ON THE SUMMATION FORMULA OF VORONOI

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Abstract. A formula involving sums of the form $\sum d(n)f(n)$ and $\sum d(n)g(n)$ is derived, where $d(n)$ is the number of divisors of $n$, and $f(x)$, $g(x)$ are Hankel transforms of each other. Many forms of such a formula, generally known as Voronoi's summation formula, are known, but we give a more symmetrical formula. Also, the reciprocal relation between $f(x)$ and $g(x)$ is expressed in terms of an elementary kernel, the cosine kernel, by introducing a function of the class $L^2(0, \infty)$. We use $L^2$-theory of Mellin and Fourier-Watson transformations.

Introduction. In 1904 Voronoi [10] published the following general formula: If $\tau(n)$ is an arithmetic function and $f(x)$ is continuous and has a finite number of maxima and minima in $a < x < b$, then analytic functions $\alpha(x)$ and $\delta(x)$, dependent on $\tau(n)$, can be determined such that

$$\frac{1}{2} \sum_{n \leq b} \tau(n)f(n) + \frac{1}{2} \sum_{n \leq a} \tau(n)f(n) = \int_{a}^{b} f(x) \delta(x) \, dx + 2\pi \sum_{n=1}^{\infty} \tau(n) \int_{a}^{b} f(x) \alpha(nx) \, dx.$$  

One of the better known special cases of this formula is when $\tau(n) = d(n)$, the number of divisors of $n$, and

$$\alpha(x) = (2/\pi)K_0(4\pi x^{1/2}) - Y_0(4\pi x^{1/2}), \quad \delta(x) = \log x + 2\gamma,$$

$\gamma$ being Euler's constant and $Y_0, K_0$ denote Bessel functions of second and third kinds respectively, of order zero. This special case is generally known as Voronoi's summation formula. Later, this formula received considerable attention as a result of which many modifications were put forth by A. L. Dixon and W. L. Ferrar [2], J. R. Wilton [13], A. P. Guinand [3] and others. Most of the authors used complex analysis and in all the new forms of the Voronoi formula, the kernel used was a combination of the Bessel functions $Y_0(x)$ and $K_0(x)$.  

Our object in this paper is to obtain a more symmetric and simplified form of Voronoi's formula, which holds under simple conditions. We state below the main result. First, a definition, due to Miller [6] and Guinand [4].
**Definition.** A function \( f(x) \in G_2^\lambda(0, \infty) \) if and only if, for a fixed \( \lambda > 1/p \) and \( p > 1 \), there exists almost everywhere a function \( f^{(\lambda)}(x) \), such that

\[
\begin{align*}
(i) & \quad f(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f^{(\lambda)}(t) \, dt, \quad x > 0, \\
(ii) & \quad x^\lambda f^{(\lambda)}(x) \in L^p(0, \infty).
\end{align*}
\]

The function \( f^{(\lambda)}(x) \) is the \( \lambda \)th derivative (apart from a factor \(-1)^{\lambda-1}\) of \( f(x) \) when \( \lambda \) is an integer. It can be shown that if \( f(x) \in G_2^\lambda(0, \infty) \), then

\[
x^r + 1/2 f^{(\lambda)}(x) \to 0 \quad \text{as} \quad x \to 0 \quad \text{or} \quad \infty, \quad 0 \leq r < \lambda,
\]

and that \( G_2^\lambda \) is a subclass of \( L^2 \). In this paper we shall use the class \( G_2^\lambda(0, \infty) \). The properties (i) and (ii), in this case, simply mean that (i) \( f(x) \) is the integral of its derivative \( f'(x) \) (apart from the factor \(-1\)) and (ii) \( xf'(x) \in L^2(0, \infty) \).

