we take $(U(A), i)$ for the universal enveloping algebra of A,

(ii) $U(A)$ is generated by $\bar{a}$ and $\bar{a}'$, $a \in A$,

(iii) if $K \lhd A$ and D is the ideal of $U(A)$ generated by $(K \oplus K')i$, then there exists an isomorphism of $U(A/K)$ onto $U(A)/D$ such that

$$\overline{a + (K \oplus K')i} \to \bar{a} + D$$

and

$$\overline{a' + (K \oplus K')i} \to \bar{a}' + D.$$

We now wish to prove that if A is finite dimensional and I is a nil ideal of A, then $(I \oplus I')i$ generates a nilpotent ideal in $U(A)$. We begin by proving

LEMMA 3.5. *Suppose $I \lhd A$ and I^* is the subalgebra of $U(A)$ generated by $(I \oplus I')i$. Then $D = I^* + I^* \otimes U(A)$ is a two-sided ideal of $U(A)$.*

Proof. It is clear that D is a right ideal of $U(A)$. To see that it is a left ideal, we observe from (3.3)(i)–(iii) that we have the following identities in $U(A)$:

$$
(3.6)\quad
\begin{aligned}
&\text{(i)} && \bar{a} \otimes \bar{b}' = \bar{b}' \otimes \bar{a} - \overline{ba} + \bar{b} \otimes \bar{a},\\
&\text{(ii)} && \bar{a}' \otimes \bar{b}' = (\overline{ba})' + \overline{ba} - \bar{b} \otimes \bar{a},\\
&\text{(iii)} && \bar{a} \otimes \bar{b} = \overline{ab} + \overline{ba} - \bar{b} \otimes \bar{a},\\
&\text{(iv)} && \bar{a}' \otimes \bar{b} = \bar{b} \otimes \bar{a}' + \bar{b} \otimes \bar{a} - \overline{ba}.
\end{aligned}
$$

Since $\bar{a}$ and $\bar{a}'$, $a \in A$, generate $U(A)$, and $\bar{b}$ and $\bar{b}'$, $b \in I$, generate I^*, it follows from these identities that D is a two-sided ideal of $U(A)$.

COROLLARY 3.7. *In the setting of the previous lemma, if I^* is nilpotent, then D is nilpotent.*

Proof. Since $U(A) \otimes I \subseteq D$, an easy induction argument shows that for all n, $D^n \subseteq (I^*)^n + (I^*)^n \otimes U(A)$, and the result follows.

LEMMA 3.8. *If A is nil and the dimension of A is one, then $(U(A))^3 = 0$.*

Proof. Since the dimension of A is one, $A = F \cdot e$ where $e^2 = 0$. By (3.3)(iv) and (v) we have $\bar{e} \otimes \bar{e} = \bar{e}' \otimes \bar{e}' = 0$. Hence by (3.6)(i), $\bar{e} \otimes \bar{e}' = \bar{e}' \otimes \bar{e}$. Since $U(A)$ is generated by $\{\bar{e}, \bar{e}'\}$, it follows easily that $(U(A))^3 = 0$.

LEMMA 3.9. *Suppose A is finite dimensional. If A is nil, then $U(A)$ is nilpotent.*

Proof. The proof is by induction on the dimension of A. By Lemma 3.8, we may assume $\dim A = n > 1$. Since A is solvable, there is an $n-1$ dimensional subspace I such that $A^2 \subseteq I \subseteq A$, whence $I \lhd A$. Consequently, as $\dim (A/I) = 1$, $U(A/I)$ is nilpotent. Now let I_1 be the ideal of $U(A)$ and I^* the subalgebra of $U(A)$ generated by $(I \oplus I')i$. Then $I_1 = I^* + I^* \otimes U(A)$. By the induction hypothesis, $U(I)$ is nilpotent, hence I^*, being a homomorphic image of $U(I)$, is nilpotent. By Corollary 3.7, I_1 is nilpotent. But by (3.4)(iii), $U(A/I)$ is isomorphic to $U(A)/I_1$, whence $U(A)/I_1$ is nilpotent. Therefore $U(A)$ is nilpotent, as required.

THEOREM 3.10. *Suppose A is finite dimensional, $(U(A), i)$ is the universal enveloping algebra of A, and I is a nil ideal of A. Then $(I \oplus I')i$ generates a nilpotent ideal in $U(A)$.*

Proof. Suppose I^* is the subalgebra of $U(A)$ generated by $(I \oplus I')i$. By Lemma 3.9, $U(I)$ is nilpotent, thus I^* is nilpotent, and the result follows from Corollary 3.7.

