TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

F-MINIMAL SETS

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Abstract. An F-minimal set is the simplest proximal extension of an equicontinuous minimal set. It has one interesting proximal cell and all the points in this proximal cell are uniformly asymptotic. The Sturmian minimal sets are the best known examples of F-minimal sets. Our analysis of them is in terms of their maximal equicontinuous factors. Algebraically speaking F-minimal sets are obtained by taking an invariant $*$-closed algebra of almost periodic functions and adjoining some suitable functions to it. Our point of view is to obtain these functions from the maximal equicontinuous factor. In §3 we consider a subclass of F-minimal sets which generalize the classical Sturmian minimal sets, and in §4 we examine the class of minimal sets obtained by taking the minimal right ideal of an F-minimal set and factoring by a closed invariant equivalence relation which is smaller than the proximal relation.

0. **Preliminaries.** Let (X, T) be a transformation group. We will denote the action of t, an element of T, on x, an element of X, by xt and we will denote the orbit of x by $O(x)$. We say T acts *freely* on X or (X, T) is *free* if for each x in X the map from T to X which sends t to xt is injective. We will always assume that our transformation groups have compact Hausdorff phase spaces.

Two points x and y of X are *proximal* if given an element α of the uniformity on X there exists $t \in T$ such that $(xt, yt) \in \alpha$. We will let P denote the pairs of proximal points in $X \times X$, and we will let $P[x] = \{y : (x, y) \in P\}$. We say that x is a distal point if $P[x] = \{x\}$.

A point x of X is *locally almost periodic* if given any neighborhood U of x there exists a neighborhood V of x and a left syndetic set A such that $VA \subset U$. If x is a locally almost periodic point, then every point in $\mathrm{Cl}\,(O(x))$ is locally almost periodic [8, 3.25]. We say a transformation group is locally almost periodic if every point is locally almost periodic. If (X, T) is locally almost periodic, then the proximal relation P is a closed invariant equivalence relation on X and $(X/P, T)$ is the equicontinuous structure transformation group of (X, T); i.e., the maximal equicontinuous factor of (X, T) [4, Theorem 3].

Let $\mathscr{B}(T)$ denote the bounded complex valued functions on T and let $\mathscr{U}(T)$ denote the bounded right uniformly continuous functions on T. Equip $\mathscr{B}(T)$ with the supremum norm. Note that $\mathscr{U}(T) \subset \mathscr{B}(T)$ and $\mathscr{U}(T_d) = \mathscr{B}(T)$ where T_d denotes

Presented to the Society, March 26, 1971; received by the editors November 25, 1970.

AMS 1970 *subject classifications*. Primary 54H20; Secondary 54H15.

Key words and phrases. Minimal set, proximal relation, locally almost periodic, almost automorphic, Sturmian minimal sets, enveloping semigroup.

(1) This research was supported by the National Science Foundation GP-12038.

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T with the discrete topology. Let $C(X)$ denote the continuous complex valued functions on X. For each $f \in C(X)$ and x in X we define $f_x \in \mathscr{B}(T)$ by $f_x(t) = f(xt)$. Suppose that (X, T_d) is a transformation group and $O(x')$ is dense in X, then the map $(x, t) \to xt$ is continuous on $X \times T$ if and only if $\{f_{x'} : f \in C(X)\} \subset \mathscr{U}(T)$.

A point x in X is an *almost automorphic point* if given any net $\{t_{\alpha'}\}$, $\alpha' \in \Lambda'$, there exists a subnet $\{t_\alpha\}$, $\alpha \in \Lambda$, such that $\lim_{\alpha \in \Lambda} xt_\alpha = z$ and $\lim_{\alpha \in \Lambda} zt_\alpha^{-1} = x$. We say a minimal set is almost automorphic if it contains at least one almost automorphic point. It is known that a minimal transformation group is almost automorphic if and only if it is locally almost periodic and has distal points.

Similarly a complex valued function f on T is *almost automorphic* if given any net $\{t_{\alpha'}\}$, $\alpha' \in \Lambda'$, there exists a subnet $\{t_\alpha\}$, $\alpha \in \Lambda$, and $g \in \mathscr{B}(T)$ such that $\lim_{\alpha \in \Lambda} f(t_\alpha t) = g(t)$ and $\lim_{\alpha \in \Lambda} g(t_\alpha^{-1} t) = f(t)$ [13]. It is easy to see that x is an almost automorphic point if and only if f_x is an almost automorphic function for each f in $C(X)$. The above definition makes sense for any range space not just the complex numbers. When the range space is C^α where α is some cardinal number then f is almost automorphic if and only if each coordinate function is almost automorphic. (Use [13, Lemma 1.3.1].)

1. **Constructing F-minimal sets.** We begin with the abstract definition of an F-minimal set, and proceed to show how they can be constructed from a certain type of function on an equicontinuous flow.

DEFINITION. A minimal transformation group (X, T) on a compact Hausdorff space is *F-minimal* if it satisfies the following conditions:

(a) It is locally almost periodic.

(b) There exist distal points.

(c) There exists x_0 in X such that $F = P[x_0] \neq \{x_0\}$ and such that $P[x] \neq \{x\}$ implies $O(x) \cap P[x_0] \neq \varnothing$.

(d) Given an index α of the uniformity on X, there exists a compact subset K of T such that $(xt, yt) \in \alpha$ when $x, y \in P[x_0]$ and $t \notin K$.

(e) If $t \neq e$ (e = identity of T), then $(x, xt) \notin P$ for all $x \in X$.

If (X, T) is an F-minimal set, then the proximal relation P is a closed invariant equivalence relation and $(X/P, T)$ is the equicontinuous structure transformation group of (X, T). We will denote $(X/P, T)$ by (Y, T) and say that (X, T) is an F-minimal set over (Y, T). Also D will denote the distal points of (X, T) and p will denote the canonical homomorphism of (X, T) onto (Y, T).

REMARKS. (1) An F-minimal set is almost automorphic.

(2) An F-minimal set is uniquely ergodic when T is the integers.

(3) If (X, T) is an F-minimal set, then the action of T on both X and Y is free.

(4) If (X, T) is an F-minimal set, then p maps D homeomorphically onto $p(D) = Y \backslash O(p(x_0))$.

Proof. (1) Conditions (a) and (b) are equivalent to almost automorphy.

(2) An argument used by Veech works here [14, p. 8].

(3) This follows from condition (e).

(4) If C is a closed subset of X then $p(C)$ is closed in Y and $p(C \cap D) = p(C) \cap p(D)$.

The Sturmian minimal sets [9] and the minimal set of Jones [8, 14.16 to 14.24] are F-minimal sets. A discrete substitution minimal set is an F-minimal set if the cardinality of J_1 is one [2]. Floyd's example of a nonhomogeneous minimal set [5] is not an F-minimal set only because it fails to satisfy condition (d) [7, p. 712]. Condition (d) is essential in our analysis of F-minimal sets.

THEOREM 1.1. *Let T be a topological group which can be covered by a countable family of compact subsets, and let (X, T) be an F-minimal set over (Y, T). Then X is a metric space if and only if both F and Y are metric spaces.*

Proof. Suppose that Y and F are metric spaces. It suffices to find a countable family of continuous real valued functions on X which separate points. Since Y is metric, we can find a countable family of continuous real valued functions which separate points of Y. Composing them with p we get a countable family of continuous real valued functions on X which separate points that are not proximal. Since the map $t \to yt$ is injective for a fixed y, we see that every compact subset C of T is metrizable and $F \times C$ is metrizable. The closed subset FC of X is homeomorphic to $F \times C$. By extending continuous real valued functions on FC to X and by covering T with a countable family of compact subsets we can obtain a countable family of continuous real valued functions on X which separate proximal points. Therefore, X is metrizable.