**Main Theorem.** Let \( \phi(x) \in G_2^\lambda(0, \infty) \). Then there exist functions \( f(x) \) and \( g(x) \), both \( \in G_2^\lambda(0, \infty) \), defined by

\[
\begin{align*}
(f) & \quad f(x) = 2 \int_0^\infty \phi(t) \cos 2\pi xt \, dt, \quad x > 0, \\
(g) & \quad g(x) = 2 \int_0^\infty \frac{1}{t} \phi \left( \frac{1}{t} \right) \cos 2\pi xt \, dt, \quad x > 0,
\end{align*}
\]

such that

\[
\lim_{N \to \infty } \left\{ \sum_{n=1}^N d(n) f(n) - \int_0^N (\log t + 2\gamma) f(t) \, dt \right\} = \lim_{N \to \infty } \left\{ \sum_{n=1}^N d(n) g(n) - \int_0^N (\log t + 2\gamma) g(t) \, dt \right\},
\]

where \( \gamma \) is Euler's constant.

This symmetric form of Voronoi's formula could be derived from a general formula [3] of A. P. Guinand, if we had used the kernel \(-Y_0(4\pi nx^{1/2}) + (2/\pi) K_0(4\pi x^{1/2})\) and employed sophisticated order results. In our proof we make use of easily derived and elementary results, using the theory of mean convergence of functions of \( L^2(0, \infty) \).

**Definition 2.** A kernel \( k(x) \in D^2 \) if and only if

(i) there is defined a.e. in \((-\infty, \infty)\) a function \( K(\frac{1}{2} + it) \), such that \( |K(\frac{1}{2} + it)| = 1 \), \( K(\frac{1}{2} + it)K(\frac{1}{2} - it) = 1 \);

(ii) the function \( k_1(x) \), defined a.e. by

\[
k_1(x) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{1/2 - iT}^{1/2 + iT} K(s) \frac{1}{s - x^2} \, ds,
\]
may be chosen, so that
(a) $k_1(x)$ is differentiable, $k_1(x) = \int k(t) \, dt$,
(b) $k_1(x)$ is $O(x^{1/2})$, $x \to \infty$, and $O(x^{1/2})$, $x \to 0$,
(c) $k(x) \in L(1/n, n)$, for all finite $n > 0$.
Such a class of kernels is due to J. B. Miller [7].

The following results can be deduced from the functional relations and expansions of Bessel functions $Y_n(x)$ and $K_n(x)$ [12, pp. 62–80]. If $L_n(x) = -Y_n(x) - (2/\pi)K_n(x)$ and $M_n(x) = -Y_n(x) + (2/\pi)K_n(x)$, then
\begin{equation}
(1.2) \quad (d/dx)(xL_1(x)) = xM_0(x).
\end{equation}
\begin{equation}
(1.3) \quad L_1(x) = O(x^{-1/2}), \quad \text{as } x \to \infty,
\end{equation}
and $= O(x \log x)$, as $x \to 0$.

2. Preliminary results. Consider the function
\begin{equation}
(2.1) \quad h(x) = \left\{ \sum_{n \leq x} d(n) - x \log x + 2y - 1 \right\} x^{-1}.
\end{equation}
Since [8, p. 262]
\begin{equation}
\sum_{n \leq x} d(n) - x \log x + 2y - 1 = O(x^{1/2}), \quad x \to \infty,
\end{equation}
therefore
\begin{equation}
(2.2) \quad h(x) = O(x^{-1/2}), \quad x \to \infty,
\end{equation}
\begin{equation}
= O(\log x), \quad x \to 0.
\end{equation}
Then its Mellin transform
\begin{equation}
H(s) = \int_0^\infty h(x)x^{s-1} \, dx \quad (s = \sigma + it)
\end{equation}
exists for $0 < \sigma < \frac{1}{2}$. Or
\begin{equation}
H(s) = \int_1^\infty h(x)x^{s-1} \, dx + \int_1^\infty h(x)x^{s-1} \, dx, \quad 0 < \sigma < \frac{1}{2},
\end{equation}
\begin{equation}
= \frac{1}{s^2 - \frac{2\gamma - 1}{s}} + \int_1^\infty h(x)x^{s-1} \, dx, \quad \sigma < \frac{1}{2}.
\end{equation}
This gives the analytic continuation into $\sigma < 0$. Now
\begin{equation}
\int_1^\infty h(x)x^{s-1} \, dx = \int_1^\infty \sum_{n \leq x} d(n)x^{s-2} \, dx - \int_1^\infty (\log x + 2y - 1)x^{s-1} \, dx.
\end{equation}
By splitting the range of integration $(1, \infty)$ into $(1, 2), (2, 3), \ldots$ and solving, we get
\begin{equation}
\int_1^\infty \sum_{n \leq x} d(n)x^{s-2} \, dx = \frac{1}{1-s} \sum_{n=1}^{\infty} d(n)n^{s-1} = \frac{\xi(s)(1-s)}{1-s},
\end{equation}
where $\xi(x)$ is the Riemann-zeta function.
Now, for $\sigma < 0$,
\[
\int_{1}^{\infty} (\log x + 2\gamma - 1)x^{s-1} \, dx = \frac{1}{s^2} - \frac{2\gamma - 1}{s}.
\]
Hence, by analytic continuation, we obtain
\begin{equation}
H(s) = \zeta^2(1-s)/(1-s) \quad (0 < \sigma < \frac{1}{2}).
\end{equation}

