SOME INVARIANT \( \sigma \)-ALGEBRAS FOR MEASURE-PRESERVING TRANSFORMATIONS

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Abstract. For an invertible measure-preserving transformation \( T \) of a Lebesgue measure space \( (X, \mathcal{B}, m) \) and a sequence \( N \) of integers, a \( T \)-invariant partition \( a_N(T) \) of \( (X, \mathcal{B}, m) \) is defined. The relationship of these partitions to spectral properties of \( T \) and entropy theory is discussed and the behaviour of the partitions \( a_N(T) \) under group extensions is investigated. Several examples are discussed.

0. Introduction. For an invertible measure-preserving transformation \( T \) of a Lebesgue space \( (X, \mathcal{B}, m) \) and a sequence \( N=\{n_k\}_{k=1}^{\infty} \) of integers we define a \( \sigma \)-algebra by \( \mathcal{A}_N(T) = \{ A \in \mathcal{B} | m(T^{n_k}A \Delta A) \to 0 \} \). Our aim is to study these \( \sigma \)-algebras. In §1 we evaluate those elements of \( L^2(X, \mathcal{B}, m) \) which are measurable with respect to \( \mathcal{A}_N(T) \). The connections the algebras \( \mathcal{A}_N(T) \) have with discrete spectrum and entropy theory are discussed in §2. Every ergodic \( T \) with discrete spectrum has \( \mathcal{A}_N(T) = \mathcal{B} \) for some sequence \( N \) and every \( T \) with \( \mathcal{A}_N(T) = \mathcal{B} \) for some sequence \( N \) has zero entropy. It turns out that the algebras \( \mathcal{A}_N(T) \) have properties in common with the \( \sigma \)-algebra generated by the eigenfunctions of \( T \) and also properties in common with the \( \sigma \)-algebra generated by all the finite algebras having zero entropy relative to \( T \). Some of these properties are noted in §3 which also contains remarks on the relationship of the \( \sigma \)-algebras \( \mathcal{A}_N(T) \) to mixing properties of \( T \). The behaviour of the \( \sigma \)-algebras \( \mathcal{A}_N(T) \) under group extensions is discussed in §4 and in §5 we use Gaussian processes to give examples of weak mixing transformations with \( \mathcal{A}_N = \mathcal{B} \) for some sequence \( N \). §6 is devoted to a discussion of further properties of the algebras \( \mathcal{A}_N(T) \).

Throughout \( T \) will denote an invertible measure-preserving transformation of a Lebesgue space \( (X, \mathcal{B}, m) \) [14]. \( \mathcal{N} \) will denote the trivial \( \sigma \)-algebra consisting of those members of \( \mathcal{B} \) with measure 0 or 1. \( \nu \) will denote the trivial partition and \( \epsilon \) will denote the partition into points of any Lebesgue space. Greek letters \( \xi, \eta, \zeta \) etc. will be used to denote measurable partitions. We shall use partitions and their associated \( \sigma \)-algebras interchangeably. The factor space of \( X \) by \( \xi \) will be denoted by \( X/\xi \) and if \( T_\xi = \xi \) the factor transformation induced by \( T \) on \( X/\xi \) will be denoted by \( T \). If \( \mathcal{A} \) denotes the \( \sigma \)-algebra generated by the members of \( \xi \) then \( L^2(\xi) \) and \( L^2(\mathcal{A}) \) will both denote the collection of all elements of \( L^2(X, \mathcal{B}, m) \) measurable.
with respect to $\mathcal{A}$. In particular $L^2(\mathcal{A})$ and $L^2(\mathcal{B})$ will stand for $L^2(\mathcal{X}, \mathcal{B}, m)$. $U_T$ will denote the unitary operator of $L^2(\mathcal{B})$ defined by $f \mapsto f \circ T$ and $\|\cdot\|_2$ will denote the norm on an $L^2$-space. We shall repeatedly use the spectral theorem which implies that for each $f \in L^2(\mathcal{B})$ there is a Borel measure $\sigma_T$ on the unit circle $K$ with $(U_T^*f,f) = \int_X \lambda^* \mathrm{d}\sigma_T(\lambda) \forall \lambda \in \mathbb{Z}$. $K$ will always denote the unit circle. $\sigma_T$ is called the spectral measure of $f$.

I would like to thank W. Parry for valuable discussions.

