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SOME CHARACTERIZATIONS OF THE SPACES $L^1(\mu)$

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Abstract. Answers are given to the question of when the so-called hom and tensor functors in categories of Banach spaces preserve certain short exact sequences. The answers characterize the spaces of integrable, real-valued functions $L^1(\mu)$.

Many questions can be raised concerning preservation properties of functors and their dual functors (see [8]) in categories of Banach spaces. Some are answered. For example, it can be shown if a functor preserves compact operators so does its dual functor. Similarly, we can ask the question: If a functor in categories of Banach spaces preserves certain types of exact sequences, does its dual functor do the same? Investigating this problem in particular for the hom and tensor functors, which are dual to each other, leads to the characterization of the spaces $L^1(\mu)$. The purpose of this paper is to give these characterizations.

1. **Preliminaries.** The letter $\boldsymbol{B}$ will denote the category in which the objects are real Banach spaces A, B, C, X or Y. The morphism set $\boldsymbol{B}(A, B)$ corresponding to an ordered pair (A, B) of objects is the Banach space of all continuous linear mappings f, h, or k from A to B.

Special objects of $\boldsymbol{B}$ are the spaces $L^p(\mu)$ $(1 \leqq p < \infty)$. In order to proceed with the investigation at hand, some preliminary knowledge of these spaces is needed. A detailed description of these spaces can be found in [1] and [11].

In the following discussion, E will always denote some fixed locally compact topological space and μ a fixed positive (Radon) measure on E. The letter I will always denote the space of real numbers. The letter μ^* will denote the extension of μ to all positive extended real-valued functions.

Let S be any set. Two functions $f, g: E \to S$ are *equivalent* (with respect to μ) if $X = \{x \in E \mid f(x) \neq g(x)\}$ is *negligible*, that is, $\mu^*(\chi_X) = 0$. The relation "f is related to g if f is equivalent to g" is an equivalence relation on the set S^E of functions from E to S. The equivalence class of f is denoted by $\tilde{f}$. If the set S is a vector space over I, by defining

$$\tilde{f} + \tilde{g} = (f+g)^{\sim}, \quad f, g \in S^E, \quad \text{and} \quad \alpha\tilde{f} = (\alpha f)^{\sim}, \quad \alpha \in I,$$

a vector space structure is obtained on the set of equivalence classes.

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(1) These results are part of those contained in the author's doctoral dissertation written at Western Michigan University under the direction of Professor Kung-Wei Yang.

Now let B be a Banach space over I with norm $|\cdot|$. If $f\colon E \to B$ is a function, for each integer $1 \leqq p < +\infty$ let $N_p(f)$ be the finite or infinite positive number given by

$$N_p(f) = \left(\int^* |f(\cdot)|^p \, d\mu\right)^{1/p}$$

where $|f(\cdot)|$ is the positive function $E \to I$ given by $x \to |f(x)|$, and $\int^* |f(\cdot)|^p \, d\mu$ is another way of writing $\mu^*(|f(\cdot)|^p)$. It can be shown that the function $N_p(\tilde{f}) = N_p(f)$ is a well-defined, real-valued function on the set of equivalence classes of elements of B^E; that is, if $\tilde{f} = \tilde{g}$ then $N_p(f) = N_p(g)$.

DEFINITION 1.1. For $1 \leqq p < +\infty$, let $\mathscr{F}^p(B)$ (or $\mathscr{F}^p(B, E, \mu)$) be the seminormed vector space of all elements f in B^E such that $N_p(f) < +\infty$. Let $K_B(E)$ denote the vector space of all continuous functions $f\colon E \to B$ with compact support. Define $\mathscr{L}^p(B)$ (or $\mathscr{L}^p(B, E, \mu)$) to be the closure in the space $\mathscr{F}^p(B)$ of $K_B(E)$. Define $L^p(B)$ (or $L^p(B, E, \mu)$) to be the Banach space of equivalence classes of functions in $\mathscr{L}^p(B)$ with norm N_p. Elements of $\mathscr{L}^p(B)$ are called *pth-power integrable functions.*

EXAMPLE 1.2. Take the special case where E is any space with the discrete topology. Then $K_I(E) = \{f\colon E \to I \mid f(x) = 0$ for all but finitely many $x \in E\}$. Define the Radon measure $\mu\colon K_I(E) \to I$ by $\mu(f) = \sum_{x \in E} f(x)$, a finite sum. If B is a Banach space and $f \in K_B(E)$, $N_1(f) = \sum_{x \in E} |f(x)|$, a finite sum. A function $f\colon E \to B$ is in $\mathscr{L}^1(B)$ if and only if $\sum_{x \in E} |f(x)|$ is finite. The space $L^1(B)$ in this case is denoted by $l^1(B)$, and the measure is called *discrete.*

DEFINITION 1.3. A function $f\colon E \to B$ is *measurable* if for each compact subset K of E there exists a negligible set $N \subset K$ ($\mu^*(\chi_N) = 0$) and a partition of $K \cap (E \backslash N)$, formed from a sequence (K_n) of compact sets, such that the restriction of f to each K_n is continuous. A subset A of E is *measurable* if its characteristic function χ_A is measurable.