Since $x^{\sigma-1}h(x) \in L^2(0, \infty)$, $0 < \sigma < \frac{1}{2}$, by Mellin's inversion formula
\[
\frac{1}{\zeta^2(1-s)l(1-s)} = \frac{1}{2\pi i} \lim_{t \to \infty} \int_{\sigma-it}^{\sigma+it} x^{-s} \, ds.
\]

Next we shall show that $H(s) \in L^2(-\infty, \infty)$ on $s = \frac{1}{4} + it$ and deduce that
\[
h(x) \in L^2(0, \infty).
\]

Now [8, p. 92]
\[
\zeta(\frac{1}{2} + it) = O(t^{1/6} \log t), \quad t \to \infty.
\]
Therefore $\zeta^2(1-s)/(1-s) \in L^2(\frac{1}{2} - \infty, \frac{1}{2} + \infty)$ and has a Mellin transform $h_1(x)$, say, belonging to $L^2(0, \infty)$, defined by
\[
h_1(x) = \frac{1}{2\pi i} \lim_{t \to \infty} \int_{\frac{1}{2} + it}^{\frac{1}{2} - it} \frac{\xi^2(1-s)}{1-s} x^{-s} \, ds
\]
a.e. for $x > 0$. Let $C$ be the contour $(\sigma - iT, \frac{1}{2} - iT, \frac{1}{2} + iT, \sigma + iT, \sigma - iT)$. By Cauchy's Theorem
\[
\int_{C} \frac{\xi^2(1-s)}{1-s} x^{-s} \, ds = 0, \quad 0 < \sigma < \frac{1}{2},
\]
the integrals along the lines $(\sigma - iT, \frac{1}{2} - iT)$ and $(\frac{1}{2} + iT, \sigma + iT)$ vanish as $T \to \infty$, since [8, p. 82] $\zeta(\sigma + it) = O(t^{1/2 - \sigma/2})$, $0 < \sigma < 1$.

We have then
\[
\lim_{T \to \infty} \int_{\frac{1}{2} + iT}^{\frac{1}{2} - iT} \frac{\xi^2(1-s)}{1-s} x^{-s} \, ds = \lim_{T \to \infty} \int_{\sigma - iT}^{\sigma + iT} \frac{\xi^2(1-s)}{1-s} x^{-s} \, ds
\]
a.e. Or, $h_1(x) = h(x)$ a.e. and hence $h(x) \in L^2(0, \infty)$.

Let us define a function
\begin{equation}
A(x) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \frac{\zeta(s)}{1-s} x^{1-s} \, ds,
\end{equation}
where $\zeta(s) = \psi(1-s)/\psi(s)$ and $\psi(s) = \sum_{n=1}^{\infty} d(n)n^{-s}$.