1. The $\sigma$-algebras $\mathcal{A}_N(T)$. For a sequence $N=\{n_i\}$ of integers let $\mathcal{A}_N(T) = \{A \in \mathcal{B} \mid m(T^{n_i}A \Delta A) \rightarrow 0\}$. We show below that $\mathcal{A}_N(T)$ is a $\sigma$-algebra. The corresponding partition will be denoted by $\alpha_N(T)$. Let $\mathcal{A}(T) = \bigvee_N \mathcal{A}_N(T)$ (the refinement is taken over all sequences $N$ of integers) and let $\alpha(T)$ denote the corresponding partition. We have $\tau \alpha_N(T) = \alpha_N(T)$, $\tau \alpha_N(T) = \alpha_N(T)$, $\tau \alpha(T) = \alpha(T)$ and $\tau \alpha(T) = \alpha(T)$. When $T$ is understood we shall write $\mathcal{A}_N$, $\alpha_N$, $\mathcal{A}$ and $\alpha$.

**Theorem 1.** $\mathcal{A}_N(T)$ is a $\sigma$-algebra.

**Proof.** Clearly $\emptyset \in \mathcal{A}_N$. Since $T^{n_i}(X \setminus A)\Delta (X \setminus A) = T^{n_i}A \Delta A \mathcal{A}_N$ is closed under complementation. $\mathcal{A}_N$ is finitely additive since if $A_1, A_2 \in \mathcal{A}_N$ then

$$T^{n_i}(A_1 \cup A_2)\Delta (A_1 \cup A_2) \subset (T^{n_i}A_1 \Delta A_1) \cup (T^{n_i}A_2 \Delta A_2)$$

implies $A_1 \cup A_2 \in \mathcal{A}_N$. It remains to show that if $A_j \in \mathcal{A}_N (j \geq 1)$ and $A_1 \subset A_2 \subset \cdots$ then $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_N$. Let $e > 0$ be given. Choose $j_0$ so that $m(A \setminus A_{j_0}) < e$. Choose $I$ so that $I > j_0 m(T^{n_i}A_{j_0} \Delta A_{j_0}) < e$. Then

$$I > J \Rightarrow m(T^{n_i}A \Delta A) \leq m(T^{n_i}A \Delta T^{n_i}A_{j_0}) + m(T^{n_i}A_{j_0} \Delta A_{j_0}) + m(A_{j_0} \Delta A) < 3e.$$ 

Therefore $A \in \mathcal{A}_N$ and $\mathcal{A}_N$ is countably additive.

Our next aim is to show $L^2(\mathcal{A}_N(T)) = \{f \in L^2(\mathcal{B}) \mid \|U_T^*f - f\|_2 \rightarrow 0\}$.

**Lemma 1.** Let $f \in L^2(\mathcal{B})$ be real valued and nonconstant. Let $N=\{n_i\}$ be a sequence of integers. If $\|U^{n_i}f - f\|_2 \rightarrow 0$ then $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$, where $\mathcal{C}$ denotes the $\sigma$-algebra of Borel subsets of $R$.

**Proof.** Let $b \in R$. Put $B = \{x \mid f(x) \leq b\}$ and $B_\varepsilon = \{x \mid f(x) \leq b + \varepsilon\}$. Let $\delta > 0$ be given. On $T^{-n_i}B \setminus B_\varepsilon$ we have $|f(T^{-n_i}x) - f(x)| \geq \varepsilon$ and therefore $m(T^{-n_i}B \setminus B_\varepsilon) \rightarrow 0$ as $i \rightarrow \infty$. Since $B_\varepsilon \setminus B$ decreases with $\varepsilon$ and $\bigcap_{\varepsilon > 0} (B_\varepsilon \setminus B) = \emptyset$, choose $\varepsilon_0$ so that $m(B_\varepsilon \setminus B) < \delta$. Choose $i_0$ so that $i > i_0$ implies $m(T^{-n_i}B \setminus B_\varepsilon) < \delta$. Then $m(T^{-n_i}B \setminus B) \leq m(T^{-n_i}B \setminus B_\varepsilon) + m(B_\varepsilon \setminus B) < 2\delta$ if $i > i_0$. Therefore $m(T^{-n_i}B \Delta B) \rightarrow 0$. We have shown $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$ and by Theorem 1 $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$.