The following proposition follows easily from a result in [1, p. 191].

PROPOSITION 1.4. *In order that a function $f\colon E \to B$ be measurable, it is necessary and sufficient that*

(1) *the set $f^{-1}(U)$ is measurable where U is any closed ball in B; and*

(2) *for each compact set $K \subset E$, there is a negligible set S in K such that $f(K \backslash S)$ is separable.*

PROPOSITION 1.5. *In order that $f\colon E \to B$ be pth-power integrable for $1 \leqq p < +\infty$, it is necessary and sufficient that f be measurable and $N_p(f)$ finite.*

Proof. See [1, p. 194].

Listed below are the functors from category $\boldsymbol{B}$ to $\boldsymbol{B}$ that will be prominent in the ensuing discussion.

(i) *The hom functor $\Omega_X\colon \boldsymbol{B} \to \boldsymbol{B}$ for some fixed object X in $\boldsymbol{B}$.* It assigns to each object A in $\boldsymbol{B}$ the Banach space $\boldsymbol{B}(X, A)$.

(ii) *The tensor functor* Σ_X: $\boldsymbol{B} \to \boldsymbol{B}$ *for some fixed* X *in* $\boldsymbol{B}$. It assigns to each object A in $\boldsymbol{B}$, the completion $X \hat{\otimes} A$ of the projective tensor product $X \otimes A$. (The norm on $X \otimes A$ is the greatest crossnorm of Schatten [10, p. 206].) If f: $A \to B$ is in $\boldsymbol{B}$, $\Sigma_X f = 1_X \hat{\otimes} f$: $X \hat{\otimes} A \to X \hat{\otimes} B$ is the continuous extension of $1_X \otimes f$: $X \otimes A \to X \hat{\otimes} B$ given by $1_X \otimes f(\Sigma x_i \otimes a_i) = \Sigma f(x_i) \otimes a_i$.

(iii) *The functors* L^p *and* l^p. They assign to each A in $\boldsymbol{B}$ the spaces $L^p(A)$ and $l^p(A)$ respectively.

(iv) *The contravariant dual functor* $*$. It assigns to each Banach space A its dual space A^*.

(v) *Compositions of the above functors.* In particular, if F: $\boldsymbol{B} \to \boldsymbol{B}$ is a functor, $F^{\#}$: $\boldsymbol{B} \to \boldsymbol{B}$ is a functor $* \circ F \circ *$. Also, the "double-dual" functor $**$ is the composition $* \circ *$.

PROPOSITION 1.6. *If* A, B, *and* C *are Banach spaces, then* $\boldsymbol{B}(A \hat{\otimes} B, C)$ *and* $\boldsymbol{B}(B, \boldsymbol{B}(A, C))$ *are isometrically isomorphic.*

Proof. The function ξ: $\boldsymbol{B}(A \hat{\otimes} B, C) \to \boldsymbol{B}(B, \boldsymbol{B}(A, C))$ given by $(\xi(f)b)a = f(a \otimes b)$ for $f \in \boldsymbol{B}(A \hat{\otimes} B, C)$, $b \in B$, and $a \in A$ can be shown to be an isometric isomorphism.

PROPOSITION 1.7. *The functors* $\Omega_X \circ **$ *and* $(\Sigma_X)^{\#}$ *is naturally equivalent. This means that for any* A *and* B *in* $\boldsymbol{B}$ *and for any* f: $A \to B$ *in* $\boldsymbol{B}$, *the diagram*

$$\begin{array}{ccc} (X \hat{\otimes} A^*)^* & \xrightarrow{(i_X \hat{\otimes} f^*)^*} & (X \hat{\otimes} B^*)^* \\ \xi_A \big\downarrow & & \big\downarrow \xi_B \\ \boldsymbol{B}(X, A^{**}) & \xrightarrow{\Omega_X(f^{**})} & \boldsymbol{B}(X, B^{**}) \end{array}$$

commutes for some equivalences (isometric isomorphisms) ξ_A *and* ξ_B.

Proof. The fact that for each A in $\boldsymbol{B}$, $\boldsymbol{B}(X, A^{**})$ is isometrically isomorphic to $(X \hat{\otimes} A^*)^*$ by an equivalence ξ_A is given by (1.6). Using these equivalences, the diagram can easily be seen to be commutative.

PROPOSITION 1.8. *The contravariant functors* $* \circ \Sigma_X$ *and* $\Omega_X \circ *$ *are naturally equivalent. This means if* f: $A \to B$ *is in* $\boldsymbol{B}$, *the diagram*

$$\begin{array}{ccc} \boldsymbol{B}(X, B^*) & \xrightarrow{\Omega_X(f^*)} & \boldsymbol{B}(X, A^*) \\ \mu_B \big\downarrow & & \big\downarrow \mu_A \\ (X \hat{\otimes} B)^* & \xrightarrow{(i_X \hat{\otimes} f)^*} & (X \hat{\otimes} A)^* \end{array}$$

commutes for some isometric isomorphisms μ_A *and* μ_B.