Thus $\psi(s) = \xi^2(s)$ and using the functional equation
\[
\zeta(s) = 2^s\pi^{s-1}(\sin \frac{1}{2}s\pi)\Gamma(1-s)\zeta(1-s),
\]
where $\Gamma(z)$ is the gamma function.
we obtain

\[ \mathcal{K}(s) = 4(2\pi)^{-2s} \Gamma^2(s) \cos^2 \frac{1}{2}s\pi. \]

Now

\[ |\mathcal{K}(\frac{1}{2} + it)| = 1, \quad |\mathcal{K}(\frac{1}{2} + it)| \mathcal{K}(\frac{1}{2} - it) = 1 \]

and consequently, on the line \( s = \frac{1}{2} + it \), \( |\mathcal{K}(s)/(1 - s)| = O(t^{-1}) \) and thus belongs to \( L^2(-\infty, \infty) \) when integrated with respect to \( t \). Hence the integral (2.4) converges in mean square. Also, \( x^{-1} A(x) \in L^2(0, \infty) \) and \( A(x) \) is a Fourier kernel in Watson's sense [11].

Substituting the value of \( \mathcal{K}(s) \), obtained above, in (2.4), we have

\[ A(x) = \lim_{T \to \infty} \int_{1/2-iT}^{1/2+iT} 4(2\pi)^{-2s} \Gamma(s) \Gamma(s-1) \cos^2 \frac{1}{2}s\pi \cdot x^{1-s} \, ds. \]

We shall now evaluate the above integral. It is known [9, p. 195] that for \( 1 < \sigma < \frac{3}{4} \)

\[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} (\pi)^{-1} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) \cos \pi s \cdot x^{-s} \, ds = x^{-1/2} Y_1(4\pi x^{1/2}). \]

Moving the line of integration to \( \sigma = \frac{1}{4} \) and by applying the theory of residues we get

\[ \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} (\pi)^{-1} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) \cos \pi s \cdot x^{-s} \, ds = x^{-1/2} Y_1(4\pi x^{1/2}) + (2\pi^2 x)^{-1}. \]

Also, [9, p. 197] for \( \sigma > 1 \)

\[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} (\pi)^{-1} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) x^{-s} \, ds = \frac{2}{\pi} x^{-1/2} K_1(4\pi x^{1/2}). \]

Moving the line of integration to \( \sigma = \frac{1}{2} \), we have

\[ \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} (\pi)^{-1} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) x^{-s} \, ds = \frac{2x^{-1/2}}{\pi} K_1(4\pi x^{1/2}) - (2\pi^2 x)^{-1}. \]

Now from (2.8) and (2.9),

\[ -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{2}{\pi} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) \cos^2 \pi s \cdot x^{-s} \, ds = -x^{-1/2} \{ Y_1(4\pi x^{1/2}) + (2/\pi) K_1(4\pi x^{1/2}) \}. \]

Thus (2.7) yields \( A(x) = x^{1/2} L_1(4\pi x^{1/2}) \).

Note that \( A(x) \) is differentiable, and let \( \lambda(x) = \int_0^x \chi(t) \, dt \), from whence \( \chi(x) = 2\pi M_0(4\pi x^{1/2}) \) by (1.2). From (1.3) and (2.6) we see that all relevant conditions are satisfied and therefore \( \chi(x) \) belongs to the kernel class \( D^2 \).

Further, let

\[ F(x) = \lim_{T \to \infty} \int_{1/2-iT}^{1/2+iT} \frac{5\mathcal{K}(s)}{(1-s)(2-s)} x^{1-s} \, ds. \]
From (2.6), \(|s \mathcal{K}(s)/(1-s)(2-s)| = O(t^{-1})\), therefore the integral (2.10) converges in mean square and \(x^{-1}F(x) \in L^2(0, \infty)\). Thus \(F(x)\) is a generalized Hankel kernel [11].

**Lemma 2.1.** Let \(h(x)\) be defined by (2.1). Then

\[
\int_0^x th(t) \, dt = x \int_0^\infty h(t) \frac{F(xt)}{t} \, dt,
\]

where \(F(x)\) is the generalized Hankel kernel defined by (2.10).

