THE SIGN OF LOMMEL'S FUNCTION

BY

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Abstract. Lommel's function $s_{\mu,v}(x)$ is a particular solution of the differential equation $x^2y'' + xy' + (x^2 - \nu^2)y = x^{\mu+1}$. It is shown here that $s_{\mu,v}(x) > 0$ for $x > 0$, if $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$, or if $\mu > \frac{1}{2}$ and $|\nu| \leq \mu$. This includes earlier results of R. G. Cooke's. The sign of $s_{\mu,v}(x)$ for other values of $\mu$ and $\nu$ is also discussed.

1. Introduction. In 1876, Lommel [6] considered the inhomogeneous differential equation

$$z^2y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1},$$

where $\mu$ and $\nu$ are complex parameters. He obtained two particular solutions: the Lommel functions of the first kind, $J_{\mu,v}(z)$, and of the second kind, $S_{\mu,v}(z)$. The homogeneous equation associated with (1) is

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0,$$

Bessel's equation.

The function $s_{\mu,v}(z)$ is defined for all pairs $\mu, \nu$ such that neither $\mu - \nu$ nor $\mu + \nu$ is an odd negative integer, and for all $z$ with $-\pi < \arg z \leq \pi$, by the series

$$s_{\mu,v}(z) = \frac{1}{2}z^{\mu+1} \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2})^{2n}\Gamma((\mu - \nu + 1)/2)\Gamma((\mu + \nu + 1)/2)}{\Gamma((\mu - \nu + n + 3)/2)\Gamma((\mu + \nu + n + 3)/2)}.$$}

The symmetry property

$$s_{\mu,v}(z) = s_{\mu,-v}(z)$$

is obvious from (3).

We shall consider $s_{\mu,v}(z)$ for $\mu$ and $\nu$ real, and positive $z$. Its importance arises from the formula [1, §3.20], [9, §10.74]

$$\int x^\nu C_\nu(x) \, dx = x[C_\nu(x)s_{\nu}\mu(x) - s_{\mu,v}(x)C_\nu'(x)],$$

in which $C_\nu(x)$ denotes any real solution of equation (2), that is,

$$C_\nu(x) = \alpha J_\nu(x) + \beta Y_\nu(x),$$

where $\nu$, $\alpha$, and $\beta$ are real, $x > 0$, and $J_\nu(x)$ and $Y_\nu(x)$ denote the usual Bessel functions.

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A case of particular interest is that in which \( \nu = \nu \). Then, unless \( \nu \) is half of an odd negative integer, we have

\[
(6) \quad s_{\nu,v}(x) = 2^{v-1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})H_v(x),
\]

where \( H_v(x) \) is Struve's function of order \( \nu \) [1, (3.127)]. Now it is known [9, §10.45] that \( H_v(x) > 0 \) for all \( x > 0 \) if \( \nu > \frac{1}{2} \), that \( H_{1/2}(x) = (2/\pi x)^{1/2}(1-\cos x) \), and that \( H_v(x) \) has an infinity of changes of sign if \( \nu < \frac{1}{2} \). The corresponding problem for \( s_{\nu,v}(x) \) is more difficult. R. G. Cooke [4] found conditions on \( \nu \) and \( \nu \) sufficient to ensure that \( s_{\nu,v}(x) > 0 \) for all \( x > 0 \). We may state his results as

**Theorem A.** If \( \nu \geq 0 \) and \( \mu \geq \nu + 1 \), then \( s_{\nu,v}(x) > 0 \) for \( x > 0 \). If \( \nu \geq \frac{1}{2} \) and \( \mu \geq \nu \), then \( s_{\nu,v}(x) > 0 \) for \( x > 0 \), unless \( \mu = \nu = \frac{1}{2} \), when \( s_{\nu,v}(x) \geq 0 \) for \( x > 0 \).

The symmetry relation (4) gives the corresponding results for \( \nu < 0 \).

Cooke's proof requires an expression for \( s_{\nu,v}(x) \) as a fractional integral involving \( J_v(x) \), and a previous result of his ([2] and [3]) which implies that if \( \mu = 0 \) and \( \nu > 1 \), or if \( \mu = 1 - \nu \) and \( \nu > 1 \), then

\[
\int_0^\xi t^2J_v(t) \, dt > 0, \quad \xi > 0.
\]

We shall here consider the same problem. It will often be convenient to refer to a \((\mu, \nu)\)-plane, and to associate with each pair \( \mu, \nu \) the point with those coordinates.