Proof. Let μ_A and μ_B be inverses for ξ_A and ξ_B in (1.6). Let $g \in \boldsymbol{B}(X, B^*)$. Then for $x \in X$ and $a \in A$,

$$\begin{aligned}[((i_X \hat{\otimes} f)^* \circ \mu_B)g](x \otimes a) &= [\mu_B(g) \circ (i_X \hat{\otimes} f)](x \otimes a) \\ &= [\mu_B(g)](x \otimes f(a)) = g(x)(f(a)).\end{aligned}$$

Also

$$\begin{aligned}[(\mu_A \circ \Omega_X f^*)g](x \otimes a) &= [\mu_A(f^* \circ g)](x \otimes a) \\ &= [(f^* \circ g)x](a) = [g(x) \circ f](a) = g(x)(f(a)).\end{aligned}$$

A. Grothendieck proved the following result in [3, p. 59].

PROPOSITION 1.9. *The functors $\Sigma_{L^1(I)}$ and L^1 are naturally equivalent.*

2. **Results concerning exact sequences in $\boldsymbol{B}$.** This section contains results necessary to prove the main results given in the next section.

DEFINITION 2.1. A morphism $f: A \to B$ in $\boldsymbol{B}$ is a *normal morphism* if the induced continuous linear function from $A/\operatorname{Ker} f \to f(A)$ is an isometry. In addition, it is a *strictly normal morphism* if, for each b in $f(A)$, there is an $a \in A$ such that $f(a)=b$ and $|a|=|b|$.

DEFINITION 2.2. A sequence

$$0 \to A \to B \to C \to 0$$

is a *normal exact sequence* if it is exact (f is a monomorphism, g is an epimorphism, and $\operatorname{Ker} g = \operatorname{Im} f$) and each morphism is normal. It is *strictly normal exact* if it is exact and each morphism is strictly normal.

PROPOSITION 2.3. *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence in $\boldsymbol{B}$ with f normal, then*

$$0 \longrightarrow L^p(A) \xrightarrow{L^p(f)} L^p(B) \xrightarrow{L^p(g)} L^p(C)$$

is exact $1 \leqq p < \infty$ with $L^p(f)$ normal.

Proof. Since f is isometric, for $x \in \mathscr{L}^p(A)$,

$$|f \circ x(\cdot)| = |f(x(\cdot))| = |x(\cdot)|.$$

Therefore,

$$\begin{aligned}N_p(L^p f(\tilde{x})) &= N_p((f \circ x)^\sim) = N_p(f \circ x) \\ &= \left(\int^* |f(x(\cdot))|^p\right)^{1/p} = \left(\int^* |x(\cdot)|^p\right)^{1/p} = N_p(x) = N_p(\tilde{x}).\end{aligned}$$

This means $L^p(f)$ is isometric.

Now suppose $\tilde{y} \in L^p(B)$ such that $L^p(g)(\tilde{y}) = (g \circ y)^\sim = \tilde{0}$. Then $g \circ y = 0$ almost everywhere on E so that $y(t) \in \operatorname{Ker} g = \operatorname{Im} f$ almost everywhere for $t \in E$. It can be supposed that $y(t) \in \operatorname{Ker} g$ for all $t \in E$. Define $\tilde{x}$ in $L^p(A)$ to be the equivalence

class of $x: E \to A$ defined by the rule: if $t \in E$, $x(t)$ is the unique element in A so that $f(x(t)) = y(t)$ and $|x(t)| = |y(t)|$. It must be shown that $x \in \mathscr{L}^p(A)$. By (1.5) it suffices to show x is measurable and $N_p(x)$ is finite. Since $|x(\cdot)| = |y(\cdot)|$, $N_p(x) = N_p(y)$ is finite. To show x is measurable the condition of (1.4) must be verified. Let U be any closed ball in A. Then $x^{-1}(U) = y^{-1}(f(U))$ since f is isometric. Suppose $U = \{a : |a - a_0| \leqq r\}$. Then

$$f(U) = \{f(a) : |f(a) - f(a_0)| \leqq r\}.$$

Let $V = \{b : b \in B \text{ and } |b - f(a_0)| \leqq r\}$. Then $f(U) = V \cap f(A)$. Hence,

$$y^{-1}(f(U)) = y^{-1}(V \cap f(A)) = y^{-1}(V) \cap y^{-1}(f(A)) = y^{-1}(V) \cap E = y^{-1}(V).$$

Since y is measurable, $y^{-1}(V)$ is a measurable set and hence $x^{-1}(U)$ is measurable.