**Proof.** Applying Parseval’s theorem to \(L^2\)-functions \(h(x)\) and \(x^{-1}F(x)\), we have

\[
x \int_0^\infty h(t) \frac{F(xt)}{t} \, dt = \frac{x}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{H(1-s)\xi(s)}{(1-s)(2-s)} x^{1-s} \, ds,
\]

which, by (2.3) and (2.5), is

\[
\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\xi^2(1-s)}{(1-s)(2-s)} x^{1-s} \, ds = \int_0^x th(t) \, dt,
\]

as required.

Thus we can say that \(h(x)\) is the \(F\)-transform of itself.

**Lemma 2.2.** Let \(f(x) \in G_2^2(0, \infty)\). Then there exists \(g(x) \in G_4^2(0, \infty)\), such that

\[
g(x) = 2\pi \int_0^\infty f(t) \chi(xt) \, dt, \quad x > 0,
\]

and

\[
f(x) = 2\pi \int_0^\infty g(t) \chi(xt) \, dt, \quad x > 0.
\]

Further \(xf'(x)\) and \(xg'(x)\) are \(F\)-transforms of each other. Here \(\chi(x) = 2\pi M_0(4\pi x^{1/2})\).

**Proof.** The first part is immediate by a result due to J. B. Miller [6], since the kernel \(\chi(x) \in D^2\). The second part can be proved by the same method as used in the proof of Lemma 2.1.

**Lemma 2.3.** Let \(\phi(x) \in G_2^2(0, \infty)\) and define \(f(x)\) by the equation

\[
(2.11) \quad f(x) = 2\int_0^\infty \phi(t) \cos 2\pi xt \, dt, \quad x > 0.
\]

Then \(f(x) \in G_2^2(0, \infty)\). Further, if a function \(g(x)\) is defined by

\[
(2.12) \quad g(x) = 2\int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) \cos 2\pi xt \, dt, \quad x > 0,
\]

then \(g(x) \in G_4^2(0, \infty)\).

**Proof.** It can be seen that \(2 \cos 2\pi x \in D^2\). Thus by Theorem I of J. B. Miller [6], \(f(x) \in G_2^2(0, \infty)\), since \(\phi(x) \in G_2^2(0, \infty)\). Similarly \(g(x) \in G_4^2(0, \infty)\), provided we
can show that \( (1/x)\phi(1/x) \in G^2_2(0, \infty) \) when \( \phi(x) \) does. Now,

\[
x \frac{d}{dx} \left\{ \frac{1}{x} \phi \left( \frac{1}{x} \right) \right\} = -\frac{1}{x} \phi \left( \frac{1}{x} \right) - \frac{1}{x^2} \phi \left( \frac{1}{x} \right).
\]

Since \( \phi(x) \in G^2_2 \), by property (ii), \( (1/x)\phi(1/x) \) and \( (1/x^2)\phi'(1/x) \) belong to \( L^2(0, \infty) \), and using Minkowski’s inequality, we can show that \( x(d/dx)((1/x)\phi(1/x)) \) also belongs to \( L^2(0, \infty) \).

Also,

\[
\phi(x) = \frac{1}{x} \int_0^x \frac{d}{dt} \{t\phi(t)\} \, dt.
\]

Or,

\[
\frac{1}{x} \phi \left( \frac{1}{x} \right) = \int_0^{1/x} \{\phi(t) + t\phi'(t)\} \, dt
\]

\[
= \int_x^{\infty} \left\{ \frac{1}{u} \phi \left( \frac{1}{u} \right) + \frac{1}{u^2} \phi' \left( \frac{1}{u} \right) \right\} \, du = -\int_x^{\infty} \frac{d}{du} \left\{ \frac{1}{u} \phi \left( \frac{1}{u} \right) \right\} \, du.
\]

Thus \( (1/x)\phi(1/x) \) is the integral of its derivative, and hence \( (1/x)\phi(1/x) \in G^2_2(0, \infty) \).
This proves the lemma.