REMARK. Schafer has obtained what is essentially this result for finite-dimensional alternative algebras over fields of characteristic zero [7].

As a corollary to Theorem 3.10, we prove the following result which we will require later in the paper.

COROLLARY 3.11. *Suppose A is finite dimensional, A_1 is a subalgebra of A, and I is a nil ideal of A_1. If C is the subalgebra of* $\mathrm{Hom}_F(A, A)$ *generated by R_a, L_a, $a \in A_1$, then R_b, L_b, $b \in I$, generate a nilpotent ideal in C.*

Proof. Let ρ be a linear map from $A_1 \oplus A_1'$ into C where, if $a \in A_1$, $a\rho = R_a$ and $a'\rho = L_a$. Then ρ is a representation of A_1, hence there is a homomorphism from $U(A_1)$ into C. Since C is generated by R_a and L_a, $a \in A_1$, it follows that this homomorphism is onto C and maps the ideal of $U(A_1)$ generated by $(I \oplus I')i$ onto the ideal of C generated by R_b, L_b, $b \in I$. Since $(I \oplus I')i$ generates a nilpotent ideal in $U(A_1)$, R_b and L_b, $b \in I$, generate a nilpotent ideal in C, as desired.

4. **The theory of a-nilpotence.**

(4.1) Suppose A is an alternative algebra over a field F, and $b_1, b_2, b_3 \in A$. We set

$$a(b_1, b_2, b_3) = b_3 \cdot b_1 b_2 + b_2 b_1 \cdot b_3 - b_2 b_3 \cdot b_1 - b_1 \cdot b_3 b_2.$$

Since A is alternative, we can also write

$$a(b_1, b_2, b_3) = [b_2, [b_1, b_3]] - 4(b_1, b_2, b_3).$$

We define $a^1(b_1, b_2, b_3) = a(b_1, b_2, b_3)$ and inductively for $k > 1$ and elements $b_1, \ldots, b_{2k+1} \in A$

$$a^k(b_1, \ldots, b_{2k+1}) = a(a^{k-1}(b_1, \ldots, b_{2k-1}), b_{2k}, b_{2k+1}).$$

When no ambiguity arises, we will write the left-hand side of this expression $a^k(\ )$. We say that A is *a-nilpotent* if there is a $k > 0$ such that $a^k(\ ) = 0$ for all choices $b_i \in A$. Clearly any subalgebra and any homomorphic image of an a-nilpotent alternative algebra is a-nilpotent. Furthermore, since $a^k(\ )$ is multilinear for all k, A is a-nilpotent if and only if A_K is a-nilpotent for all extensions K of F. If A contains a unity and G is a subfield of the centre of A containing the unity of A, then A can be regarded as an algebra over G. In this case, we have that A is a-nilpotent if and only if A/G is a-nilpotent.

(4.2) Suppose $b, c \in A$. We define a linear transformation on A as follows: for all $x \in A$, $xS(b, c) = a(x, b, c)$. Note that $S(b, c)$ is linear in each of its arguments. If $b \in A$ and for some positive integer $n = n(b)$, $S(b, b)^n = 0$, then b is called a-nilpotent. If every element of A is a-nilpotent, A is called *a-nil*. We note that if A is a-nil and $\dim A = m < \infty$, then $S(b, b)^m = 0$ for all $b \in A$.

Throughout the rest of this section and §5, we will assume that A is an $m < \infty$ dimensional alternative algebra with unity 1 over a field F.

Our determination of the structure of a-nilpotent alternative algebras is parallel to the determination of the structure of associator nilpotent Jordan algebras [6]. Hence where proofs are not provided, the corresponding results in the Jordan case will be cited, the proofs in the alternative case being the same (mutatis mutandis).

(4.3) Suppose A contains a nil ideal N such that A/N is an associative, purely inseparable algebraic field over F. Then A is called *purely inseparable*. If F is perfect, this condition becomes $A/N \cong F$, or $A = F \cdot 1 + N$.

LEMMA 4.4. *If A is purely inseparable and K is the algebraic closure of F, then $A_K = K \cdot 1 + N$ where N is a nil ideal of A_K.*

Proof [6, LEMMA 1, p. 344].

(4.5) If $b \in A$, then b is *separable* if the subalgebra $F[b]$ of A generated by b and 1 is separable, i.e. for all extensions P of F, $F[b]_P$ is a direct sum of simple ideals. If e is an idempotent of A, then e is separable. If b is separable and K is the algebraic closure of F, then $F[b]_K = K \cdot e_1 + \cdots + K \cdot e_t$ where $\{e_1, \ldots, e_t\}$ is a set of pairwise orthogonal idempotents whose sum is 1.