We shall see that \( s_{\nu,v}(x) > 0 \) for \( x > 0 \) if \( \mu \geq \frac{1}{2} \) and \( |\nu| \leq \mu \), except when \( \mu = |\nu| = \frac{1}{2} \), in which case \( s_{\nu,v}(x) \geq 0 \). These inequalities define a larger region in the \((\mu, \nu)\)-plane than Cooke's. If \( \mu < \frac{1}{2} \), or if \( \mu = \frac{1}{2} \) and \( |\nu| > \frac{1}{2} \), then \( s_{\nu,v}(x) \) changes sign infinitely often on \((0, \infty)\). If \( \mu > \frac{1}{2} \) and \( \mu < |\nu| - 1 \), \( s_{\nu,v}(x) \) has an odd number of changes of sign on \((0, \infty)\). Finally, if \( \mu > \frac{1}{2} \) and \( |\nu| - 1 < \mu < |\nu| \), \( s_{\nu,v}(x) \) has an even number of changes of sign on \((0, \infty)\); but I have not been able to decide whether this number is always positive. I shall show, however, that there are points \((\mu, \nu)\) in this region such that the corresponding \( s_{\nu,v}(x) \) has changes of sign (an arbitrarily large number of them, in fact).

Our main tool is an oscillation theorem of Makai's [7] for second order differential equations. With it, we can determine the sign of the function

\[
h(\xi) = \int_0^\xi t^2C_v(t) \, dt, \quad \xi > 0,
\]

for certain pairs \( \mu, \nu \). Then, by using (5), we can deduce results on the sign of \( s_{\nu,v}(x) \).

2. Oscillation theorems. Our starting point is

**Theorem B.** Let \( y = y(x) \) be a solution of the differential equation \( y'' + \varphi(x)y = 0 \), where \( \varphi \) is continuous and increasing on \((x_0, x_2)\). Let \( x_1 \) be the only zero of \( y(x) \) on \((x_0, x_2)\). Further, assume that

\[
(7) \quad \lim_{x \to x_2+0} y(x) = \lim_{x \to x_2-0} y(x) = 0.
\]
Then,
\[ \int_{x_0}^{x_1} |y(x)| \, dx \geq \int_{x_1}^{x_2} |y(x)| \, dx, \]
with strict inequality if \( \phi \) is strictly increasing.

If \( \phi \) is decreasing on \((x_0, x_2)\), then (8) is reversed.

Theorem B is due essentially to E. Makai [7], [8, §1.82], but the endpoint conditions (7) have been introduced to avoid difficulties which may arise when \( \phi \) is discontinuous at \( x_0 \), or at \( x_2 \). The monotonicity condition on \( \phi \) could be relaxed somewhat [7, §2(d)], but we will not need this here.

From Theorem B we deduce

**Theorem 1.** Let \( y = y(x) \) be a solution of the differential equation
\[ (r(x)y')' + p(x)y = 0, \]
where \( r \) and \( p \) are continuous, \( pr \) is increasing and \( r \) is positive, on \((x_0, x_2)\). Assume further that \( \int_{x_0}^{x_2} (r(u))^{-1} \, du \) converges. Let
\[ \lim_{x \to x_0 + 0} y(x) = \lim_{x \to x_2 - 0} y(x) = 0, \]
and let \( x_1 \) be the only zero of \( y \) on \((x_0, x_2)\). Then,
\[ \int_{x_0}^{x_1} \frac{|y(x)|}{r(x)} \, dx \geq \int_{x_1}^{x_2} \frac{|y(x)|}{r(x)} \, dx, \]
with strict inequality if \( pr \) is strictly increasing. And if \( pr \) is decreasing (11) is reversed.

**Proof.** We transform the independent variable in (9) by setting
\[ t = f(x) = \int_{x_0}^{x} \frac{du}{r(u)}. \]
Equation (9) can then be written as
\[ d^2y/dt^2 + r(g(t))p(g(t))y = 0, \]
where \( g \) denotes the inverse of the (increasing) function \( f \) [5, p. 235]. Inequality (11) now follows by applying Theorem B to (12), and then changing the independent variable back to \( x \).