3. The Main Theorem. Applying Parseval’s theorem [1] for the two pairs \( h(x), h(x) \) and \( xf'(x), xg'(x) \) of F-transforms of the class \( L^2(0, \infty) \), we have

(3.1) \[
\int_0^\infty xh(x)f'(x) \, dx = \int_0^\infty xh(x)g'(x) \, dx.
\]

The left-hand side is

\[
\int_0^\infty \left\{ \sum_{n \in \mathbb{N}} d(n)x(\log x + 2\gamma - 1) \right\} f'(x) \, dx
\]

\[
= \lim_{N \to \infty} \left\{ \sum_{n \in \mathbb{N}} d(n)x(\log x + 2\gamma - 1) \right\} \left[ f(x) \right]_0^N
\]

\[
= \left[ f(x) \right]_0^N d \left( \sum_{n \in \mathbb{N}} d(n) \right) + \int_0^N (\log x + 2\gamma) f(x) \, dx.
\]

Since \( f(x) \) and \( h(x) \) satisfy (1.1) and (2.2) respectively, the integrated term vanishes at both limits, and the above expression reduces to

\[
\lim_{N \to \infty} \left\{ -\sum_{n=1}^N d(n)f(n) + \int_0^N (\log x + 2\gamma) f(x) \, dx \right\}.
\]

Treating the right-hand side of (3.1) in the same manner, we obtain

**Theorem 3.1.** Let \( f(x) \in G^2_2(0, \infty) \). If \( g(x) \) is defined by

\[
g(x) = 2\pi \int_0^\infty f(t)\chi(xt) \, dt
\]
then \( g(x) \) belongs to \( G^2(0, \infty) \), where \( \chi(x) = 2\pi M_0(4\pi x^{1/2}) \). Further

\[
\lim_{N \to \infty} \left\{ N \sum_{n=1}^{N} d(n)f(n) - \int_{0}^{\infty} (\log x + 2\gamma)f(x) \, dx \right\} = \lim_{N \to \infty} \left\{ N \sum_{n=1}^{N} d(n)g(n) - \int_{0}^{\infty} (\log x + 2\gamma)g(x) \, dx \right\}.
\]

**Theorem 3.2.** Let \( \phi(x) \in G^2(0, \infty) \). If there exist functions \( f(x) \) and \( g(x) \) defined by the equations (2.11) and (2.12), then the equations

\[
f(x) = \int_{0}^{\infty} g(t)x(t) \, dt, \quad g(x) = \int_{0}^{\infty} f(t)x(t) \, dt
\]

hold for \( x > 0 \), where \( \chi(x) = 2\pi M_0(4\pi x^{1/2}) \).

**Proof.** Integrating by parts the integral in (2.11), we get

\[
f(x) = \left[ \phi(t) \frac{\sin 2\pi xt}{\pi x} \right]_{-\infty}^{\infty} - \int_{0}^{\infty} \phi'(t) \frac{\sin 2\pi xt}{\pi x} \, dt
\]

(3.2)

The integrated term vanishes by (1.1) since \( \phi(x) \in G^2(0, \infty) \). If \( \Phi(s) \) denotes the Mellin transform of \( \phi(x) \), then \( -s\Phi(s) \) is the Mellin transform of \( x\phi'(x) \). Now, we know that \( x\phi'(x) \) and \( \sin 2\pi xt \) both belong to \( L^2(0, \infty) \). Therefore, by applying, to the right side of (3.2), the Parseval theorem for Mellin transforms of \( L^2 \)-functions, we obtain

\[
f(x) = -\frac{1}{\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} s\Phi(s)(2\pi x)^{s-1-1/2} \Gamma(-s) \sin \frac{s\pi x}{2} \, ds.
\]

(3.3)

Now, from (2.12),

\[
\int_{0}^{\infty} g(u) \, du = \frac{1}{\pi} \int_{0}^{\infty} \phi \left( \frac{1}{t} \right) \frac{\sin 2\pi xt}{t^2} \, dt.
\]

Let \( G(s) \) be the Mellin transform of \( g(x) \). It can be shown easily that \( \phi(1-s) \) is the Mellin transform of \( (1/x)\phi(1/x) \) and \( x^s/s \) is the Mellin transform of the function \( 1, 0 < u < x; 0, u > x \). Applying the Parseval theorem for Mellin transforms to both sides of the last equation, we get

\[
\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} G(s) \frac{x^{1-s}}{1-s} \, ds = -\frac{1}{\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \Phi(s)(2\pi)^{-s}\Gamma(s-1) \cos \frac{s\pi x}{2} \, ds.
\]