**Theorem 2.** $L^2(\mathcal{A}_N(T)) = \{f \in L^2(\mathcal{B}) \mid \|U_T^*f - f\|_2 \rightarrow 0\}$.

**Proof.** Let $\mathcal{H}$ denote the right-hand side. Certainly $L^2(\mathcal{A}_N(T)) \subset \mathcal{H}$. Suppose $f \in \mathcal{H} \setminus L^2(\mathcal{A}_N(T))$. We can assume $f$ is real valued since either the real or imaginary part of $f$ does not belong to $L^2(\mathcal{A}_N(T))$ but belongs to $\mathcal{H}$. By Lemma 1 $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$ (where $\mathcal{C}$ = Borel subsets of $R$) and hence $f \in L^2(\mathcal{A}_N(T))$, a contradiction.
Our aim is to study the algebras \(A_n(T)\). Of particular interest are those transformations with \(A_n(T) = B\) (\(a_n(T) = e\)) for some sequence \(N\), those with \(A(T) = B\) (\(a(T) = e\)) and those with \(A(T) = N\) (\(a(T) = \nu\)). The condition \(A_n(T) = B\) means \(T^n\) converges to the identity in the space of invertible measure-preserving transformations of \((X, \mathcal{B}, m)\) with the weak topology [6] or equivalently \(U^n\) converges to \(I\) in the space of unitary operators of \(L^2(\mathcal{B})\) with the weak (or strong) topology. The following result relates the property \(A_n(T) = B\) to the maximal spectral type of \(T\). For the theory of spectral measures and types see [13].

**Theorem 3.** \(A_n(T) = B \iff \int K |\lambda^n - 1|^2 \, d\sigma(\lambda) \to 0\) where \(\sigma\) denotes a finite measure on \(K = \{z \mid |z| = 1\}\) whose type is the maximal spectral type of \(T\).

**Proof.** Suppose the right-hand side holds and \(h \in L^1(\sigma)\). We shall show \(\int K |\lambda^n - 1|^2 h(\lambda) \, d\sigma(\lambda) \to 0\). Let \(\delta > 0\) be given and choose \(h_1, h_2\) so that \(h = h_1 + h_2\), \(h_1\) is bounded (\(|h_1(A)| \leq c_\delta\) say) and \(\int |h_2(\lambda)| \, d\sigma(\lambda) < \delta\).

\[
\left| \int K |\lambda^n - 1|^2 h(\lambda) \, d\sigma(\lambda) \right| \leq 2 \left( \int |\lambda^n - 1|^2 |h_1(\lambda)| \, d\sigma(\lambda) + \int |\lambda^n - 1|^2 |h_2(\lambda)| \, d\sigma(\lambda) \right) < c_\delta \int |\lambda^n - 1|^2 \, d\sigma(\lambda) + 4\delta < 5\delta
\]

if \(i > i_0\) and \(i_0\) is chosen so that \(i > i_0\) implies \(\int |\lambda^n - 1|^2 \, d\sigma(\lambda) < \delta/c_\delta\). Hence \(\int |\lambda^n - 1|^2 h(\lambda) \, d\sigma(\lambda) \to 0\). If \(f \in L^2(\mathcal{B})\) the spectral measure \(\sigma_f\) of \(f\) is absolutely continuous with respect to \(\sigma\) and by the above \(\|U^n f - f\|^2 = \int |\lambda^n - 1|^2 \, d\sigma_f(\lambda) \to 0\). Hence \(f \in L^2(A_n(T))\) and \(A_n(T) = B\).

Conversely if \(A_n(T) = B\) then choosing \(f \in L^2(\mathcal{B})\) with spectral measure \(\sigma_f\) of maximal type we have \(\int |\lambda^n - 1|^2 \, d\sigma_f(\lambda) = \|U^n f - f\|^2 \to 0\). By the above, if \(\sigma\) is any measure whose type is the maximal spectral type of \(T\) then \(\int |\lambda^n - 1|^2 \, d\sigma(\lambda) \to 0\).

2. Some properties of \(A_n(T)\). The simplest examples of transformations with \(A_n = B\) are given by

**Theorem 4.** If \(T\) is ergodic with discrete spectrum there exists a sequence \(N = \{n_i\}\) with \(A_n(T) = B\).

**Proof.** We can suppose \(T\) is an ergodic rotation \(T x = ax\) on a compact abelian group \(G\) [6, p. 48]. Choose \(N = \{n_i\}\) so that \(a^{n_i} \to e\) the identity element of \(G\). If \(\gamma\) is a character of \(G\) \(\|U^n_\gamma \gamma - \gamma\|_2^2 = |\gamma(a^{n_i}) - 1|^2 \to 0\). Since the characters generate \(L^2(G)\) we have \(A_n(T) = B\).