LEMMA 4.6. *Suppose A is a-nilpotent. Then every separable element of A is contained in the centre of A.*

Proof [6, LEMMA 2, p. 345].

THEOREM 4.7. *The algebra A is a-nilpotent if and only if $A = A_1 \oplus \cdots \oplus A_s$ where the A_i are ideals of A, and each A_i contains a separable subfield G_i of its centre containing the identity of A_i such that A_i/G_i is purely inseparable.*

Proof [6, THEOREM 18, p. 345].

COROLLARY 4.8. *Suppose F is algebraically closed. Then A is a-nilpotent if and only if A is expressible as a direct sum of ideals A_i where $A_i = F \cdot 1_i + N_i$, 1_i is the identity of A_i and N_i is a nil ideal of A_i.*

Proof [6, COROLLARY 2, p. 346].

To prove the analogue of Engel's theorem, we will require

LEMMA 4.9. *For any $b \in A$ and nonnegative integers i, j, k,*

$$S(b^i, b^j)^k = S(b, b)^k \varphi_k(R_b, L_b)$$

where $\varphi_k(i, j)$ is a polynomial in i and j over the integers. Consequently, if b is a-nilpotent, then $S(b^i, b^j)$ is a nilpotent linear transformation of A for all $i, j \geqq 0$.

Proof. Since the subalgebra of A generated by any two elements from A is associative, it is clear that

(a) $$S(b^i, b^j) = S(b^j, b^i) \quad \text{for all } i, j \geqq 0.$$

We recall that in an associative algebra

(b) $$[xy, z] = x[y, z] + [x, z]y.$$

Applying this to (4.1), and since the subalgebra of A generated by $\{x, b\}$ is associative for $x \in A$, we find $xS(b^i, b^j) = [b^{i-1}b, [x, b^j]]$, whence

(c) $$S(b^i, b^j) = S(b, b^j)L_{b^{i-1}} + S(b^{i-1}, b^j)R_b, \qquad i \geqq 2.$$

From (a) and (c) an easy induction gives us

(d) $$S(b^i, b^j) = S(b, b)\varphi_1(R_b, L_b), \qquad i, j \geqq 1,$$

where $\varphi_1(i, j)$ is a polynomial in i and j over the integers. Next, using (b), we obtain

(e) $$R_b S(b^i, b^j) = S(b^i, b^j)R_b, \qquad i, j \geqq 0,$$

and similarly

(f) $$L_b S(b^i, b^j) = S(b^i, b^j)L_b, \qquad i, j \geqq 0.$$

Now (e) and (f) together with (2.3) imply that for all i and j, $S(b^i, b^j)$ commutes with any polynomial in R_b and L_b. Thus from (d), we have

(g) $$S(b^i, b^j)^k = S(b, b)^k \varphi_k(R_b, L_b)$$

where $i, j, k > 0$ and $\varphi_k = \varphi_k(i, j)$ is a polynomial over the integers. The case when one or both of i and j is zero is trivial. The final assertion now follows from (g).

We now give

THEOREM 4.10 [ENGEL]. *The algebra A is a-nil if and only if it is a-nilpotent.*

Proof [6, THEOREM 19, p. 347].

5. **Cartan subalgebras of alternative algebras.** Suppose L is a nilpotent Lie algebra of linear transformations of an $n < \infty$ dimensional vector space W where the ground field of both L and W is F. Then we can write $W = W_0 \oplus W_1$ where W_0 and W_1 are respectively the Fitting null and one component of W relative to L. We have $W_0 = \{w \in W \mid wh^n = 0 \text{ for all } h \in L\}$, and $W_1 = \bigcap_i W(L^*)^i$ where L^* is the subalgebra of the enveloping associative algebra $C(L)$ of L generated by L [5, p. 39]. Furthermore, Jacobson [4, p. 599] has shown that W_1 can be characterized as any complementary subspace of W_0 that is invariant under L.

Now suppose P is an arbitrary extension of F. Then L_P is a nilpotent Lie algebra of linear transformations of W_P. Since every element of $(L_P)^*$ is a P-linear combination of elements of L^*, and since $W_0(L^*)^n = 0$, we have $(W_P)_0 = (W_0)_P$. Similarly, we have $(W_1)_P = (W_P)_1$.

(5.1) Suppose $b \in A$. We set $B_b = \{x \in A \mid xS(b, b)^m = 0\}$. Note that both b and 1 are contained in B_b. B_b is called *minimal Engel in A* if for $c \in A$ and $B_c \subseteq B_b$, $B_b = B_c$.