In the sequel, we shall consider the equation
\[ (x^{1-2\mu}y')' + x^{-2\mu-1}(x^2 + \mu^2 - \nu^2)y = 0, \]
which is of the form (9), with
\[ (p(x)r(x))' = 2x^{-4\mu-1}[(1-2\mu)x^2 + 2\mu(\nu^2 - \mu^2)]. \]
The integrand in (5), \( x^{\mu}C_{\nu}(x) \), is a solution of equation (13) [9, §4.31, (19)-(20)]. We shall often choose \( \mu = \frac{1}{2} \); (13) then becomes
\[ y'' + (1 + (1 - 4\nu^2)/4x^2)y = 0. \]
3. **Lommel functions.** The two Lommel functions $s_{\mu,v}(x)$ and $S_{\mu,v}(x)$ are related by the identity

$$s_{\mu,v}(x) = S_{\mu,v}(x) + 2^{\mu-1} \Gamma\left(\frac{\mu-v+1}{2}\right) \Gamma\left(\frac{\mu+v+1}{2}\right) \left[ \cos \left(\frac{(\mu-v)\pi}{2}\right) Y_{v}(x) - \sin \left(\frac{(\mu-v)\pi}{2}\right) J_{v}(x) \right],$$

(16)

whenever $s_{\mu,v}(x)$ is defined [9, §10.71].

Now it can be shown [9, §10.75] that $S_{\mu,v}(x) \sim x^{\mu-1}$, as $x \to +\infty$. On combining this with the relations

$$J_{v}(x) = \frac{2}{\pi x^{1/2}} \left[ \cos (x - \frac{1}{2} \pi (\mu - v)) + O(1/x) \right]$$

and

$$Y_{v}(x) = \frac{2}{\pi x^{1/2}} \left[ \sin (x - \frac{1}{2} \pi (\mu + v)) + O(1/x) \right]$$

[9, §7.22], we see that as $x \to +\infty$, the dominant term on the right-hand side of

(16) is

$$2^{\mu-1} \left(\frac{2}{\pi x}\right)^{1/2} \Gamma\left(\frac{\mu-v+1}{2}\right) \Gamma\left(\frac{\mu+v+1}{2}\right) \sin \left(\frac{1}{2} \mu \pi - \frac{1}{2} \pi \right) \text{ or } x^{\mu-1},$$

according as $\mu < \frac{1}{2}$ or $\mu > \frac{1}{2}$. It follows that $s_{\mu,v}(x)$ has an infinity of changes of sign if $\mu < \frac{1}{2}$, but is positive for all sufficiently large $x$, if $\mu > \frac{1}{2}$. We now proceed to make this simple observation more precise. We begin by establishing

**Theorem 2.** If $\mu < \frac{1}{2}$, or if $\mu = \frac{1}{2}$ and $|v| > \frac{1}{2}$, then $s_{\mu,v}(x)$ has an infinity of changes of sign on $(0, \infty)$.

**Proof.** For $\mu < \frac{1}{2}$ and $v$ unrestricted, the result follows as above. For $\mu = \frac{1}{2}$ and $|v| > \frac{1}{2}$, we apply Theorem B to the particular solution $y(x) = x^{1/2} J_{v}(x)$ of equation

(15). It is clear from (14) that $pr$ is increasing on $(0, \infty)$ for such pairs $\mu, v$. Now let $j_{v,k}$ denote the $k$th positive zero of $J_{v}(x)$. In (7), we may take $x_{0} = j_{v,k}$, for any $k$. And we may also take $x_{0} = 0$, since $J_{v}(x) = O(x^{1/2})$ as $x \to 0+$. Since $J_{v}(x) > 0$ for $0 < x < j_{v,1}$ if $v > -1$, it follows from (8) that

$$\int_{0}^{\xi} x^{1/2} J_{v}(x) \, dx > 0, \quad \xi > 0, \, v > \frac{1}{2}.$$

Together with (5), this shows that $J_{v}(x)s_{1/2,v}(x) - s_{1/2,v}(x)J_{v}(x) > 0$, for the right-hand side of (5) vanishes at $x = 0$ if $\beta = 0$ and $\mu + v + 1 > 0$, by (3). In particular,

$$s_{1/2,v}(j_{v,k})J_{v}(j_{v,k}) < 0, \quad k = 1, 2, \ldots, \, v > \frac{1}{2}.$$

But $\text{sgn} \, J_{v}(j_{v,k}) = (-1)^{k}$ for $v \geq 0$; thus $s_{1/2,v}(x)$ must have an odd number of changes of sign between consecutive positive zeros of $J_{v}(x)$, if $v > \frac{1}{2}$. Because of (4), this is also true if $v < -\frac{1}{2}$, and the proof is complete.

Next, we turn to the case $\mu \geq \frac{1}{2}$, and prove
THEOREM 3. If \( \mu \geq \frac{1}{2} \) and \(|\nu| \leq \mu\), then \( s_{\mu,\nu}(x) > 0 \) for all \( x > 0 \), except if \( \mu = |\nu| = \frac{1}{2} \), when \( s_{\mu,\nu}(x) \geq 0 \).