It must still be shown that for any compact set $K \subset E$, there is a negligible set S so that $x(K \backslash S)$ is separable. Since $y: E \to B$ is measurable, a negligible set S does exist so that $y(K \backslash S)$ is separable. Let H be a countable dense subset of $y(K \backslash S)$. Let H' be the subset of $x(K \backslash S)$ in a one-to-one correspondence with H via f. H' is countable. Also

$$\overline{H} \text{ (closure in } y(K \backslash S)) = y(K \backslash S) \cap \overline{H} \text{ (closure in } B) = y(K \backslash S).$$

Hence,

$$\begin{aligned} x(K \backslash S) &= f^{-1}(y(K \backslash S)) = f^{-1}(y(K \backslash S) \cap \overline{H}) \\ &= x(K \backslash S) \cap f^{-1}(\overline{H}) = x(K \backslash S) \cap \overline{H}' \text{ (closure in } A) \\ &= \overline{H}' \text{ (closure in } x(K \backslash S)). \end{aligned}$$

This means H' is dense in $x(K \backslash S)$ so that $x(K \backslash S)$ is separable.

PROPOSITION 2.4. *If $B \xrightarrow{g} C \to 0$ is strictly normal exact in $\mathbf{B}$, then for any X in $\mathbf{B}$,*

$$X \hat{\otimes} B \xrightarrow{i_X \hat{\otimes} g} X \hat{\otimes} C \longrightarrow 0$$

is normal exact.

Proof. Since $i_X \otimes g: X \otimes B \to X \otimes C$ is a surjection, $i_X \hat{\otimes} g(X \hat{\otimes} B)$ is dense in $X \hat{\otimes} C$. To show $i_X \hat{\otimes} g$ is a surjection, it will be shown that the induced map from $X \hat{\otimes} B/\mathrm{Ker}\,(i_X \hat{\otimes} g)$ to $X \hat{\otimes} C$ is an isometric map. Let $\sum_{i=1}^n x_i \otimes c_i$ be in $X \otimes C$. By assumption, for each $i = 1, \ldots, n$, there is a $b_i \in B$ such that $|b_i| = |c_i|$ and

$$i_X \hat{\otimes} g\left(\sum_{i=1}^n x_i \otimes b_i\right) = \sum_{i=1}^n x_i \otimes g(b_i) = \sum_{i=1}^n x_i \otimes c_i.$$

Therefore, $\sum_{i=1}^n |x_i|\,|b_i| = \sum_{i=1}^n |x_i|\,|c_i|$. Considering $[\sum_{i=1}^n x_i \otimes b_i]$ as an element of $X \hat{\otimes} B/\mathrm{Ker}\,(i_X \hat{\otimes} g)$,

$$\left|\left[\sum_{i=1}^n x_i \otimes b_i\right]\right| = \inf_{u \in [\Sigma x_i \otimes b_i]} |u| \geqq \left|\sum_{i=1}^n x_i \otimes c_i\right|.$$

Now let $\sum_{j=1}^{m} x_j' \otimes c_j' = \sum_{i=1}^{n} x_i \otimes c_i$ in $X \otimes C$. By assumption, for each $j=1, \ldots, m$, there exists b_j' such that $|b_j'| = |c_j'|$ and $g(b_j') = c_j'$. Therefore,

$$i_X \hat{\otimes} g\left(\sum_{j=1}^{m} x_j' \otimes b_j'\right) = \sum_{j=1}^{m} x_j' \otimes c_j' = \sum_{i=1}^{n} x_i \otimes c_i,$$

and

$$\left|\sum_{j=1}^{m} x_j' \otimes b_j'\right| \leqq \sum_{j=1}^{m} |b_j'|\,|x_j'| = \sum_{j=1}^{m} |x_j'|\,|c_j'|.$$

Since this is true for each element of $X \otimes C$ equal to $\sum_{i=1}^{n} x_i \otimes c_i$, $|[\sum_{i=1}^{n} x_i \otimes b_i]| \leqq |\sum_{i=1}^{n} x_i \otimes c_i|$.

Now let y be any element of $i_X \hat{\otimes} g(X \hat{\otimes} B)$. Then there is a sequence of elements y_i in $X \otimes C$ such that $y_i \to y$ in $X \hat{\otimes} C$. By the preceding argument, then there exist z_i in $X \hat{\otimes} B$ such that $|[z_i]| = |y_i|$ and $i_X \hat{\otimes} g(z_i) = y_i$. Since $|[z_i] - [z_j]| = |[z_i - z_j]| = |y_i - y_j|$, $[z_i]$ is a Cauchy sequence in $X \hat{\otimes} B/\mathrm{Ker}\,(i_X \hat{\otimes} g)$ converging to some $[z]$ and $i_X \hat{\otimes} g(z) = y$. Also, $|[z]| = \lim |[z_i]| = \lim |y_i| = |y|$. Thus $X \hat{\otimes} B/\mathrm{Ker}\,(i_X \hat{\otimes} g) \to X \hat{\otimes} C$ is an isometric map.

PROPOSITION 2.5. *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a strictly normal exact sequence in* $\boldsymbol{B}$, *then*

$$0 \to L^1(I) \hat{\otimes} A \to L^1(I) \hat{\otimes} B \to L^1(I) \hat{\otimes} C \to 0$$

is normal exact in $\boldsymbol{B}$.