Suppose X is a metric space. Then clearly F is also a metric space. The space Y is metrizable because it is the quotient space of a separable metric space by upper semicontinuous decomposition with compact members [11, p. 149].

Standing hypothesis. The topological group T is locally compact but not compact.

Let (Y, T) be a free equicontinuous minimal transformation group on a compact Hausdorff space Y. Let F be a closed subset of the cube $Q_\alpha = [0, 1]^\alpha$ where α is some cardinal number. Thus F can be any compact Hausdorff space. We will always assume that F contains more than one point. Let y_0 be a point in Y and let U_0 be a compact neighborhood of e in T. Suppose that we are given a continuous function f mapping $Y \backslash y_0 U_0$ into Q_α satisfying the following conditions:

(α) The oscillation of f at y_0 is F, i.e.,

$$\omega(f, y_0) = \bigcap_{U \in \mathcal{N}} \mathrm{Cl}\,(f(U \backslash y_0 U_0)) = F$$

where $\mathcal{N}$ is the neighborhood filter at y_0.

(β) For each $y \notin \boldsymbol{O}(y_0)$ the function $f_y(t) = f(yt)$ is right uniformly continuous on T.

(γ) Let t be in U_0 and let $\{y_\alpha\}$ and $\{y_\beta\}$ be nets in $Y \backslash \boldsymbol{O}(y_0)$ converging to $y_0 t$. If

$$\lim_\alpha f(y_\alpha t^{-1}) = \lim_\beta f(y_\beta t^{-1}),$$

then

$$\lim_{\alpha} f(y_\alpha) = \lim_{\beta} f(y_\beta).$$

We will call a function satisfying these conditions an *F-almost automorphic function.*

LEMMA 1.2. *Let f be an F-almost automorphic function on $Y\backslash y_0U_0$. Then $z\in\omega(f, y_0)$ if and only if there exists a net $\{y_n\}\subset Y\backslash \boldsymbol{O}(y_0)$ such that y_n converges to y_0 and $f(y_n)$ converges to z.*

The proof is left to the reader.

Form the transformation group $(\beta(Y\backslash \boldsymbol{O}(y_0)), T_d)$ where $\beta(\cdot)$ denotes the Stone-Čech compactification. Let $\tilde{p}$ denote the extension of the inclusion map of $Y\backslash \boldsymbol{O}(y_0)$ into Y to $\beta(Y\backslash \boldsymbol{O}(y_0))$. It is a homomorphism of $(\beta(Y\backslash \boldsymbol{O}(y_0)), T_d)$ onto (Y, T_d).

LEMMA 1.3. (1) *The transformation group $(\beta(Y\backslash \boldsymbol{O}(y_0)), T_d)$ is locally almost periodic and minimal.*

(2) *Two points x and y in $\beta(Y\backslash \boldsymbol{O}(y_0))$ are proximal if and only if $\tilde{p}(x)=\tilde{p}(y)$.*

(3) *A point x in $\beta(Y\backslash \boldsymbol{O}(y_0))$ is distal if and only if $\tilde{p}(x)\notin \boldsymbol{O}(y_0)$.*

Proof. (1) Let h denote the homeomorphism which imbeds $Y\backslash \boldsymbol{O}(y_0)$ into $\beta(Y\backslash \boldsymbol{O}(y_0))$, and let U be a neighborhood of $h(y)$. There exists a neighborhood V of y in $Y\backslash \boldsymbol{O}(y_0)$ and a left syndetic set A of T_d such that $VA\subset h^{-1}(U)$ because (Y, T_d) is an almost periodic transformation group. Thus $h(V)A\subset U$ and $\mathrm{Cl}\,(h(V)A)\subset \mathrm{Cl}\,(U)$. Since $\mathrm{Cl}\,(h(V))A\subset \mathrm{Cl}\,(h(V)A)$ and $\beta(Y\backslash \boldsymbol{O}(y_0))$ is regular it suffices to show that $h(y)$ is in the interior of $\mathrm{Cl}\,(h(V))$. There exists an open set W of $\beta(Y\backslash \boldsymbol{O}(y_0))$ such that $h(V)=h(Y\backslash \boldsymbol{O}(y_0))\cap W$. Certainly $h(y)\in W$ and

$$W\subset \mathrm{Cl}\,(h(V)).$$

The minimality is clear.

(2) If x and y are proximal, then $\tilde{p}(x)=\tilde{p}(y)$ because (Y, T_d) is equicontinuous. Suppose $\tilde{p}(x)=\tilde{p}(y)$. Observe that if $y\notin \boldsymbol{O}(y_0)$, then $\tilde{p}^{-1}(y)=h(y)$. Let $\{t_n\}$ be a net such that $\tilde{p}(x)t_n$ converges to $y\notin \boldsymbol{O}(y_0)$ and both xt_n and yt_n converge. Let x' and y' be their limits. Then $\tilde{p}(x')=\tilde{p}(y')=y\notin \boldsymbol{O}(y_0)$. Hence $x'=y'$ and x and y are proximal.

(3) From (2) we see that if $\tilde{p}(x)\notin \boldsymbol{O}(y_0)$, then x is distal. Suppose that x is distal and $\tilde{p}(x)\in \boldsymbol{O}(y_0)$. We can assume that $\tilde{p}(x)=y_0$. By (2) we have $\tilde{p}^{-1}(y_0)=x$. Because $\omega(f, y_0)=F\neq$ point $\tilde{p}^{-1}(y_0)\neq x$ which is a contradiction.

Let $\tilde{f}$ be the extension of f to $\beta(Y\backslash \boldsymbol{O}(y_0))$. Using $\tilde{f}$ we define a relation R_f on $\beta(Y\backslash \boldsymbol{O}(y_0))$ by $(x, y)\in R_f$ if either $x=y$ or, for some t, $\tilde{p}(xt)=y_0=\tilde{p}(yt)$ and $\tilde{f}(xt)=\tilde{f}(yt)$. Note that if such a t exists it is unique because $yt=y$ implies $t=e$ for y in Y.

LEMMA 1.4. *The relation R_f is a closed invariant equivalence relation.*

Proof. It is easy to show that it is an invariant equivalence relation and we leave this to the reader. Suppose that $\{(x_n, y_n)\}$, $n\in\Lambda$, is a net in R_f converging to (x, y).

If we can show that $(x, y) \in R_f$, then it follows that R_f is closed. We may as well assume that $x \neq y$. It is clear that, for some t, $\tilde{p}(xt) = y_0 = \tilde{p}(yt)$. Because $(x_n, y_n) \in R_f$ and $x \neq y$ we can assume $w_n = x_n t_n$ and $z_n = y_n t_n$ where $\tilde{p}(w_n) = y_0 = \tilde{p}(z_n)$ and $\tilde{f}(w_n) = \tilde{f}(z_n)$. When t_n has a subnet converging to τ, it follows that $\tau = t$, $\tilde{f}(xt) = \tilde{f}(yt)$, and $(x, y) \in R_f$. Suppose that $\{t_n\}$ does not have a convergent subnet. Then there exists $N \in \Lambda$ such that $n > N$ implies $t_n \notin tU_0^{-1}$ which in turn implies that $\tilde{f}(x_n t) = \tilde{f}(w_n t_n^{-1} t) = \tilde{f}(z_n t_n^{-1} t) = \tilde{f}(y_n t)$ for $n > N$. Since $\{(x_n t, y_n t)\}_{n \in \Lambda}$ converges to (xt, yt), $\tilde{f}(xt) = \tilde{f}(yt)$ and $(x, y) \in R_f$.