(5.2) Suppose R is a subalgebra of A. By $L_A(R)$ we mean the Lie algebra of linear transformations of A generated by the set $\{S(b, c) \mid b, c \in R\}$.

We have

THEOREM 5.3. *Suppose R is an a-nilpotent subalgebra of A and $1 \in R$. Then $L_A(R)$ is nilpotent, and if $A = A_0 \oplus A_1$ is the Fitting decomposition of A relative to $L_A(R)$, then*

(i) *A_0 is a subalgebra of A containing R,*

(ii) *$A_0 A_1 \subseteq A_1$ and $A_1 A_0 \subseteq A_1$.*

Moreover, if F is algebraically closed and $\{e_1, \ldots, e_t\}$ is a set of pairwise orthogonal primitive idempotents of R whose sum is 1, then $A_0 = \bigcap \{B_b \mid b \in R\} = \bigcap_{i=1}^t B_{e_i}$.

Proof. By the remarks preceding (5.1), we see that we may assume F is algebraically closed. By virtue of Corollary 3.11 and Corollary 4.8, the proof that $L_A(R)$ is nilpotent is the same as Jacobson's proof of the analogous result in the Jordan case (mutatis mutandis) [4, p. 600].

We will now prove the rest of the theorem. Since F is algebraically closed, we may write $R = \sum_{i=1}^t (F \cdot e_i + N_i)$ where $\{e_1, \ldots, e_t\}$ is a set of pairwise orthogonal primitive idempotents whose sum is 1 and the N_i are nil ideals of R. Suppose $\sum_{i,j} A_{ij}$ is the Pierce decomposition of A relative to the set $\{e_1, \ldots, e_t\}$. Then for all integers $i, p, q > 0$, $A_{ii} S(e_p, e_q) = 0$. It follows easily that if $a_{ii} \in A_{ii}$ and $b, c \in R$, then $a_{ii} S(b, c) = a_{ii} S(n_i, n_i')$ where n_i, n_i' are the components of b and c respectively

in N_i. Thus $a_{ii}S(b, c) \in A_{ii}$, and hence $a_{ii}S(b, c)^m = a_{ii}S(n_i, n_i')^m = 0$ by Corollary 3.11. Thus each element of $L_A(R)$ acts nilpotently on A_{ii}, and we have $R \subseteq \sum A_{ii} \subseteq A_0$. If $b \in R$, $S(b, b) \in L_A(R)$, hence $A_0 \subseteq \bigcap \{B_b \mid b \in R\} \subseteq \bigcap_{i=1}^t B_{e_i}$. Now let $x \in \bigcap B_{e_i}$. By (2.5), we write $x = \sum_i x_{ii} + \sum_{i \neq j} x_{ij}$. We compute $xS(e_k, e_k) = \sum_{i=k \neq j; i \neq j=k} x_{ij}$. But $x \in B_{e_k}$ means that $xS(e_k, e_k)^m = \sum_{i=k \neq j; i \neq j=k} x_{ij} = 0$, and as (2.5) is a direct sum, $x_{ik} = x_{ki} = 0$ for $i \neq k$. Since $x \in B_{e_k}$ for $k = 1, \ldots, t$, it follows that $x = \sum x_{ii}$, i.e. $\bigcap B_{e_i} \subseteq \sum A_{ii}$. Consequently, $R \subseteq A_0 = \sum A_{ii} = \bigcap \{B_b \mid b \in R\} = \bigcap B_{e_i}$. As $\sum_{i \neq j} A_{ij}$ is invariant under $L_A(R)$, we have $A_1 = \sum_{i \neq j} A_{ij}$.

That A_0 is a subalgebra of A, and $A_0 A_1 \subseteq A_1$ and $A_1 A_0 \subseteq A_1$ follow from (2.6), and the proof is complete.

COROLLARY 5.4. *For any $b \in A$, B_b is a subalgebra of A.*

Proof. We define $B_b^* = \{x \in A \mid xS(b^i, b^j)^m = 0, i, j = 0, 1, \ldots\}$. From Lemma 4.9, we have immediately that $B_b = B_b^*$. We recall that $F[b]$ is the subalgebra of A generated by b and 1. Clearly $F[b]$ is a-nilpotent. Thus $L_A(F[b])$ is nilpotent, and the Fitting null component A_0 of A relative to $L_A(F[b])$ is a subalgebra of A containing $F[b]$. But elements of $L_A(F[b])$ are sums of products of $S(b^i, b^j)$, $i, j = 0, 1, \ldots$, and it follows that $B_b = A_0$, as required.