Or,

\[
\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \{ G(s) - 2(2\pi)^{-s}\Phi(s)\Gamma(s) \cos \frac{s\pi}{2} \frac{x^{1-s}}{1-s} \} \, ds = 0,
\]

and, by Mellin inversion formula,

\[
G(s) = 2(2\pi)^{-s}\Phi(s)\Gamma(s) \cos \frac{s\pi}{2}.
\]

(3.4)
a.e. on $R(s) = \frac{1}{2}$. Substituting the value of $\Phi(s)$ in (3.4) and using the functional equation $\Gamma(s)\Gamma(1-s) = \pi \csc \pi s$, we obtain from (3.3)

$$f(x) = -\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{2}{\pi} (2\pi)^{2s-1} \Gamma(-s)\Gamma(1-s)\sin^2 \frac{\pi x^4 - 1}{2} s\Gamma(1-s) s G(s) \, ds$$

say, where

$$\mathcal{L}(s) = \frac{(2/\pi)(2\pi)^{1-2s}\Gamma(s-1)}{\Gamma(s)} \cos^2 \frac{\pi x^{-s}}{2}.$$

It can be easily deduced from the value of the integral in (2.7) that $\mathcal{L}(s)$ is the Mellin transform of $-(x^2)^{1/2} L_1(4\pi(x^2)^{1/2})$, when considered as a function of $t$. Now $x^2(x^2)^{1/2} L_1(4\pi(x^2)^{1/2})$ both belong to $L^2(0, \infty)$ due to (1.1), as $g(x) \in G_2^2$, and (1.3). Thus applying Parseval's theorem to the above pair of $L^2$-functions, we obtain

$$f(x) = -\left[ x^{-1/2} \int_0^\infty g(t) M_0(4\pi(x^2)^{1/2}) \, dt \right]_{\infty}^0 + 2\pi \int_0^\infty g(t) M_0(4\pi(x^2)^{1/2}) \, dt = f(x).$$

Integrating the left-hand side by parts, we can write (3.5) as

$$f(x) = -\left[ x^{-1/2} \int_0^\infty g(t) M_0(4\pi(x^2)^{1/2}) \, dt \right]_{\infty}^0 + 2\pi \int_0^\infty g(t) M_0(4\pi(x^2)^{1/2}) \, dt.$$

The integrated term vanishes at both the limits by (1.1) and (1.3). Hence

$$f(x) = 2\pi \int_0^\infty g(t) M_0(4\pi(x^2)^{1/2}) \, dt, \quad x > 0,$$

$$= \int_0^\infty g(t) x(t) \, dt,$$

as required. Similarly

$$g(x) = \int_0^\infty f(t) x(t) \, dt, \quad x > 0.$$

Finally, the main theorem stated in the introduction follows by combining the results obtained in Theorems 3.1 and 3.2.