Proof. By (2.4), $L^1(I) \hat{\otimes} B \to L^1(I) \hat{\otimes} C \to 0$ is normal exact. By (1.9), the functors $\Sigma_{L^1(I)}$ and L^1 are naturally equivalent by an equivalence τ. Hence the diagram

$$\begin{tikzcd}
(*) \quad 0 \arrow[r] & L^1(I) \hat{\otimes} A \arrow[r] \arrow[d, "\tau_A"'] & L^1(I) \hat{\otimes} B \arrow[r] \arrow[d, "\tau_B"'] & L^1(I) \hat{\otimes} C \arrow[r] \arrow[d, "\tau_C"'] & 0 \\
(**) \quad 0 \arrow[r] & L^1(A) \arrow[r, "L^1(f)"] & L^1(B) \arrow[r] & L^1(C) \arrow[r] & 0
\end{tikzcd}$$

commutes. Since $L^1(f)$ is isometric and (**) is exact at $L^1(B)$ by (2.3), sequence (*) is normal exact.

COROLLARY 2.6. *Let* $0 \to A \to B \to C \to 0$ *be a strictly normal exact sequence in* $\boldsymbol{B}$. *If* X *is a retract of* $L^1(I)$ (*meaning there exists* $f\colon X \to L^1(I)$ *with* $|f| \leqq 1$ *and* $g\colon L^1(I) \to X$ *with* $|g| \leqq 1$ *such that* $g \circ f = i_X$), *then*

$$0 \to X \hat{\otimes} A \to X \hat{\otimes} B \to X \hat{\otimes} C \to 0$$

is normal exact in $\boldsymbol{B}$.

LEMMA 2.7. *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is normal exact in* $\boldsymbol{B}$, *then* $0 \to C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \to 0$ *is strictly normal exact.*

Proof. $B^* \xrightarrow{f^*} A^* \to 0$ is strictly normal exact due to the Hahn-Banach theorem [2, p. 63]. The remainder of the proof is straightforward and is omitted.

COROLLARY 2.8. *If* $0 \to A \to B \to C \to 0$ *is a normal exact sequence with* B *reflexive, then it is strictly normal exact.*

The proof of the following proposition can be found in [5].

PROPOSITION 2.9. *X is equivalent to a space $L^1(I)$ (for some Radon measure μ on a locally compact space E) if and only if for each Banach space B and closed linear subspace A of B, the map $i_X \hat{\otimes} f\colon X \hat{\otimes} A \to X \hat{\otimes} B$ is an isometric (into) map where $f\colon A \to B$ is the insertion map.*

Proposition 2.9 can be used to prove the next lemma, given here without proof.

LEMMA 2.10. *If X is a Banach space such that X^{**} is (equivalent to) a space $L^1(I)$, then X is also a space $L^1(I)$.*

LEMMA 2.11. *Suppose X is a Banach space satisfying the conditions illustrated in the following diagram:*

$$\begin{tikzcd} & X \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ B \arrow[r, "h"] & C \arrow[r] & 0 \end{tikzcd}$$

The morphism h is a normal epimorphism while $\dim B = 3$ *and* $\dim C = 2$. *It is assumed that for all such B, C, and $h\colon B \to C$, for any $f \in \boldsymbol{B}(X, C)$ the diagram can be filled in with $g \in \boldsymbol{B}(X, B)$ so that $h \circ g = f$ and $|g| = |f|$. Then X^* satisfies the condition that for any Banach space $Y \subset Z$ with* $\dim Z = 3$ *and* $\dim Y = 2$, *every element in $\boldsymbol{B}(Y, X^*)$ has a norm preserving extension from Z to X^*.*

$$\begin{tikzcd} & X^* & \\ 0 \arrow[r] & Y \arrow[u, "f'"] \arrow[r, "h'"] & Z \arrow[ul, dashed, "g'"'] \end{tikzcd}$$

Proof. A morphism $g' \in \boldsymbol{B}(Z, X)$ must be found so that $|g'| = |f'|$ and $g' \circ h' = f'$ where h' is the insertion map. Consider the dual diagram (2.11a) where $c_X\colon X \to X^{**}$ is the canonical embedding.

(2.11a)
$$\begin{tikzcd} & X \arrow[d, "c_X"] \arrow[dl, dashed, "g"'] \\ & X^{**} \arrow[d, "f'^*"] \\ Z^* \arrow[r, "h'^*"] & Y^* \end{tikzcd}$$

By assumption, $g\colon X \to Z^*$ exists so that $h'^* \circ g = f'^* \circ c_X$ and $|f'^* \circ c_X| = |g|$. (See (2.11b).)

(2.11b)
$$\begin{tikzcd}
X^* & \\
X^{***} \arrow[u, "c_X^*"] & \\
Y^{**} \arrow[u, "f'^{**}"] \arrow[r] & Z^{**} \arrow[ul, "g^*"'] \\
Y \arrow[u, "c_Y"] \arrow[r, "h'"] & Z \arrow[u, "c_Z"']
\end{tikzcd}$$

The morphism f' can be shown to be $c_X^* \circ f'^{**} \circ c_Y$. Letting $g' = g^* \circ c_Z$, the proposition is proved.

3. **The main results.** To prove the first of the three main results listed in this section, the following information is needed.