(5.5) A subalgebra H of A is a *Cartan subalgebra* if

(i) H is a-nilpotent,

(ii) H coincides with the Fitting null component of A relative to $L_A(H)$.

We know that Cartan subalgebras exist in some alternative algebras. Indeed, if F is algebraically closed, by Theorem 5.3 we have that H is a Cartan subalgebra of A if and only if H contains a set $\{e_1, \ldots, e_t\}$ or pairwise orthogonal primitive idempotents whose sum is 1, and $H = \sum A_{ii}$ where $\sum A_{ij}$ is the Pierce decomposition of A relative to the e_i. However, to construct Cartan subalgebras in a more general situation, we will follow the work of Barnes in the Lie case [1].

Thus let $\{u_1, \ldots, u_m\}$ be a basis for A over F, and $P = F\{\xi_1, \ldots, \xi_m\}$ be the field of rational expressions where the ξ_i are independent transcendentals over F. Then $x = \sum_{i=1}^m \xi_i u_i \in A_P$ is a generic element of A, and the characteristic polynomial $\varphi(t, x)$ of $S(x, x)$ in A_P can be written

$$\varphi(t, x) = t^m + \beta_1(x)t^{m-1} + \cdots + \beta_s(x)t^{m-s} \tag{5.6}$$

where $\beta_i(x) = \beta_i(\xi_1, \ldots, \xi_m)$ is a homogeneous polynomial of total degree $2i$ or the zero polynomial, and $\beta_s(x)$ is the last coefficient which is not the zero polynomial. Since x is an eigenvector of $S(x, x)$, we have that $s < m$. If $a = \sum_{i=1}^m \alpha_i u_i \in A$, then the specialization $\xi_i = \alpha_i$ in (5.6) gives

$$\varphi(t, a) = t^m + \beta_1(a)t^{m-1} + \cdots + \beta_s(a)t^{m-s}. \tag{5.7}$$

We note that $\varphi(t, a)$ is unaffected by an extension of the ground field. The element $a \in A$ is called *a-regular* if and only if $\beta_s(a) \neq 0$. Clearly if a is an a-regular element of A, then B_a is minimal Engel in A, but not necessarily vice versa. Also the property

of being a-regular is preserved under field extensions. Moreover, if F is infinite, the set of a-regular elements of A forms a dense Zariski open subset of A.

The proofs of the following two lemmas are virtually identical with the proofs given by Barnes in the Lie case [1, p. 351].

LEMMA 5.8. *If F has at least $2(m-1)$ elements, then A contains a-regular elements.*

LEMMA 5.9. *Suppose F has at least $2m$ elements. If R is a subalgebra of A, B_b is minimal with respect to dimension in $\{B_c \mid c \in R\}$, and $R \subseteq B_b$, then $B_b \subseteq B_c$ for all $c \in R$.*

To prove the existence theorem, we will require the following lemma. First, we recall that if K is an extension of F and $\{u_1, \ldots, u_m\}$ is a basis for A over F, then $\{1 \otimes u_1, \ldots, 1 \otimes u_m\}$ is a basis for A_K over K. If $b \in A_K$, we will write $b = \sum_{i=1}^{m} \alpha_i u_i$, $\alpha_i \in K$.

LEMMA 5.10. *Suppose F has at least $2m$ elements, R is a subalgebra of A, and for all $b \in R$, $S(b, b)$ is nilpotent of A. Then for any extension K of F, $S(b', b')$ is nilpotent on A_K for all $b' \in A_K$.*

Proof. Suppose $\{b_1, \ldots, b_{m'}\}$ is a basis for R over F, $b \in R$, and $b = \sum_{i=1}^{m'} \beta_i b_i$ where $\beta_i \in F$. Since S is linear in each of its arguments, we have

$$\begin{aligned} 0 &= S(b, b)^m \\ &= \left(\sum_{j_1, j_2 = 1}^{m'} \beta_{j_1} \beta_{j_2} S(b_{j_1}, b_{j_2}) \right)^m. \end{aligned}$$

Let T'_{i_1} equal the sum of the terms of T where β_1 appears to the i_1 power, and set $T'_{i_1} = (\beta_1)^{i_1} T_{i_1}$. Then $T = \sum_{i_1=0}^{2m} (\beta_{i_1})^{i_1} T_{i_1}$. However, $T_{2m} = S(b_1, b_1)^m = 0$ and $T_0 = S(\beta_2 b_2 + \cdots + \beta_{m'} b_{m'}, \beta_2 b_2 + \cdots + \beta_{m'} b_{m'})^m = 0$. Therefore,

$$T = \sum_{i_1=1}^{2m-1} (\beta_1)^{i_1} T_{i_1}.$$

Choosing $2m-1$ different nonzero values for β_1, we obtain a system of $2m-1$ homogeneous equations in T_{i_1}, $i_1 = 1, \ldots, 2m-1$, whose matrix of coefficients has a nonzero determinant. Thus we conclude that $T_{i_1} = 0$ for $i_1 = 0, \ldots, 2m$.