Let $(X_f, T_d) = (\beta(Y \backslash \boldsymbol{O}(y_0))/R_f, T_d)$ and let π denote the canonical projection. It is locally almost periodic. There exists a homomorphism p from (X_f, T_d) to (Y, T_d) such that $\tilde{p} = p \circ \pi$ and $p(x) = p(y)$ if and only if x and y are proximal. The distal points are given by $p^{-1}(Y \backslash \boldsymbol{O}(y_0))$. From condition (γ) we obtain a map $\hat{f}$ from X_f to Q_α such that $\tilde{f} = \hat{f} \circ \pi$ and $\hat{f}(x) = f \circ p(x)$ when $p(x) \notin y_0 U_0$. Note that $\hat{f}$ is injective on $p^{-1}(y_0)$ and $\hat{f}$ is constant on $p^{-1}(y)$ if $y \notin y_0 U_0$. Let x_0 be some point in $p^{-1}(y_0)$.

THEOREM 1.5. *The transformation group (X_f, T) is an F-minimal set over (Y, T).*

Proof. We must show that $(x, t) \to xt$ is continuous on $X_f \times T$, that $p^{-1}(y_0)$ is homeomorphic to F, and that condition (d) of the definition holds. From the remarks in the preceding paragraph it is clear that the rest of the conditions are satisfied.

Let x be a distal point in X_f and let $\mathscr{A} = \{g \in C(X_f) : g_x \in \mathscr{U}(T)\}$. Then $\mathscr{A}$ is a $*$-closed subalgebra of $C(X_f)$ containing the constants. If we can show that $\mathscr{A}$ separates points, then $\mathscr{A} = C(X_f)$ and $(x, t) \to xt$ is continuous on $X_f \times T$. Let y and z be distinct points in X_f. We consider two cases. First suppose $p(y) \neq p(z)$. There exists $h \in C(Y)$ such that $h(p(y)) \neq h(p(z))$ and $h_{p(x)}$ is right uniformly continuous on T. Let $g = h \circ p$ and note that $g_x = h_{p(x)}$. Next suppose that $p(y) = p(z)$ or equivalently that y and z are proximal. For some τ, $\hat{f}(y\tau) \neq \hat{f}(z\tau)$. Let $f_i, \hat{f}_i$ denote the coordinate functions of f and $\hat{f}$ respectively. Since $(f_i)_{p(x)} = (\hat{f}_i)_x$, $\hat{f}_i$ and its translates are in $\mathscr{A}$. We know that, for a suitable i and τ, $\hat{f}_i(y\tau) \neq \hat{f}_i(z\tau)$. In this case we let $g(x) = \hat{f}_i(x\tau)$.

To show that $p^{-1}(y_0)$ is homeomorphic to F it suffices to show that $\tilde{f}[\tilde{p}^{-1}(y_0)] = F$. Let x be in $\tilde{p}^{-1}(y_0)$. There exists a net x_n converging to x such that $p(x_n) \notin y_0 U_0$. Observe that $p(x_n)$ converges to y_0. Because $\tilde{f}(x_n) = f \circ \tilde{p}(x_n)$ it follows that $\tilde{f}(x) = \lim f \circ p(x_n) \in \omega(f, y) = F$. Conversely if $z \in F$ there exists a net $\{x_n\}$ in $Y \backslash \boldsymbol{O}(y_0)$ such that x_n converges to y_0 and $f(x_n)$ converges to z. By picking a convergent subnet of $\{\tilde{p}^{-1}(x_n)\}$ we obtain a point x in $p^{-1}(y_0)$ such that $\tilde{f}(x) = z$.

The compact subsets of T form a directed set with $K_1 \leqq K_2$ if $K_1 \subset K_2$. Let α be an index of X_f and suppose that given any compact set K of T there exists $x, y \in P[x_0]$ and $t \notin K$ such that $(xt, yt) \notin \alpha$. Thus there exist nets $\{x_n\}$, $\{y_n\}$ in $P[x_0]$ and $\{t_n\}$ in T with $n \in \Lambda$ such that $x_n t_n \to x'$, $y_n t_n \to y'$, $x' \neq y'$, and given K there exists $N \in \Lambda$ such that $n \geqq N$ implies $t_n \notin K$. Clearly $(x', y') \in P$ and, for some τ, $x'\tau, y'\tau \in P[x_0]$.

Hence $\hat{f}(x'\tau)\neq\hat{f}(y'\tau)$. There exists t_n such that $\hat{f}(xt_n\tau)\neq\hat{f}(yt_n\tau)$ and $t_n\notin U_0\tau^{-1}$ which is impossible. Therefore, condition (d) holds, and the proof is finished.

THEOREM 1.6. *Let (X, T) be an F-minimal set over (Y, T). Then for a suitable $y_0\in Y$ and any compact neighborhood U_0 of e there exists an F-almost automorphic function f such that (X, T) is isomorphic to (X_f, T). Moreover, the imbedding of F in Q_α can be arbitrarily chosen.*

Proof. Let U_0 be a compact neighborhood of e in T and let V be an open neighborhood of e contained in U_0. Define an equivalence relation on X by $x\sim y$ if $x=y$ or if $(x, y)\in P$ and $x, y\notin P[x_0]V$. Using condition (d) of the definition it can be shown that this is a closed equivalence relation. Let $X'=X/\sim$, let $\phi\colon X\to X'$, $\theta\colon X'\to Y$, and $p\colon X\to Y$ be the canonical maps such that $p=\theta\circ\phi$, and let $y_0=p(x_0)$. It is easy to see that ϕ maps $P[x_0]$ homeomorphically onto $\theta^{-1}(y_0)$ and θ maps $\theta^{-1}(Y\backslash y_0U_0)$ homeomorphically onto $Y\backslash y_0U_0$. Let F be imbedded in Q_α. There exists a homeomorphism f' from $\theta^{-1}(y_0)$ onto F. Because Q_α is an absolute retract f' extends to X'. Let $f=f'\circ\theta^{-1}$ on $Y\backslash y_0U_0$. Using Lemma 1.2 we obtain $\omega(f, y_0)=F$. Since $f_{p(x)}=(f'\circ\phi)_x$ when $p(x)\notin \boldsymbol{O}(y_0)$, f_y is right uniformly continuous for $y\notin \boldsymbol{O}(y_0)$. An easy calculation shows that condition (γ) also holds. We will complete the proof by showing that (X, T) is isomorphic to (X_f, T).

As before form $(\beta(Y\backslash \boldsymbol{O}(y_0)), T_d)$ and observe that there exists a unique homomorphism π of it onto (X, T_d) such that $p\circ\pi=p_f\circ\pi_f$. Let $R=\{(x, y) : \pi(x)=\pi(y)\}$. It suffices to show that $R=R_f$. Note that $R\subset P$ and $\tilde{f}=f'\circ\phi\circ\pi$. Because $f'\circ\phi$ a homeomorphism on $P[x_0]$ we see that $\pi(x)=\pi(y)$ if and only if $f'\circ\phi\circ\pi(x)=f'\circ\phi\circ\pi(y)$ where $\tilde{p}(x)=y_0=\tilde{p}(y)$. Thus when $\tilde{p}(x)=y_0=\tilde{p}(y)$, $(x, y)\in R$ if and only if $\tilde{f}(x)=\tilde{f}(y)$. It follows that $R=R_f$.