Proof. For \( \mu = |\nu| = \frac{1}{2} \) we have, from (6),

(17) \[ s_{1/2,1/2}(x) = s_{1/2,-1/2}(x) = x^{-1/2}(1 - \cos x). \]

For \( \mu = \frac{1}{2} \) and \( |\nu| < \frac{1}{2} \) we shall use Theorem B and (5). From (3), and familiar facts about the behavior of \( J_\nu(x) \) and \( Y_\nu(x) \) as \( x \to 0^+ \), it is easily seen that we may take 0 as lower limit of integration in (5), and that the right-hand side of (5) vanishes at \( x = 0 \), if \( \mu > |\nu| - 1. \) Hence, for \( \mu > |\nu| - 1 \) and \( \xi > 0 \), we have

\[ \int_0^\xi x^\nu C_\nu(x) \, dx = \xi [C_\nu(\xi) s_{\mu,\nu}(\xi) - s_{\mu,\nu}(\xi) C_\nu'(\xi)]. \]

The particular choice \( \xi = c_{\nu,k} \), the \( k \)th positive zero of \( C_\nu(x) \), yields

(18) \[ \int_0^{c_{\nu,k}} x^\nu C_\nu(x) \, dx = -c_{\nu,k} s_{\mu,\nu}(c_{\nu,k}) C_\nu'(c_{\nu,k}). \]

Now \( y(x) = x^\nu C_\nu(x) \) is a solution of (13). Since \( \lim_{x \to 0^+} x^\nu C_\nu(x) = 0 \) if \( \mu > |\nu| \), (10) is satisfied by \( x_0 = 0 \). Moreover, it is clear from (14) that \( pr \) is strictly decreasing on \((0, \infty)\), if \( \mu = \frac{1}{2} \) and \( |\nu| < \frac{1}{2} \). Thus, the hypotheses of Theorem B are satisfied by \( y(x) = x^{1/2} C_\nu(x) \) on each of the intervals \((0, c_{\nu,2})\) and \((c_{\nu,k} c_{\nu,k+2}), k \geq 1, \) if \( |\nu| < \frac{1}{2} \). It follows that

(19) \[ \left\{ \int_0^{c_{\nu,k}} x^{1/2} C_\nu(x) \, dx \right\} \left\{ \int_0^{c_{\nu,k+1}} x^{1/2} C_\nu(x) \, dx \right\} < 0, \quad k \geq 1, \ |\nu| < \frac{1}{2}. \]

Since \( C_\nu(c_{\nu,k}) C_\nu'(c_{\nu,k+1}) < 0 [9, \S 15.21] \), it follows from (18) and (19) that

(20) \[ s_{1/2,\nu}(c_{\nu,k}) > 0, \quad k \geq 1, \ |\nu| < \frac{1}{2}. \]

But \( x^{1/2} C_\nu(x) \) is an arbitrary nonnull solution of (15). Therefore, (20) implies that \( s_{1/2,\nu}(x) > 0 \) for all \( x > 0 \) if \( |\nu| < \frac{1}{2} \), since any \( x > 0 \) is a zero of some nonnull solution of (15). Together with (17), this proves Theorem 3 for \( \mu = \frac{1}{2} \). And if we use the integrals

\[ s_{\mu + \sigma,\nu + \sigma}(x) = 2x^\sigma \frac{\Gamma((\mu + \nu + 2\sigma + 1)/2)}{\Gamma(\sigma)\Gamma((\mu + \nu + 1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin \theta) \sin^{\nu+1} \theta \cos^{2\sigma-1} \theta \, d\theta, \]
\[ s_{\mu + \sigma,\nu - \sigma}(x) = 2x^\sigma \frac{\Gamma((\mu + \nu + 2\sigma + 1)/2)}{\Gamma(\sigma)\Gamma((\mu - \nu + 1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin \theta) \sin^{\nu-1} \theta \cos^{2\sigma-1} \theta \, d\theta, \]

both valid for \( \sigma > 0 \) and \( \mu + \nu > -1 [1, \S 3.20] \), the theorem's truth for \( \mu > \frac{1}{2} \) is an immediate consequence of its truth for \( \mu = \frac{1}{2} \).