Now let T'_{i_1, i_2} be the sum of the terms of T_{i_1} where β_2 appears to the i_2 power, and set $T'_{i_1, i_2} = (\beta_2)^{i_2} T_{i_1, i_2}$. Repeating the above process, we conclude $T_{i_1, i_2} = 0$. Continuing in this fashion, we find that $T_{i_1, \ldots, i_{m'}} = 0$ for all m'-tuples $(i_1, \ldots, i_{m'})$ such that $\sum_{j=1}^{m'} i_j = 2m$.

Finally, if $b' \in R_K$, we write $b' = \sum_{i=1}^{m'} \alpha_i b_i$ where $\alpha_i \in K$. Then

$$S(b', b')^m = \sum_{i_1, \ldots, i_{m'}; i_1 + \cdots + i_{m'} = 2m} \alpha_1^{i_1} \cdots \alpha_{m'}^{i_{m'}} T_{i_1, \ldots, i_{m'}} = 0,$$

as required.

We have

THEOREM 5.11. *Suppose F has at least $2m$ elements. Then H is a Cartan subalgebra of A if and only if H is minimal Engel in A.*

Proof. Suppose $H=B_b$ is minimal Engel in A. Then B_b is minimal with respect to dimension in $\{B_c \mid c \in H\}$, consequently, by Lemma 5.9, $B_b \subseteq B_c$ for all $c \in H$. Thus $S(c, c)$ is nilpotent on H for all $c \in H$, and by Theorem 4.10, H is a-nilpotent. Since $L_A(H)$ is nilpotent, $H \subseteq A_0$ where A_0 is the Fitting null component of A relative to $L_A(H)$. However, $A_0 \subseteq B_b = H$, and $H = A_0$ as desired.

To prove the converse, we first make the following observations:

(a) if R is a subalgebra of A and $R \subseteq B_c$ for some $c \in A$, then $R_K \subseteq (B_c)_K$ for all extensions K of F,

(b) if R is a Cartan subalgebra of A, then R_K is a Cartan subalgebra of A_K for all extensions K of F (see remarks preceding (5.1)).

Now suppose H is a Cartan subalgebra of A and B_b is minimal with respect to dimension in $\{B_h \mid h \in H\}$. We claim $H=B_b$. Since H is a-nilpotent, $H \subseteq B_b$, and consequently by Lemma 5.9, $B_b \subseteq B_h$ for all $h \in H$. If K is the algebraic closure of F then $H_K = \bigcap \{B_{h'} \mid h' \in H_K\}$. But $B_b \subseteq B_h$ for all $h \in H$ means $S(h, h)$ acts nilpotently on B_b for all $h \in H$. By Lemma 5.10, $S(h', h')$ is nilpotent on $(B_b)_K$ for all $h' \in H_K$. Therefore $(B_b)_K \subseteq H_K$, and by (a) it follows that $(B_b)_K = H_K$. Hence the dimension of $(B_b)_K$ and H_K over K are equal, which implies the dimension of H and B_b over F are equal. Since $H \subseteq B_b$, $H = B_b$. That B_b is minimal Engel in A follows easily, as required.

COROLLARY 5.12. *If a is an a-regular element of A, then B_a is a Cartan subalgebra of A.*

COROLLARY 5.13. *Suppose F has at least $2(m-1)$ elements. Then A contains Cartan subalgebras.*

We conclude this section with some remarks on the conjugacy of Cartan subalgebras. Our discussion parallels the Jordan case, and requires many results that can be found in Chevalley [2]. We will make the assumption that the characteristic of F is zero.

In this situation, $D(b, c) = [L_b, L_c] + [L_b, R_c] + [R_b, R_c]$ is a derivation of A for all $b, c \in A$. The set $\{D(b, c) \mid b, c \in A\}$ of these derivations generates an ideal $D'(A)$ in the derivation algebra $D(A)$ of A. Such derivations are called *inner*. Now let $\mathfrak{G} = \{G \mid G$ is an algebraic group of linear transformations whose Lie algebra G_L contains $D'(A)\}$, and set $I = \bigcap \{G \mid G \in \mathfrak{G}\}$. Since the group of automorphisms of A is an algebraic group whose Lie algebra is $D(A)$, elements in I are automorphisms of A. We call I the *inner automorphism* group of A. We have

THEOREM 5.14. *Suppose F is algebraically closed and of characteristic zero. If H_1 and H_2 are two Cartan subalgebras of A, then there is an $\sigma \in I$ such that $H_1 = H_2\sigma$.*

Proof. The proof is patterned after Chevalley's proof in the Lie case [2, III, pp. 215–219] and follows the proof in the Jordan case [4, p. 603]. We sketch it here for the reader's convenience.