COROLLARY 1.7. *Let (X, T) be an F-minimal set over (Y, T). If F is an absolute retract {absolute neighborhood retract}, then there exists an F-almost automorphic function on $Y\backslash y_0U_0$ such that $f(Y\backslash y_0U_0)\subset F$ $\{f(U\backslash y_0U_0)\subset F$ for some $U\in\mathcal{N}\}$ and (X, T) is isomorphic to (X_f, T).*

Let F be fixed not just as a compact Hausdorff space but as a closed subset of some Q_α. Let $\mathcal{F}(Y, T, y_0, F)$ denote the set of F-almost automorphic functions f mapping $Y\backslash y_0U_0$ into Q_α with $\omega(f, y_0)=F$ for some compact neighborhood U_0 of e. Here as before (Y, T) is minimal, free, and equicontinuous. We know that any F-minimal set over (Y, T) can be represented in the form (X_f, T) for some f in $\mathcal{F}(Y, T, y_0, F)$ and $y_0\in Y$.

Let $f, g\in\mathcal{F}=\mathcal{F}(Y, T, y_0, F)$. We say f is equivalent to g, written $f\sim g$, if there exists a homeomorphism h of F onto F satisfying the following condition: If $\{y_n\}$ is a net in $Y\backslash \boldsymbol{O}(y_0)$, $\{y_n\}$ converges to y_0, and $\{f(y_n)\}$ converges to z, then $\{g(y_n)\}$ converges to $h(z)$. This defines an equivalence relation on $\mathcal{F}$.

THEOREM 1.8. *Let f and g be in $\mathcal{F}(Y, T, y_0, F)$. Then the following are equivalent:*

(1) *There exists an isomorphism ϕ of (X_f, T) onto (X_g, T) such that $p_f=p_g\circ\phi$.*

(2) *The functions f and g are equivalent.*

(3) *The relations R_f and R_g in $\beta(Y\backslash \boldsymbol{O}(y_0))$ are equal.*

Proof. First we assume (1) and obtain (2) by presenting a suitable h. Let $h=[\hat{g}|p_g^{-1}(y_0)]\circ\phi\circ[\hat{f}|p_f^{-1}(y_0)]^{-1}$. Next we show that (2) implies (3). Let $(x,y)\in R_f$ with $x=y$. By the definition of R_f we have $\tilde{p}(xt)=y_0=\tilde{p}(yt)$ and $\tilde{f}(xt)=\tilde{f}(yt)=z$. We can find nets $\{x_n\}$ and $\{y_n\}$ converging to xt and yt respectively such that $\tilde{p}(x_n), \tilde{p}(y_n)\notin \boldsymbol{O}(y_0)$. Then $\tilde{f}(x_n)\to z$ and $\tilde{f}(y_n)\to z$. Since f is equivalent to g, $\{\tilde{g}(x_n)\}$ converges to both $\tilde{g}(xt)$ and $h(z)$, and $\{\tilde{g}(y_n)\}$ converges to both $\tilde{g}(yt)$ and $h(z)$. Therefore, $(x,y)\in R_g$ and $R_f\subset R_g$. By symmetry $R_g\subset R_f$ and $R_f=R_g$.

Finally (3) implies (1) is trivial.

2. **Constructing F-almost automorphic functions.** In this section we present some methods for constructing F-almost automorphic functions. Our methods are not very general, but they clearly demonstrate that for a fairly broad class of equicontinuous minimal sets there are many F-minimal sets over them.

As usual (Y,T) will be a free equicontinuous minimal set with Y compact Hausdorff. We say (Y,T) has a local cross section at y_0 if there exists a closed set $S\subset Y$ and a compact neighborhood V of e in T such that $y_0\in S$, the map $(y,t)\to yt$ is injective on $S\times V$, and $y_0\in \operatorname{Int}(SV)$.

THEOREM 2.1. *Let S be a local cross section of (Y,T) at y_0 and let $g\colon S\backslash\{y_0\}\to Q_\alpha$ such that $\omega(g,y_0)=F$. Then $\mathscr{F}(Y,T,y_0,F)\neq\varnothing$.*

Proof. It suffices to find an extension f of g to $Y\backslash y_0V$ so that f_y is in $\mathscr{U}(T)$ when $y\notin \boldsymbol{O}(y_0)$ and condition (γ) holds. Thus we need only consider the coordinate functions of g, i.e., we can assume $g\colon S\backslash y_0\to[0,1]$.

First let $f'\colon SV\to[0,1]$ by $f'(yv)=g(y)$ where $y\in S$ and $v\in V$. Choose subsets $S_1\subset\bar{S}_1\subset S_2\subset\bar{S}_2\subset S$ and $V_1\subset\bar{V}_1\subset V_2\subset\bar{V}_2\subset V$ such that $y_0\in S_1$, $e\in V_1$, and S_iV_i is open in Y, $i=1,2$ [8, 1.18]. It follows that $S_1V_1\subset\bar{S}_1\bar{V}_1\subset S_2V_2\subset\bar{S}_2\bar{V}_2\subset SV$. Let $h\colon Y\to[0,1]$ such that $h(y)=1$ for $y\in\bar{S}_1\bar{V}_1$ and $h(y)=0$ for $y\notin S_2V_2$. Finally define f by

$$\begin{aligned} f(y) &= f'(y)h(y) && \text{if } y\in SV\backslash y_0V,\\ &= 0 && \text{if } y\notin S_2V_2\cup y_0V. \end{aligned}$$

Next we show that f_y is right uniformly continuous on T for $y\notin \boldsymbol{O}(y_0)$. If ys and yt are not in $S_2V\cup y_0V$, then $|f(ys)-f(yt)|=0$. There exists a symmetric neighborhood U of e such that when ys is in S_2V_2 and $ts^{-1}\in U$ we have yt is in SV, $|f'(ys)-f'(yt)|<\varepsilon/2$, and $|h(ys)-h(yt)|<\varepsilon/2$. We can find such a U because (Y,T) is equicontinuous, f' is uniformly continuous on SV, h_y is right uniformly continuous on T, and we can assume that $\bar{S}_2\bar{V}_2\subset\operatorname{Int}(SV)$. It follows that for $ts^{-1}\in U$ we have $|f(ys)-f(yt)|<\varepsilon$.

Finally condition (γ) follows from the observation that $f(y_\alpha)$ converges to $zh(y_0t)$ if y_α converges to y_0t, $t\in V$, and $f(y_\alpha t^{-1})$ converges to z.

THEOREM 2.2. *If Y is the Cantor discontinuum, if T is a discrete group, and if F is a compact metric space, then $\mathscr{F}(Y, T, y_0, F) \neq \varnothing$.*

Proof. Think of Y in the usual way in $[0, 1]$ with $y_0 = 0$. Let $\{(a_i, b_i)\}$ be a sequence of complementary intervals of Y such that $b_{i+1} < a_i$ and $a_i \to 0$. Since we can map $[b_{i+1}, a_i]$ onto F, we are done.

THEOREM 2.3. *If Y is a topological n-manifold, if T is a discrete group, and if F is a compact metric space which is connected and locally connected, then*

$$\mathscr{F}(Y, T, y_0, F) \neq \varnothing.$$

Proof. Here we can apply Theorem 2.1 with S homeomorphic to

$$\{x \in \boldsymbol{R}^n : \|x\| \leqq 1\}$$

and $V = \{e\}$. By the Hahn-Mazurkiewicz Theorem [15, Theorem 2.5, p. 76] we can find a continuous function g_0 from $(0, 1]$ into F such that $g_0([1/n, 1/(n-1)]) = F$ for all $n \geqq 2$. Let $g(x) = g_0(\|x\|)$.