LEMMA 3.1. *If X is a Banach space, $S_X(x_0, r_0)$ denotes $\{x \in X \mid |x - x_0| \leqq r_0\}$. If two balls $S_X(x_1, r_1)$ and $S_X(x_2, r_2)$ intersect in X, then $S_X(x_1, r_1) \cap S_X(x_2, r_2) \cap A \neq \varnothing$, where A is any two-dimensional subspace of X containing x_1 and x_2.*

Proof. Let $C = (1-\alpha)x_1 + \alpha x_2$, $0 \leqq \alpha \leqq 1$, be the curve in X connecting x_1 and x_2. C is homeomorphic to $[0, 1]$. Hence there is a maximum α_0 in $[0, 1]$ so that $(1-\alpha_0)x_1 + \alpha_0 x_2$ is in $S_X(x_1, y_1)$. It can be shown that $|(1-\alpha_0)x_1 + \alpha_0 x_2 - x_1| = r_1$ and $|(1-\alpha_0)x_1 + \alpha x_2 - x_2| \leqq r_2$.

Facts 3.2. (1) If X is a Banach space, let $\{S_X(x_\alpha, r_\alpha)\}$ be a collection of mutually intersecting balls in X. Then there is a Banach space $Z \supset X$ with $\dim Z/X = 1$ such that $\bigcap_\alpha S_Z(x_\alpha, r_\alpha) \neq \varnothing$. For proof see [6, p. 51] and [9].

(2) Let X be a Banach space such that $S_X(0, 1)$ has at least one extreme point and such that X has the following property:

For every collection of four mutually intersecting balls $\{S_X(x_i, r_i)\colon i = 1, 2, 3, 4\}$ such that $\{x_i : i = 1, 2, 3, 4\}$ span a two-dimensional subspace of X, $\bigcap_{i=1}^4 S(x_i, r_i + \varepsilon) \neq \varnothing$ for every $\varepsilon > 0$.

Then X^* is a space $L^1(I)$. For proof see [6, p. 71].

THEOREM 3.3. *The following statements are equivalent:*

(1) *X is equivalent to a space $L^1(I)$.*

(2) *If B is reflexive and $0 \to A \to B \to C \to 0$ is a normal exact sequence in $\boldsymbol{B}$, then*

$$0 \to \boldsymbol{B}(X, A) \to \boldsymbol{B}(X, B) \to \boldsymbol{B}(X, C) \to 0$$

is strictly normal exact.

(3) *Same as* (2) *with B finite dimensional.*

(4) *Same as* (2) *with* $\dim B = 3$ *and* $\dim C = 2$.

Proof. Statement (1) implies (2). By (2.7)

$$0 \to C^* \to B^* \to A^* \to 0$$

is strictly normal exact. By (2.5)

$$0 \to X \hat{\otimes} C^* \to X \hat{\otimes} B^* \to X \hat{\otimes} A^* \to 0$$

is normal exact. Again by (2.7)

$$0 \to (X \hat{\otimes} A^*)^* \to (X \hat{\otimes} B^*)^* \to (X \hat{\otimes} C^*)^* \to 0$$

is strictly normal exact. By (1.7) the following diagram is commutative with ξ_A, ξ_B and ξ_C equivalence:

$$\begin{array}{llllllll}
(*) & 0 \longrightarrow & (X \hat{\otimes} A^*)^* & \longrightarrow & (X \hat{\otimes} B^*)^* & \longrightarrow & (X \hat{\otimes} C^*)^* & \longrightarrow 0 \\
 & & \downarrow \xi_A & & \downarrow \xi_B & & \downarrow \xi_C & \\
(**) & 0 \longrightarrow & \boldsymbol{B}(X, A^{**}) & \longrightarrow & \boldsymbol{B}(X, B^{**}) & \longrightarrow & \boldsymbol{B}(X, C^{**}) & \longrightarrow 0
\end{array}$$

This means sequence (**) is strictly normal exact, or since B is reflexive (hence A and C are reflexive),

$$0 \to \boldsymbol{B}(X, A) \to \boldsymbol{B}(X, B) \to \boldsymbol{B}(X, C) \to 0$$

is strictly normal exact.

The proofs that (2) implies (3) and (3) implies (4) are trivial.

Statement (4) implies (1). (See also [7, p. 496].) From the hypothesis of (4), X satisfies the conditions of (2.11). It will be established that X^* (in which the unit ball always has an extreme point by the Kreĭn-Milman Theorem [2, p. 440]) satisfies (2) of (3.2) so that X^{**} will be a space $L^1(I)$. By (2.10) X will then be a space $L^1(I)$. It is sufficient to show that for every collection of four mutually intersecting balls $\{S_{X^*}(x_i, r_i): i=1, 2, 3, 4\}$ such that the centers span a two-dimensional subspace of X^*, there exists an x in X^* such that $|x-x_i| \leqq r_i$ for $i=1, 2, 3, 4$. Let Y be the two-dimensional subspace spanned by the set $\{x_i : i=1, 2, 3, 4\}$. By (3.1) the balls in $\{S_Y(x_i, r_i) : i=1, 2, 3, 4\}$ are mutually intersecting. By (1) of (3.2) there exists $Z \supset Y$ with $\dim Z/Y=1$ and a point z in Z such that $|z-x_i|=r_i$ for $i=1, \ldots, 4$. Let $g: Z \to X^*$ be the operator whose restriction to Y is the insertion $f: Y \to X^*$ and for which $|g|=|f|$ (see (2.11)). Then $x=g(z)$ satisfies for each $i=1, \ldots, 4$,