We begin by showing that H_1 and H_2 each contain relatively open subsets (in the Zariski topology) O_1 and O_2 such that $T(\Omega_i:a)=A$ for all $a \in O_i$, where Ω_i is the orbit of H_i under $I(A)$ and $T(\Omega_i:a)$ is the tangent space to Ω_i at a, $i=i, 2$. By the remark made after (5.5), there is a set $\{e_1, \ldots, e_t\}$ of pairwise orthogonal primitive idempotents whose sum is 1 and $H_1=\sum A_{ii}$ where $\sum A_{ij}$ is the Pierce decomposition relative to this set, and each $A_{ii}=F\cdot e_i+N_i$ where N_i is a nil ideal of H_1. Let $O_1=\{a \in H_1 \mid a=\sum(\alpha_i e_i+z_i), \alpha_i \in F, z_i \in N_i, \text{ and } \prod_{i\neq j}(\alpha_i-\alpha_j)\neq 0\}$. Then O_1 is an open subset of H_1. If $b_{kp} \in A_{kp}$, $k\neq p$, and $a=\sum(\alpha_i e_i+z_i) \in O_1$, we compute

$$aD(e_k, b_{kp}) = (\alpha_k-\alpha_p)b_{kp}+z_k b_{kp}-b_{kp}z_p.$$

Since $D(e_k, b_{kp}) \in D'(A)$, $aD(e_k, b_{kp}) \in T(\Omega_1:a)$ [2, III, pp. 192–193]. Define $S_{kp}=(\alpha_k-\alpha_p)M+L_{z_k}-R_{z_p}$ where M is the identity transformation on A. Hence $b_{kp}S_{kp} \in T(\Omega_1:a)$. Now S_{kp} maps A_{kp} onto itself by (2.6). Also, by Corollary 3.11, $L_{z_k}-R_{z_p}$ is nilpotent, and as $\alpha_k-\alpha_p\neq 0$, S_{kp} is invertible. Therefore, $A_{kp}\subseteq T(\Omega_1:a)$ for all $k\neq p$, and it follows that $T(\Omega_1:a)=A$. Similarly $T(\Omega_2:b)=A$ for all $b \in O_2$, where O_2 is defined in a manner analogous to O_1. Hence Ω_1 and Ω_2 contain nonempty open subsets of A [2, III, Proposition 13, p. 180 and the corollary on p. 192]. By (5.7), $\Omega_1 \cap \Omega_2$ contains an a-regular element b. Hence there are elements $\sigma_i \in I$ with $b \in H_i\sigma_i$, $i=1, 2$. Then $H_i\sigma_i \subseteq B_b$, and since B_b and $H_i\sigma_i$ are Cartan subalgebras of A, $B_b=H_1\sigma_1=H_2\sigma_2$. Thus $H_1=H_2\sigma$ where $\sigma=\sigma_2\sigma_1^{-1} \in I$, as desired.

6. **Conclusion.** Let $\mathfrak{A}$ be a class of finite-dimensional alternative, Jordan, and Lie algebras over a field F where

(i) if $A \in \mathfrak{A}$ and $\Theta \in \operatorname{Hom} A$, then $A\Theta \in \mathfrak{A}$,

(ii) if $A \in \mathfrak{A}$ and I is a subalgebra of A, then $I \in \mathfrak{A}$,

(iii) if $A \in \mathfrak{A}$, then the ground field of A is either F or an extension of F; if the ground field of A is K, then $A_P \in \mathfrak{A}$ for all extensions P of K,

(iv) if $A \in \mathfrak{A}$ and A is either alternative or Jordan, then A contains a unity element,

(v) if $A \in \mathfrak{A}$ and A is either Jordan or Lie, then the characteristic of F is different from 2.

We remark that (i)–(iii) means that $\mathfrak{A}$ is a universal class of algebras.