An intermediate situation between those of the last two theorems is described by

THEOREM 4. Let \( \mu > \frac{1}{2} \). If \( \mu < |\nu| - 1 \), then \( s_{\mu,\nu}(x) \) has an odd number of changes of sign on \((0, \infty)\). If \( |\nu| - 1 < \mu < |\nu| \), then \( s_{\mu,\nu}(x) \) has an even number of changes of sign (perhaps none) on \((0, \infty)\).
Proof. From (3) we have $s_{\mu,\nu}(x) \sim x^{\mu+1}/((\mu + 1)^2 - \nu^2)$, as $x \to 0^+$. On the other hand, it follows from our discussion of (16) that when $\mu > \frac{1}{2}$, $s_{\mu,\nu}(x) \sim x^{\mu-1}$, as $x \to +\infty$. The conclusion is now obvious.

I have not been able to decide whether the even number alluded to in Theorem 4 is in fact always positive. However, I can show that if the point $(\mu, \nu)$, with $\mu > \frac{1}{2}$ and $|\nu| - 1 < \mu < |\nu|$, is close enough to the line $\mu = \frac{1}{2}$, or to one of the lines $\mu = |\nu| - 1$, then the corresponding $s_{\mu,\nu}(x)$ changes sign. In fact, by choosing the point $(\mu, \nu)$ close enough to the line $\mu = \frac{1}{2}$ in the region $\mu > \frac{1}{2}$, $|\nu| - 1 < \mu < |\nu|$, we can find Lommel functions with an arbitrarily large number of changes of sign.

This last statement can be verified as follows. We may assume $\nu > 0$. By applying Theorem 1 to (13), we see that if

$$ (1 - 2\mu)x^2 + 2\mu(\nu^2 - \mu^2) > 0 \quad \text{for} \quad 0 < x < \bar{x}, $$

and if $j$ is the largest zero of $J_\nu(x)$ in $(0, \bar{x}]$, then

$$ \int_0^x t^{\nu-1}J_\nu(t)\,dt > 0 \quad \text{for} \quad 0 < x \leq j, $$

since (10) is satisfied by $y(x) = x^\mu J_\nu(x)$ and $x_0 = 0$, if $\mu + \nu > 0$. For $\mu \geq \frac{1}{2}$, this implies that

$$ \int_0^x t^{\nu}J_\nu(t)\,dt > 0 \quad \text{for} \quad 0 < x \leq j, $$

by the second mean value theorem.

Thus, if $(0, \bar{x}]$ contains at least two positive zeros of $J_\nu(t)$, then (22) holds with $j = j_{\nu,2}$, and hence also with $j = j_{\nu,3}$. As before, if $\mu > \nu - 1$, this will imply that $s_{\mu,\nu}(x)$ changes sign on $(j_{\nu,1}, j_{\nu,3})$ and on $(j_{\nu,2}, j_{\nu,3})$. Now for $\mu > \frac{1}{2}$, (21) holds if and only if $\mu < |\nu|$ and $|\bar{x}| \leq (2\mu(\nu^2 - \mu^2)/(2\mu - 1))^{1/2}$. Therefore, $s_{\mu,\nu}(x)$ will change sign if $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$, and

$$ j_{\nu,2} \leq (2\mu(\nu^2 - \mu^2)/(2\mu - 1))^{1/2}. $$

Now if we choose some $\nu > \frac{1}{4}$, and keep it fixed, then the left-hand side of (23) is fixed. But the right-hand side tends to $+\infty$, as $\mu \to \frac{1}{2} + 0$. Hence for each $\nu$, $\frac{1}{2} < \nu < \frac{3}{2}$, we can find a $\mu$ ($\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$) such that $s_{\mu,\nu}(x)$ has at least two changes of sign. In order to produce a Lommel function with at least $2n$ changes of sign, it suffices to satisfy the inequality obtained by writing $j_{\nu,2n}$ for $j_{\nu,2}$ in (23).

We now turn our attention to points near the line $\mu = \nu - 1$, in the region $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$. Here, we use (16). We fix $\nu \geq \frac{3}{2}$, and choose $x^* > 0$ such that $J_\nu(x^*) < 0$. Then, we let $\mu \to (\nu - 1) + 0$. Now $S_{\mu,\nu}(x^*)$ tends to a limit, as $\mu \to \nu - 1$ [9, §10.73]. And

$$ \lim_{\mu \to \nu - 1} \left[ \cos \frac{(\mu - \nu)\pi}{2} Y_\nu(x^*) - \sin \frac{(\mu - \nu)\pi}{2} J_\nu(x^*) \right] = J_\nu(x^*). $$
Hence, for $x = x^*$, the right-hand side of (16) tends to $-\infty$ as $\mu \to (\nu - 1) + 0$, because of the factor $\Gamma((\mu - \nu + 1)/2)$. Therefore, $s_{\nu,v}(x^*) < 0$ for an appropriate choice of $\mu$.

References


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