4. An example. Let

$$f(x) = K_0(2\pi z x), \quad R(z) > 0.$$

Then

$$\phi(x) = 2 \int_0^\infty K_0(2\pi z t) \cos 2\pi x t \, dt$$

$$= \frac{1}{2} (x^2 + x^2)^{-1/2}, \quad \text{cf. [12, p. 388]}.\]
Now define a function
\[ g(x) = 2 \int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) \cos 2\pi xt \, dt \]
\[ = \int_0^\infty t^{-1} (z^2 + t^{-2})^{-1/2} \cos 2\pi xt \, dt = \int_0^\infty (1 + z^2 t^2)^{-1/2} \cos 2\pi xt \, dt \]
\[ = z^{-1} \int_0^\infty (1 + u^2)^{-1/2} \cos \frac{2\pi xu}{z} \, du = z^{-1} K_0\left(\frac{2\pi x}{z}\right), \quad R(z) > 0, \]
cf. [12, p. 434]. Also,
\[ K_0(x) = O(x^{-1/2} e^{-x}), \quad x \to \infty, \]
\[ = O(\log x), \quad x \to 0. \]
(4.1)
Thus \( K_0(2\pi z x) \) and \( z^{-1} K_0(2\pi x/z) \), as function of \( x \), satisfy the conditions of the main theorem, which yields the formula
\[ \sum_{n=1}^\infty d(n) K_0(2\pi z n) - \int_0^\infty (\log t + 2\gamma) K_0(2\pi z t) \, dt \]
\[ = z^{-1} \sum_{n=1}^\infty d(n) K_0\left(\frac{2\pi n}{z}\right) - z^{-1} \int_0^\infty (\log t + 2\gamma) K_0\left(\frac{2\pi t}{z}\right) \, dt. \]
(4.2)
We shall now evaluate the two integrals in (4.2). First consider
\[ I_1 = \int_0^\infty (\log t + 2\gamma) K_0(2\pi z t) \, dt \]
\[ = \frac{1}{2\pi z} \left\{ (2\gamma - \log 2\pi z) \int_0^\infty K_0(u) \, du + \int_0^\infty \log u K_0(u) \, du \right\}. \]
Now [12, p. 388]
\[ \int_0^\infty K_0(u) \, du = \frac{\pi}{2}. \]
(4.3)
Let \( \int_0^\infty \log u K_0(u) \, du = I, \) say.
It is known that [12, p. 172] \( K_0(z) = \int_1^\infty e^{-zt} (t^2 - 1)^{-1/2} \, dt. \) Therefore
\[ I = \int_0^\infty \log u \, du \int_1^\infty e^{-zt} (t^2 - 1)^{-1/2} \, dt = \int_1^\infty (t^2 - 1)^{-1/2} \, dt \int_0^\infty \log u \, e^{-ut} \, du. \]
The inversion of order of integration is justified by absolute convergence.
Now
\[ \int_0^\infty \log u \, e^{-ut} \, du = -t^{-1} \log (e^t), \]
y being Euler's constant. Thus
\[ I = -\int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \log (e^t) \, dt \]
\[ = -\gamma \int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \, dt - \int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \log t \, dt \]
\[ = -\gamma \frac{\pi}{2} - \frac{\pi}{2} \log 2. \]
(4.4)
Hence from (4.3) and (4.4)

\[ I_1 = \frac{1}{2\pi z} \left\{ (2\gamma - \log 2\pi z) \frac{\pi}{2} - \frac{\pi}{2} (\gamma + \log 2) \right\} = \frac{1}{4} (\gamma - \log 4\pi z). \]

Next consider

\[
I_2 = z^{-1} \int_0^\infty (\log t + 2\gamma) K_0 \left( \frac{2\pi t}{z} \right) dt
\]

\[ = \frac{1}{2\pi} \left( 2\gamma - \log \frac{2\pi}{z} \right) \int_0^\infty K_0(u) \, du + \frac{1}{2\pi} \int_0^\infty \log u K_0(u) \, du \]

\[ = \frac{1}{4} (2\gamma - \log (2\pi/z)) - \frac{1}{4} (\gamma + \log 2), \]

by (4.3) and (4.4). Thus \( I_2 = \frac{1}{4} (\gamma - \log (4\pi/z)). \) Substituting the values of the integrals \( I_1 \) and \( I_2 \) in (4.2) and rearranging the terms, we obtain

\[
\sum_{n=1}^{\infty} d(n) K(2\pi zn) - z^{-1} \sum_{n=1}^{\infty} d(n) K_0 \left( \frac{2\pi n}{z} \right) = \frac{1}{2} z^{-1} (\gamma - \log 4\pi z) - \frac{1}{4} (\gamma - \log (4\pi/z)),
\]

which is a known formula due to N. S. Koshliakov [5].

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REFERENCES

1. I. W. Busbridge, A theory of general transforms for functions of the class \( L^p(0, \infty), \) \( (1 < p \leq 2), \) Quart. J. Math. Oxford Ser. (2) 9 (1938), 148–160.