$$|x-x_i| = |g(z-x_i)| \leqq |z-x_i| \leqq r_i. \qquad \text{Q.E.D.}$$

We may ask whether the condition that B be reflexive in (2) of (3.3) may be removed. The answer is contained in the following theorem. In order to prove this theorem the following observation is needed.

Fact 3.4. Let X be a space $L^1(I)$ where the measure is not discrete; that is, X is not a space $l^1(I)$. Then X contains a subspace isometric to $L^1(0, 1)$. ($L^1(0, 1)$ is the classical space of Lebesgue integrable functions from $(0, 1)$ to I.) See [4, p. 159].

THEOREM 3.5. *The following statements are equivalent*:

(1) *X is equivalent to a space $l^1(I)$.*

(2) *If*

$$(*) \qquad 0 \to A \to B \to C \to 0$$

is any normal exact sequence in $\boldsymbol{B}$, then

$$(**) \qquad 0 \to \boldsymbol{B}(X, A) \to \boldsymbol{B}(X, B) \to \boldsymbol{B}(X, C) \to 0$$

is normal exact.

(3) *If (*) is strictly normal exact in $\boldsymbol{B}$, then (**) is strictly normal exact.*

(4) *If $0 \to A \to B \to X \to 0$ is strictly normal exact in $\boldsymbol{B}$, then for every Banach space Y,*

$$0 \to \boldsymbol{B}(Y, A) \to \boldsymbol{B}(Y, B) \to \boldsymbol{B}(Y, X) \to 0$$

is strictly normal exact.

Proof. *Statement* (3) *implies* (1). Let A be a closed linear subspace of the Banach space B and let $f: A \to B$ be the insertion map. The sequence $0 \to A \to B \to B/A \to 0$ is a normal exact sequence. By (2.7) the sequence $0 \to (B/A)^* \to B^* \to A^* \to 0$ is strictly normal exact. Therefore by (3),

$$0 \to \boldsymbol{B}(X, (B/A)^*) \to \boldsymbol{B}(X, B^*) \to \boldsymbol{B}(X, A^*) \to 0$$

is strictly normal exact. By (1.8), the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & \boldsymbol{B}(X, (B/A)^*) & \longrightarrow & \boldsymbol{B}(X, B^*) & \longrightarrow & \boldsymbol{B}(X, A^*) & \longrightarrow 0 \\
 & & & \downarrow{\scriptstyle \mu_{B/A}} & & \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu_A} & \\
(***) & 0 & \longrightarrow & (X \hat{\otimes} B/A)^* & \longrightarrow & (X \hat{\otimes} B)^* & \longrightarrow & (X \hat{\otimes} A)^* & \longrightarrow 0
\end{array}$$

Therefore (***) is strictly normal exact which implies

$$0 \to (X \hat{\otimes} A)^{**} \to (X \hat{\otimes} B)^{**} \to (X \hat{\otimes} C)^{**} \to 0$$

is strictly normal exact. This means $X \hat{\otimes} A \to X \hat{\otimes} B$ is an isometric map, so that by (2.9), X is equivalent to a space $L^1(I)$. It must still be established that the measure μ is discrete, that is, X is a space $l^1(I)$. Consider the space $l^1(I) = l^1(I, S_X, \mu)$ as in (1.2) where $E = S_X$, the unit ball of X with the discrete topology. If $t \in l^1(I)$, t can be written as $\sum_{x_i \in S_X} \lambda_i \chi_{x_i}$ where $t(x_i) = \lambda_i$ and $\sum |\lambda_i| < \infty$. Define $h: l^1(I) \to X$ by $h(\sum_{x_i \in S_X} \lambda_i \chi_{x_i}) = \sum_{x_i \in S_X} \lambda_i x_i$. Then the sequence

$$0 \longrightarrow \operatorname{Ker} h \longrightarrow l^1(I) \xrightarrow{h} X \longrightarrow 0$$

is strictly normal exact. If

$$0 \to \boldsymbol{B}(X, \operatorname{Ker} h) \to \boldsymbol{B}(X, l^1(I)) \to \boldsymbol{B}(X, X) \to 0$$

is strictly normal exact, the map $i_X\colon X \to X$ has a norm preserving lifting $X \to l^1(I)$. This means the diagram

$$\begin{tikzcd} & X \arrow[dl, dashed, "f"'] \arrow[d, "i_X"] \\ l^1(I) \arrow[r, "h"] & X \end{tikzcd}$$

commutes for some $f\colon X \to l^1(I)$ where $|f|=1$. Therefore, X is a retract of $l^1(I)$. In $l^1(I)$ the weakly compact subsets are compact [2, p. 295]. Therefore, the same is true for X. However, if X is a space $L^1(I)$ where the measure is nondiscrete, X contains $L^1(0, 1)$ by (3.4). But in $L^1(0, 1)$, the sequence $f_n = \sin nx$ for $n=1, 2, \ldots$, for example, converges weakly but not in the norm topology. (See [12, pp. 336–337].) Therefore, X must be a space $l^1(I)$.