We will say the minimal set (Y, T) is multiplicative if we can assume that Y is a topological group and there exists a continuous homomorphism ϕ of T into Y such that $\phi(T)$ is dense in Y and yt is the group product $y\phi(t)$. It follows that multiplicative minimal sets are equicontinuous. They are free if and only if ϕ is injective. When T is abelian all equicontinuous minimal sets (Y, T) are multiplicative [8, 4.48]. Moreover, when (Y, T) is multiplicative the point y_0 can always be taken as the identity of the group Y.

THEOREM 2.4. *If (Y, T) is multiplicative with Y a finite dimensional metric space, if T is discrete, and if F is a compact metric space which is connected and locally connected, then $\mathscr{F}(Y, T, y_0, F) \neq \varnothing$.*

Proof. In view of the preceding theorem we may as well assume that Y is not a manifold. Again we use Theorem 2.1 with S a closed neighborhood of y_0 and $V = \{e\}$. We can assume that S is homeomorphic to $\{v \in \boldsymbol{R}^n : \|v\| \leqq 1\} \times C$ where C is the Cantor discontinuum [12, Theorem 69, p. 335]. Since Theorem 2.2 covers the case when $n = 0$, we will assume that $n > 0$. We can assume that y_0 is $(0, 0)$. Let ϕ be the usual Cantor function on C and let g be defined as in the previous proof. Finally let

$$\begin{aligned} f(v, x) &= g(v/\phi(x)) \quad \text{if } \|v\| \leqq \phi(x), \\ &= g(v/\|v\|) \quad \text{if } \|v\| \geqq \phi(x), \end{aligned}$$

for $(v, x) \neq (0, 0)$.

THEOREM 2.5. *If (Y, T) is multiplicative with Y a Lie group, if T is a noncompact Lie group, and if F is a compact Hausdorff space which is second countable, connected, and locally connected, then $\mathscr{F}(Y, T, y_0, F) \neq \varnothing$.*

Proof. It follows from the theory of Lie groups that we can take

$$S = \{v \in \boldsymbol{R}^n : \|v\| \leqq 1\}$$

where $n = \dim Y - \dim T$. Now apply Theorem 2.1 to the function constructed in the proof of Theorem 2.3.

It appears that in general it is difficult to obtain information about the equivalence classes of $\mathscr{F}(Y, T, y_0, F)$. In the next theorem in this section we consider a situation in which we can at least determine the number of these equivalence classes.

THEOREM 2.6. *Let Y be the Cantor discontinuum, let T be a discrete group, and let F be a finite set with the discrete topology, then there is a continuum of F-minimal sets over (Y, T) with base point y_0.*

Proof. We can assume $F = \{1/n, \ldots, 1\} \subset [0, 1] = Q_1$. Since $Y \backslash \{y_0\}$ is separable, it is clear that there are at most c of them (c = cardinality of $\boldsymbol{R}$). Each one can be obtained from an F-almost automorphic function f such that $f(U \backslash y_0) = F$ for some neighborhood U of y_0. Suppose two such functions in $\mathscr{F}(Y, T, y_0, F)$ are equivalent. Then there exists a permutation σ such that $y_n \to y_0$ and $f(y_n) \to i/n$ implies $g(y_n) \to \sigma(i/n)$. Suppose that in every neighborhood U of y_0 there exists y_n such that $\sigma(f(y_n)) \neq g(y_n)$. It follows that there exists a sequence $y_n \to y_0$ such that $f(y_n) = i/n$ and $g(y_n) \neq \sigma(i/n)$ which contradicts the equivalence of f and g. Thus there exists a neighborhood U of y_0 such that $\sigma(f(y)) = g(y)$ for all y in $U \backslash \{y_0\}$. We will use this to exhibit a continuum of F-almost automorphic functions which are not equivalent.

Let U_n be a sequence of open closed neighborhoods of y_0 such that $U_{n+1} \backslash U_n \neq \varnothing$ and $\bigcap_{n=1}^{\infty} U_n = y_0$. Each sequence in n-symbols in which each symbol occurs infinitely often gives rise to an F-almost automorphic function by the formula

$$f(x) = a_n/n \quad \text{if } x \in U_n \backslash U_{n-1}.$$

It is easy to see that there are c sequences like this. Moreover, two such sequences give rise to equivalent functions if and only if there exists a permutation σ and $N > 0$ such that $n > N$ implies $\sigma(a_n) = b_n$. There are only a countable number of sequences equivalent to a given one in this sense, which proves the theorem.

THEOREM 2.7. *There exist nonhomogeneous discrete F-minimal sets over tori.*

Proof. Let $\mathscr{T}$ denote the n-dimensional torus. Since $\mathscr{T}$ is a monothetic group, we get an equicontinuous minimal transformation group $(\mathscr{T}, \boldsymbol{Z})$ where $(x, n) \to xg^n$ and g is a generator of $\mathscr{T}$. Letting y_0 be the identity of $\mathscr{T}$ and $F = Q_r$ where r is a positive integer greater than n, we obtain an F-minimal set $(X, \boldsymbol{Z})$ over $(\mathscr{T}, \boldsymbol{Z})$ from Theorem 2.3. Clearly the dimension of X at x_0 is at least r. On the other hand, if $y' \neq g^n$ for all n, we can find $\varepsilon_m \to 0$ and as $m \to \infty$ such that $\{y : d(y, y') = \varepsilon_m\} \cap \{g^n : n \in \boldsymbol{Z}\} = \varnothing$. It follows that the dimension of X at $p^{-1}(y')$ is at most n by considering $\{x : d(p(x), y') < \varepsilon_m\}$ where m runs through the positive integers [10, Corollary 2, p. 46].

3. **Sturmian minimal sets.** A Sturmian minimal set is an F-minimal set with $F = \{0, 1\}$. This definition includes the classical Sturmian minimal sets of Hedlund

[9]. We are interested in the following question: Given the equicontinuous minimal set (Y, T), are there any Sturmian minimal sets over it? Roughly speaking this is the same as trying to find U_0 and U such that y_0U_0 separates U, which suggests that the answer will depend upon the dimensions of T and Y.

THEOREM 3.1. *Let T be a separable Lie group, and let (Y, T) be a free multiplicative minimal set where Y is a finite dimensional compact metric space. Then there exists a Sturmian minimal set over (Y, T) if and only if one of the following conditions holds*:

(1) *Y is a Lie group and* $\dim(Y)=\dim(T)+1$,

(2) *Y is not a Lie group and* $\dim(Y)=\dim(T)$.

Moreover, when (1) *holds there is exactly one Sturmian minimal set over (Y, T) and when* (2) *holds there is a continuum of them.*

Proof. Suppose that there is a Sturmian minimal set (X, T) over (Y, T). Let ϕ denote the isomorphism of T into Y which defines the action of T on Y. We can assume that y_0 is the identity element of Y.

First suppose that Y is a Lie group. Let $\mathscr{Y}$ and $\mathscr{T}$ denote the Lie algebras of Y and T respectively and let $d\phi$ denote the differential of ϕ. Clearly $\dim T \leqq \dim Y$. If equality occurs, then $\phi(T)=Y$ which is a contradiction because ϕ is injective, Y is compact, and T is not compact. Thus $\dim T<\dim Y$. Let V_0 be a compact neighborhood of 0 in $\mathscr{T}$ such that

(a) $d\phi(V_0)=\{v : \|v\| \leqq r\} \cap d\phi(\mathscr{T})$,

(b) the exponential map is a homeomorphism in spheres containing V_0 and $d\phi(V_0)$.