(6.1) Suppose $A \in \mathfrak{A}$ and $b_1, b_2, b_3 \in A$. We set $a(b_1, b_2, b_3)=b_3\cdot b_1b_2+b_2b_1\cdot b_3-b_2b_3\cdot b_1-b_1\cdot b_3b_2$ and $S(b_2, b_3)=R_{b_2}L_{b_3}+L_{b_2}R_{b_3}-L_{b_2b_3}-R_{b_3b_2}$. Let $S(A)=\{S(b_1, b_2) \mid b_1, b_2 \in A\}$. If $b \in A$ and the dimension of A is m, then

$$B_b = \{x \in A \mid xS(b, b)^m = 0\}$$

is minimal Engel in A with respect to $S(A)$ if $B_c\subseteq B_b$ for $c \in A$ implies $B_b=B_c$.

(i) If $A \in \mathfrak{A}$ and A is alternative, then $a(b_1, b_2, b_3)$ gives the "a-theory" developed in §4.

(ii) If $A \in \mathfrak{A}$ and A is Jordan, then $a(b_1, b_2, b_3) = 2(b_1, b_2, b_3)$, and it follows that $a(b_1, b_2, b_3)$ gives the "associator theory" as developed by Jacobson in [4].

(iii) If $A \in \mathfrak{A}$ and A is a Lie algebra, then $a(b_1, b_2, b_3) = -2b_1b_3 \cdot b_2$ and $S(b_1, b_2) = -2\mathrm{ad}(b_2)\,\mathrm{ad}\,(b_1)$. Thus it is clear that A is "a-nilpotent" if and only if A is nilpotent, and A is "a-nil" if and only if A is nil. By virtue of Engel's theorem, these four concepts coincide on A.

(6.2) Suppose $A \in \mathfrak{A}$ and R is an a-nilpotent subalgebra of A. Then $L_{S(A),A}(R)$, the Lie algebra of linear transformations on A generated by $S(A)$, is nilpotent. The subalgebra H of A is a *Cartan subalgebra relative to* $S(A)$ if

(i) H is a-nilpotent,

(ii) H coincides with the Fitting null component of A relative to $L_{S(A),A}(H)$.

In view of Theorem 5.11, and the results of Barnes [1] and the author [3], we have

THEOREM 6.3. *Suppose $A \in \mathfrak{A}$ and the dimension of A is m. If F has at least $2m$ elements, then H is a Cartan subalgebra relative to $S(A)$ of A if and only if H is minimal Engel with respect to $S(A)$ in A.*

Proof. If A is either alternative or Jordan, the proof is clear from Theorem 5.11 or [3]. If A is a Lie algebra, the proof follows since H is a Cartan subalgebra relative to $S(A)$ of A if and only if H is a Cartan subalgebra in the classical sense.

(6.4) Suppose $A \in \mathfrak{A}$. If A is alternative, then the definition of an a-regular element is given in §5, whereas if A is Jordan, it is given in [3]. It is clear that if A is Lie, we may define a-regular elements. If A is Lie and x is a generic element of A, then $xS(x, x) = 0$, and we see that the analogue of Lemma 5.8 is valid for the Lie algebras in $\mathfrak{A}$. Thus we have

COROLLARY 6.5. *If $A \in \mathfrak{A}$ and a is an a-regular element of A, then B_a is a Cartan subalgebra of A.*

COROLLARY 6.6. *Suppose $A \in \mathfrak{A}$, $\dim A = m$, and F has at least $2(m-1)$ elements. Then A contains Cartan subalgebras relative to $S(A)$.*

(6.7) *Notes.* (1) It should be noted that if A is an alternative algebra, then we can make A into a Jordan algebra A^+ by defining a new product $a \circ b$ of A by the equation $a \circ b = \frac{1}{2}(ab + ba)$, $a, b \in A$. We call A^+ the symmetric algebra of A. Using the associator in A^+ we would expect to develop a Cartan theory for A. However, if $b_1, b_2, b_3 \in A$, $2(b_1 \circ b_2) \circ b_3 - 2b_1 \circ (b_2 \circ b_3) = 4a(b_1, b_2, b_3) - 4(b_1, b_2, b_3)$. Therefore, there is no reason to expect that the Cartan theory developed earlier for alternative algebras arises from iterating the associator in A^+.

(2) The fact that in (6.2) we define Cartan subalgebras relative to a particular set "$S(\ )$" of linear transformations would suggest that other sets of transformations on algebras in $\mathfrak{A}$ would yield a "Cartan theory". Indeed, in the Lie case, we see that "$S(\ )$" and "ad" yield Cartan theories that coincide. It turns out that

the whole concept of a "Cartan theory" can be developed for certain universal classes of algebras, the above class $\mathfrak{A}$ being an example of one such class. A more detailed study of this fact is in preparation [3].

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