Statement (1) *implies* (2). Let Z be any Banach space. The space $\boldsymbol{B}(l^1(I, E, \mu), Z)$ is isometrically isomorphic to $l^\infty(Z)$, the space of bounded sequences (z_α), $\alpha \in E$, of elements of Z. This isomorphism is given by

$$(z_\alpha) \in l^\infty(Z) \mapsto \Big((\lambda_\alpha) \in l^1(I) \mapsto \sum_{\alpha \in E} \lambda_\alpha z_\alpha\Big).$$

Let $k\colon l^1(I) \to C$ be in $\boldsymbol{B}$ and let $\varepsilon > 0$ be arbitrary.

$$\begin{tikzcd} & l^1(I) \arrow[d, "k"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

There exists therefore a unique element $c=(c_\alpha)$ in $l^\infty(C)$ corresponding to k as above; and $|c| = \sup_{\alpha \in E} |\tau_\alpha| = |k|$. For each $\alpha \in E$, let $b_\alpha \in B$ be such that $g(b_\alpha) = c_\alpha$ and $|c_\alpha| \geqq |b_\alpha| - \varepsilon$. Set $b = (b_\alpha)_{\alpha \in E}$. Then $b \in l^\infty(B)$ and

$$|b| = \sup_{\alpha \in E} |b_\alpha| \leqq \sup_{\alpha \in E} (|c_\alpha| + \varepsilon) \leqq |c| + \varepsilon.$$

The element b corresponds to a continuous linear map $h\colon l^1(I) \to B$ with $|h| = |b| \leqq |k| + \varepsilon$. It is immediate that $g \circ h = k$. Hence

$$\boldsymbol{B}(l^1(I), B) \to \boldsymbol{B}(l^1(I), C) \to 0$$

is normal exact. It is easy to show that for any space X in $\boldsymbol{B}$ the sequence

$$0 \to \boldsymbol{B}(X, A) \to \boldsymbol{B}(X, B) \to \boldsymbol{B}(X, C)$$

is normal exact. Hence (1) implies (2).

Statement (1) *implies* (3). If $0 \to A \to B \to C \to 0$ is strictly normal exact, in the proof that (1) implies (2), $|c_\alpha| = |b_\alpha|$ with $g(b_\alpha) = c_\alpha$ so that $|b| = |c|$. Therefore $|k| = |h|$.

Statement (2) *implies* (1). The proof is similar to that of (3) implies (1) and is omitted.

Statement (4) *is equivalent to* (1). The proof can be found in [7, p. 498].

THEOREM 3.6. *The following statements are equivalent*:

(1) *X is equivalent to $L^1(I)$.*

(2) *If*

$$(*) \qquad 0 \to A \to B \to C \to 0$$

is strictly normal exact, then

$$(**) \qquad 0 \to X \hat{\otimes} A \to X \hat{\otimes} B \to X \hat{\otimes} C \to 0$$

is normal exact.

(3) *If* (*) *is normal exact with B reflexive, then* (**) *is normal exact.*

(4) *If* (*) *is normal exact with B finite-dimensional, then* (**) *is normal exact.*

(5) *If* (*) *is normal exact with B of dimension* 3 *and C of dimension* 1, *then* (**) *is normal exact.*

Proof. Statement (1) implies (2) by (2.5). Using (2.8), statement (2) implies (3). It is easy to see that (3) implies (4) and (4) implies (5). It must now be shown that (5) implies (1). Let $0 \to W \to Y \to Z \to 0$ be any normal exact sequence in $\boldsymbol{B}$ with $\dim Y = 3$ and $\dim Z = 2$. By (2.7),

$$0 \to Z^* \to Y^* \to W^* \to 0$$

is strictly normal exact with $\dim Y^* = 3$ and $\dim W^* = 1$. By assumption,

$$0 \to X \hat{\otimes} Z^* \to X \hat{\otimes} Y^* \to X \hat{\otimes} W^* \to 0$$

is normal exact. By (2.7),

$$0 \to (X \hat{\otimes} W^*)^* \to (X \hat{\otimes} Y^*)^* \to (X \hat{\otimes} Z^*)^* \to 0$$

is strictly normal exact. As in the proof of (3.3), this means that

$$0 \to \boldsymbol{B}(X, W^{**}) \to \boldsymbol{B}(X, Y^{**}) \to \boldsymbol{B}(X, Z^{**}) \to 0$$

is strictly normal exact. Hence,

$$0 \to \boldsymbol{B}(X, W) \to \boldsymbol{B}(X, Y) \to \boldsymbol{B}(X, Z) \to 0$$

is strictly normal exact. By (3.3) X is equivalent to $L^1(I)$.

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