Let $U_0=\exp V_0$ and let f be given by Corollary 1.7. Then for some δ, $0<\delta<r$, $f\circ\exp$ maps $\{v : \|v\|<\delta\}\backslash d\phi(V_0)$ onto $\{0, 1\}$. Thus $\dim(d\phi(\mathscr{T}))=\dim Y-1$. If g is another element of $\mathscr{F}(Y, T, y_0, F)$ such that $g(U\backslash y_0U_0)=F$, then we can find one δ as above for both of them. It follows that $f\sim g$ and there is at most one Sturmian minimal set over (Y, T). Conversely, if $\dim Y=\dim T+1$, there is clearly one Sturmian minimal set over (Y, T) by Theorem 2.1.

Now suppose that Y is not a Lie group. Because Y is finite dimensional we know its local structure. There exists a subset L homeomorphic with an open r-dimensional cube ($r=\dim Y$) and a 0-dimensional normal subgroup N such that

(a) every element of L commutes with every element of N;

(b) the set $V=LN$ is a neighborhood of the identity;

(c) every element $v\in V$ possesses a unique expression of the form $v=ln$, $l\in L$, and $n\in N$, and the elements l and n are continuous functions of v;

(d) Y/N is a Lie group [12, p. 335]. It follows from (c) that V is homeomorphic to $L\times N$. Since Y is metric N is the Cantor discontinuum. It is easy to see that $\sigma\circ\phi$ has a discrete kernel where σ is the canonical map of Y onto Y/N. Therefore, $\dim T=\dim Y/N=\dim Y$. For the converse first observe that $\dim T=\dim Y=\dim Y/N$ implies that the kernel of $\sigma\circ\phi$ is discrete. Next note that $\sigma\circ\phi$ is onto.

It follows that we can assume $y_0U_0=L$. Now we can apply Theorem 2.1 with $S=N$. To count them we can use Corollary 1.7 and the proof of Theorem 2.6.

The proof of the first half of the preceding theorem also yields the following result:

THEOREM 3.2. *Let T be a separable Lie group and let (Y, T) be a free multiplicative minimal set where Y is a compact manifold. If the cardinality of F is finite and greater than* 2, *then $\mathscr{F}(Y, T, y_0, F)=\varnothing$.*

4. **Weak F-minimal sets.** In this section we consider a class of minimal sets obtained by weakening condition (c) in the definition of F-minimal set. The weakened condition says that it is easy to find closed invariant equivalence relations in the proximal relation.

DEFINITION. A minimal transformation group (X, T) on a compact Hausdorff space is a weak F-minimal set if it satisfies the following conditions.

(a′) It is locally almost periodic.

(b′) For $x \in X$, $p[x]T \neq X$.

(c′) An invariant equivalence relation R in P is closed if, for all x in X, $P[x]\times P[x] \cap R$ is closed.

(d′) Given an index α of the uniformity on X and a point x_0 in X, there exists a compact subset K of T such that $(xt, yt) \in \alpha$ when $x, y \in P[x_0]$ and $t \notin K$.

(e′) If $t \neq e$, then $(x, xt) \notin P$ for all x in X.

We will characterize the weak F-minimal sets as those which have the minimal right ideal of some F-minimal set as a proximal extension. We begin by examining the minimal right ideals of F-minimal sets.

LEMMA 4.1. *Suppose the proximal relation of the minimal transformation group (X, T) is an equivalence relation, then the unique minimal right ideal I in the enveloping semigroup is given by*

$$I = \{\xi : (P[x])\xi \text{ is a point for all } x \in X\}$$

and the set of idempotents J in I is given by

$$J = \{\xi : (P[x])\xi \text{ is a point in } P[x] \text{ for all } x \in X\}.$$

Moreover, J is a closed subset of I if P is a closed subset of $X\times X$.

Proof. Let A denote the right side of the first equation. It is clear that A is a right ideal and hence $I\subset A$. Let $\xi \in A$ and let μ be an idempotent in I. Since $(P[x])\mu\subset P[x]$ for all x, we see that $\mu\xi=\xi$ and ξ is in I.

Let B denote the right side of the second equation. From the above it is clear that $B\subset I$ and $B\subset J$. Because $(x, x\mu) \in P$ for $\mu \in J$ we obtain $J\subset B$.

Let $\{\mu_n\}_{n\in\Lambda}$ be a net in J converging to η and let y be in $P[x]$. Then $\{y\mu_n\}$ converges to $y\eta$. Since $y\mu_n=x\mu_n \in P[x]$ for all n, $y\eta=x\eta$ and $\eta \in B=J$.

LEMMA 4.2. *Let (X, T) be an F-minimal set. Then $I=\mathscr{E}\backslash T$. If T is abelian, then J is homeomorphic to F.*

Proof. The first part follows from Lemma 4.1 and condition (d). Since T is abelian, $\mu t = t\mu$ for $\mu \in J$. It follows that $x_0\mu$ completely determines μ. Finally $\mu \to x_0\mu$ is a homeomorphism of J onto $P[x_0]$ because J is closed.

Henceforth we will assume that T is abelian. The assumption that T is locally compact and not compact is still in force. Let (X, T) be an F-minimal set over (Y, T). Then (Y, T) is multiplicative and we can assume that y_0 is the identity of Y. Let Γ be the structure group of (X, T) and let θ be the homomorphism of $\mathscr{E}$ onto Γ induced by p. Then $\theta(I)=\Gamma$ and, for ξ and η in I, $\theta(\xi)=\theta(\eta)$ if and only if $\xi=\eta\mu$ for some $\mu \in J$ [3, Theorem 3]. It follows that ξ and η in I are proximal if and only if $\xi=\eta\mu$ for some $\mu \in I$. The map $\xi \to y_0\xi$ is a topological isomorphism of Γ onto Y and thus $p(x_0\xi\eta)=p(x_0\xi)p(x_0\eta)$. Denote this map by ϕ. Note that $\phi(\xi)=y_0$ if and only if $\xi \in J$.

LEMMA 4.3. *Let $\{\xi_\alpha\}$, $\alpha \in \Lambda$, be a net in I, let ξ be an element of I, and let K be a compact neighborhood of e in T. Then ξ_α converges to ξ if and only if the following conditions are satisfied*:

(A) $\lim_{\alpha\in\Lambda} \phi(\xi_\alpha)=\phi(\xi)$.

(B) *If $\Lambda_1=\{\alpha \in \Lambda : \xi_\alpha \in \xi JK\}$ is cofinal, then $\lim_{\alpha\in\Lambda_1} x_0\nu_\alpha = x_0\nu$ where ν_α and ν are idempotents such that $\xi_\alpha\nu_\alpha=\xi_\alpha$ and $\xi\nu=\xi$.*

(C) *If $\Lambda_2=\{\alpha \in \Lambda : \xi_\alpha \notin \xi JK\}$ is cofinal, then $\lim_{\alpha\in\Lambda_2} x_0\xi^{-1}\xi_\alpha = x_0\nu$ where $\xi\xi^{-1}=\nu=\xi^{-1}\xi$.*

Proof. Assume $\xi_\alpha \to \xi$. Condition (A) is obvious. Suppose Λ_1 is cofinal. For $\alpha \in \Lambda_1$, $\xi_\alpha=\xi\mu_\alpha t_\alpha=\xi t_\alpha\mu_\alpha$ and hence $\mu_\alpha=\nu_\alpha$. Let μ be the limit of a convergent subnet of $\{\nu_\alpha\}$, $\alpha \in \Lambda_1$. By taking a further subnet we can assume $t_\alpha \to t$. It follows that $\xi=\xi\mu t$ and $\mu=\nu$. Therefore, $\lim_{\alpha\in\Lambda_1} \nu_\alpha=\nu$ and $\lim_{\alpha\in\Lambda_1} x_0\nu_\alpha=x_0\nu$. Next suppose that Λ_2 is cofinal. Since $\xi^{-1}\xi_\alpha \to \xi^{-1}\xi=\nu$, condition (C) holds.

To prove the converse it suffices to show that $\lim_{\alpha\in\Lambda_i} \xi_\alpha=\xi$ where Λ_i is cofinal. Suppose Λ_1 is cofinal. As before $\xi_\alpha=\xi\nu_\alpha t_\alpha$. Because $\phi(\xi)t_\alpha=\phi(\xi_\alpha) \to \phi(\xi)$ and $t_\alpha \in K$ we see that $\lim_{\alpha\in\Lambda_1} t_\alpha=e$. Thus $\lim_{\alpha\in\Lambda_1} \nu_\alpha t_\alpha=\nu$ and $\lim_{\alpha\in\Lambda_1} \xi_\alpha=\xi$. Finally suppose that Λ_2 is cofinal. From condition (A) we see that any limit point of $\xi^{-1}\xi_\alpha$ is in J, and from condition (C) we see that any limit point of $\xi^{-1}\xi_\alpha$ for $\alpha \in \Lambda_2$ is ν. Therefore, $\lim_{\alpha\in\Lambda_2} \xi^{-1}\xi_\alpha=\nu$ and $\lim_{\alpha\in\Lambda_2} \xi_\alpha=\xi$.

Ellis shows that the map $\psi: I \to \Gamma\times J$ defined by $\psi(\xi)=(\theta(\xi), \nu)$ is a bijection where ν is the idempotent in I such that $\xi\nu=\xi$. It follows that $\psi_0(\xi)=(y_0\theta(\xi), x_0\nu)$ $=(\phi(\xi), x_0\nu)$ is a bijection from I to $Y\times P[x_0]$. If f is an F-almost automorphic function representing (X, T), then there exists a continuous function $\hat{f}: X \to Q_\alpha$ such that $\hat{f}(x)=f\circ p(x)$ when $p(x) \notin y_0U_0$. Note that $\hat{f}$ maps $P[x_0]$ homeomorphically onto F. Hence $\psi_f(\xi)=(\phi(\xi), \hat{f}(x_0\nu))$ is a bijection from I to $Y\times F$. Using ψ_f we can transfer all the structure on I to $Y\times F$. We will describe this structure on $Y\times F$ in terms of the group structure on Y and the function f.

LEMMA 4.4. (1) *The action of T on $Y\times P[x_0]$ induced by ψ_0 is given by $(y, z)t=(yt, z)$.*

(2) *The semigroup structure on $Y\times P[x_0]$ induced by ψ_0 is given by $(y, z)(y', z')=(yy', z')$.*

(3) *The action of $Y\times P[x_0]$ on X induced by ψ_0 is given by*

$$\begin{aligned} x(y, z) &= x' \quad \text{if } p(x') = p(x)y \notin \boldsymbol{O}(y_0), \\ &= zt \quad \text{if } p(x)y = y_0 t. \end{aligned}$$

Proof. By definition $\psi_0(\xi t)=\psi_0(\xi)t$ or $(\phi(\xi t), \nu)=(\phi(\xi), \nu)t$ because $\xi\nu t=\xi t\nu$. Since ϕ is a homomorphism, (1) holds. Similarly $\psi_0(\xi)\psi_0(\eta)=\psi_0(\xi\eta)$ or $(\xi, \nu)(\eta, \mu)=(\phi(\xi\eta), \mu)=(\phi(\xi)\phi(\eta), \mu)$ and (2) holds.

Let $\xi\in I$ and $x\in X$. Then there exists $\eta\in I$ such that $x_0\eta=x$. From this we obtain $p(x\xi)=p(x_0\eta\xi)=\phi(\eta)\phi(\xi)=p(x)\phi(\xi)$. Now if $p(x)\phi(\xi)\notin \boldsymbol{O}(y_0)$ then there exists a unique x' in X such that $p(x')=p(x)\phi(\xi)$. It follows that $x'=x\xi$. If $\phi(\eta\xi)=p(x)\phi(\xi)=y_0t$, then $\eta\xi t^{-1}=\nu$, $x\xi t^{-1}=x_0\nu$, and $x\xi=x_0\nu t$. Now (3) follows from the definition of ψ_0.

It is now clear that ψ_f transfers ξt to $(y, z)t=(yt, z)$ and $\xi\eta$ to $(y, z)(y', z')=(yy', z')$. It remains to describe the topology on $Y\times F$ in terms of f.

Let $\mathscr{C}_f$ denote the class of pairs $((y_\alpha, z_\alpha)_{\alpha\in\Lambda}, (y, z))$ where $(y_\alpha, z_\alpha)_{\alpha\in\Lambda}$ is a net in $Y\times F$ and (y, z) is a point in $Y\times F$ such that

(A′) $\lim_{n\to\infty} y_n\to y$.

(B′) If $\Lambda_1=\{\alpha\in\Lambda : y_\alpha\in yU_0\}$ is cofinal, then $\lim_{\alpha\in\Lambda_1} z_\alpha=z$.

(C′) If $\Lambda_2=\{\alpha\in\Lambda : y_\alpha\notin yU_0\}$ is cofinal, then $\lim_{\alpha\in\Lambda_2} f(y_\alpha)=z$.

THEOREM 4.5. *Let f be an F-almost automorphic function on Y. Then $\mathscr{C}_f$ is a convergence class on $Y\times F$ yielding a compact Hausdorff topology. With this topology and $(y, z)t=(yt, z)$ we obtain a locally almost periodic minimal set which is isomorphic to (I_f, T) where I_f is the minimal right ideal in the enveloping semigroup of (X_f, T). If we let $(y, z)(y', z')=(yy', z')$, then the above map is also a semigroup isomorphism.*

Proof. Let (X_f, T) be (X, T) in the previous discussion and observe that ψ_f carries conditions (A), (B), and (C) to conditions (A′), (B′), and (C′) which proves the theorem.

Suppose that Y is the set of complex numbers of modulus one, T is the integers, and the action is multiplication by a generator g. Let $f(z)=y$ if $z\neq g$, $e^{2\pi yi}=\bar{g}z$ and $y\in(0, 1)$. Then f is an F-almost automorphic function with $F=\{0, 1\}$. Then the transformation group $(Y\times F, T)$ given by Theorem 4.5 is Ellis' example on two circles [4, Example 4], and (X_f, T) is a Sturmian minimal set.

THEOREM 4.6. *Let (X', T) be a minimal transformation group. Then (X', T) is a weak F-minimal set if and only if there exists an F-minimal set (X, T) such that (X', T) is isomorphic to $(I(X)/R, T)$ where $I(X)$ is the minimal right ideal of (X, T)*

and R is a closed equivalence relation properly contained in the proximal relation of $(I(X), T)$.

Proof. Assume that (X', T) is a weak F-minimal set. Let (Y, T) denote $(X'/P, T)$ and let p denote the canonical homomorphism of (X', T) onto (Y, T). Let I' denote the minimal right ideal in the enveloping semigroup of (X', T). Fix x' in X'. Then the map $\xi \to x'\xi$ is a homomorphism of (I', T) onto (X', T) and

$$R = \{(\xi, \eta) : x'\xi = x'\eta\}$$

is a closed invariant equivalence relation contained in the proximal relation of I' such that (X', T) is isomorphic to $(I'/R, T)$. Thus it suffices to show that (I', T) is isomorphic to the minimal right ideal of some F-minimal set.

Define R' on I' by $(\xi, \eta) \in R'$ if either $\xi = \eta$ or $\xi = \eta\mu$ for some $\mu \in J$ and $\xi \notin JT$. It is clear that R' is an invariant equivalence relation on I'. Suppose $\{(\xi_\alpha, \eta_\alpha)\}$, $\alpha \in \Lambda$, is a net in R' converging to (ξ, η) where $\xi \neq \eta$. Choose x'' in X such that $x''\xi \neq x''\eta$ and define R_0 on X' by $(x, y) \in R_0$ if either $x = y$ or, for some $t \in T$ and $\alpha \in \Lambda$, $xt, yt \in P[x''\xi_\alpha]$ and $\xi_\alpha \neq \eta_\alpha$. It follows from condition (c′) that R_0 is a closed equivalence relation on X'. Observe that $(x''\xi_\alpha, x''\eta_\alpha) \in R_0$ where $\xi_\alpha \neq \eta_\alpha$ and hence $(x''\xi, x''\eta) \in R_0$. Hence, for some $\alpha \in \Lambda$ and $\nu \in J$, $\xi = \xi_\alpha \nu t \notin JT$ and $\xi \sim \eta$. We now know that R' is closed and we will proceed to show that $(I'/R', T)$ is an F-minimal set. It is certainly locally almost periodic. Notice that $\rho(P[\xi]) = P[\rho(\xi)]$ where ρ denotes the canonical map of I' onto $X = I'/R'$. Conditions (b) and (c) follow from this observation. Condition (d′) holds on I'. Using uniform continuity and $\rho(P[\xi]) = P[\rho(\xi)]$ we obtain (d) on (X, T). Because T is abelian (e) is equivalent to (X, T) being free which it certainly is.

To complete the first half of the proof it suffices to show that (I', T) is isomorphic to (I, T) where I is the minimal right ideal associated with (X, T). This follows from the following lemma:

LEMMA 4.7. *Let (X', T) be a locally almost periodic minimal transformation group and let R be a closed invariant equivalence relation contained in the proximal relation of $(I(X'), T)$. If for some ξ_0 in $I(X')$ we have $R[\xi_0\mu] = \{\xi_0\mu\}$ for all μ in J', then $(I(X'), T)$ and $(I(I(X'))/R, T)$ are isomorphic.*

Proof. Let ρ be the canonical map of $I' = I(X')$ onto $X = I'/R$. There exists a semigroup homomorphism ψ of I' onto $I = I(X)$ such that $x_0\psi(\xi) = \rho(\xi)$ for all ξ in I' where $x_0 = \rho(\xi_0)$. The existence of ψ is established by using Lemma 2 statement (11) in [4] and the remark following Lemma 2 in [1]. Suppose $\psi(\mu) = \psi(\nu)$ where $\mu, \nu \in J'$. Then $\psi(\xi_0\mu) = \psi(\xi_0\nu)$, $\rho(\xi_0\mu) = \rho(\xi_0\nu)$, and $\mu = \nu$. Next suppose $\psi(\xi) = \psi(\eta)$. Then $\xi = \eta\mu$ and $\eta = \eta\nu$ for suitable μ, ν in J'. Thus $\psi(\eta)\psi(\nu) = \psi(\eta)\psi(\mu)$, $\psi(\nu) = \psi(\mu)$, $\nu = \mu$, and $\xi = \eta$.

The second half of the proof proceeds in two steps. First we show that the minimal right ideal I associated with an F-minimal set is a weak F-minimal set, and then we show that $(I/R', T)$ is weak F-minimal if R' is a closed invariant

equivalence relation contained in the proximal relation. In both steps the conditions for weak F-minimality with the exception of (c′) are routinely verified.

Let (X, T) be an F-minimal set and let (I, T) be the minimal right ideal in its enveloping semigroup. Let R be an invariant equivalence relation which is contained in the proximal relation of (I, T). Assume $R \cap \xi J \times \xi J$ is closed for all ξ in I, and suppose $(\xi_\alpha, \eta_\alpha)$, $\alpha \in \Lambda$, is a net in R converging to (ξ, η). We apply Lemma 4.3. Note that $\xi_\alpha \in \xi Jt$ if and only if $\eta_\alpha \in \eta Jt = \xi Jt$. For $\alpha \in \Lambda_1$ there exists t_α in K such that $\xi_\alpha, \eta_\alpha \in \xi Jt_\alpha$, and $(\xi_\alpha t_\alpha^{-1}, \eta_\alpha t_\alpha^{-1})$ is a net in $R \cap \xi J \times \xi J$ converging to (ξ, η) when Λ_1 is cofinal. When $\alpha \in \Lambda_2$, $x_0 \xi^{-1} \xi_\alpha = x_0 \eta^{-1} \eta_\alpha$. If Λ_2 is cofinal, we see by taking limits that $x_0 \nu = x_0 \mu$, $\nu = \mu$, and $\xi = \eta$. Therefore, (c′) holds in (I, T).

Let R' be a closed invariant equivalence relation in the proximal relation of (I, T), let $(X', T) = (I/R', T)$, and let ρ be the canonical map. Let R be an invariant equivalence relation contained in the proximal relation of (X', T). Suppose that $R \cap P[x] \times P[x]$ is closed for all x. It suffices to show that $\hat{\rho}^{-1}(R)$ is contained in the proximal relation of (I, T) and $\hat{\rho}^{-1}(R \cap P[\rho\xi] \times P[\rho\xi]) = \hat{\rho}^{-1}(R) \cap \xi J \times \xi J$ for all ξ where $\hat{\rho}(\xi, \eta) = (\rho\xi, \rho\eta)$. This follows because it is known that for proximally equicontinuous flows $R' \subset P_I$ if and only if $\hat{\rho}^{-1}(P_X) = P_I$ [6, Lemma 5, p. 22]. This completes the proof.

THEOREM 4.8. *Let $(X, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$ be a weak F-minimal set over $(Y, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$. If Y is a manifold and $P[x] \neq \{x\}$ and is totally disconnected for some x, then* dim $Y = m+1$ *and $P[x]$ consists of two points. Moreover, if every $P[x]$ is totally disconnected for all x in X, then there are exactly two idempotents in the minimal right ideal associated with $(X, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$.*

Proof. Using (c′) we see that there exists a Sturmian minimal set over $(Y, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$ and hence, by Theorem 3.1, dim $Y = m+1$. If $P[x]$ contains more than two points, we can use (c′) to contradict Theorem 3.2.

If $P[x]$ is totally disconnected for all x, then $P[x]$ consists of one or two points for all x. It follows that J is totally disconnected. Since $(I, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$ is a weak F-minimal set, we obtain an F-minimal set over $(Y, \boldsymbol{R}^m \times \boldsymbol{Z}^n)$ with $F = J$ and we can contradict Theorem 3.2 unless the cardinality of J is two.

THEOREM 4.9. *There exists a weak F-minimal set $(X', \boldsymbol{Z})$ such that given any compact metric space W there exists x in X such that $P[x]$ is homeomorphic to W.*

Proof. Let $(Y, \boldsymbol{Z})$ be an equicontinuous minimal transformation group where Y is the Cantor discontinuum. Let $(X, \boldsymbol{Z})$ be an F-minimal set over $(Y, \boldsymbol{Z})$ with $F = Y$. Let I be the minimal right ideal associated with $(X, \boldsymbol{Z})$. Recall the following facts: $(Y, \boldsymbol{Z})$ has a continuum of distinct orbits, there are only a continuum of distinct compact metric spaces, and every compact metric space is the continuous image of Y. It follows using (c′) and Theorem 4.6 that there exists a closed invariant equivalence relation R on I such that $(I/R, T)$ has the desired properties.

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