Later we shall give more examples of transformations with \(A_n = B\).

The algebras \(A_n(T)\) are related to the work of Katok and Stepin [7] and Chacon and Schwartzbauer [1] on approximation by periodic transformations. It is easily checked that if \(T\) admits an approximation of the second kind by periodic transformations (a.p.t.II) with speed \(o(1/n)\) in the sense of Katok and Stepin [7, p. 78] then \(A_{\text{ap}}(T) = B\). Also if \(T\) admits an approximation by periodic automorphisms in Chacon and Schwartzbauer’s sense [1] then \(A_{\text{ap}}(T) = B\).
Now we discuss the relationship of the partitions \( \alpha_N(T) \) to entropy theory. The notations for entropy are from [15]. Let \( \mathcal{D} \) denote the set of partitions with finite entropy. Pinsker [12] has defined the maximum partition with zero entropy for \( T \) as \( \pi(T) = \sqrt{\{ \xi \in \mathcal{D} \mid h(T, \xi) = 0 \}} \). We have \( T\pi(T) = \pi(T) \) and if \( \eta \in \mathcal{D} \) then \( \eta \leq \pi(T) \) if and only if \( h(T, \eta) = 0 \). Using the concept of sequence entropy introduced by Kushnirenko [8], one can define the maximum partition with zero \( N \)-entropy for \( T \) (for a sequence of integers \( N \)) by \( \pi_N(T) = \sqrt{\{ \xi \in \mathcal{D} \mid h_N(T, \xi) = 0 \}} \). It is straightforward to check that \( T\pi_N(T) = \pi_N(T) \) and if \( \eta \in \mathcal{D} \) then \( \eta \leq \pi_N(T) \) if and only if \( h_N(T, \eta) = 0 \). The main result of Kushnirenko’s paper [8] implies \( \pi_\infty(T) = \bigwedge_N \pi_N(T) \) is the maximum partition for \( T \) such that the associated factor transformation has discrete spectrum. In other words \( \pi_\infty(T) \) is the partition generated by the eigenfunctions of \( T \).

**Theorem 5.** For every sequence \( N \) of integers, \( \alpha_N(T) \leq \pi(T) \) and \( \alpha_N(T) \leq \pi_\infty(T) \). Hence \( \alpha(T) \leq \pi(T) \).

**Proof.** We first show \( \alpha_N(T) \leq \pi_N(T) \). Suppose \( \xi \) is a finite partition and \( \xi \leq \alpha_N(T) \).

We have

\[
H(T^{n_1} \xi \lor T^{n_2} \xi \lor \cdots \lor T^{n_k} \xi) \\
\leq H(T^{n_1} \xi) + H(\xi/T^{n_1-n_2} \xi) + H(\xi/T^{n_2-n_3} \xi) + \cdots + H(\xi/T^{n_k-1-n_k} \xi)
\]

so if \( H(\xi/T^{n_k-1-n_k} \xi) \to 0 \) then \( h_N(T, \xi) = 0 \) and \( \xi \leq \pi_N(T) \). But \( \mathcal{A}_{\pi_N}(T) = \mathcal{B} \) implies \( \mathcal{A}_{\alpha_N}(T) = \mathcal{B} \) and this readily implies \( H(\xi/T^{n_k-1-n_k} \xi) \to 0 \). Hence \( \alpha_N(T) \leq \pi_N(T) \).

Similarly \( \alpha_N(T) \leq \pi(T) \) since if \( \xi \leq \alpha_N(T) \) is finite \( h(T, \xi) = H(\xi/T\xi) \leq \lim_{n \to \infty} H(\xi/T^n \xi) = 0 \).

**Corollary 5.1.** \( h(T_{\alpha_N(T)}) = 0, h_N(T_{\alpha_N(\pi)}) = 0 \) and \( h(T_{\alpha(\pi)}) = 0 \).

By Theorem 4 we know there exists a sequence \( N \) such that \( \pi_\infty(T) \leq \alpha_N(T) \). This and Theorem 5 indicate that the partitions \( \alpha_N(T) \) may inherit some of the properties of \( \pi_\infty(T) \) and some of the properties of \( \pi(T) \). This is shown to be so in the later sections.

**Theorem 6.** (a) \( \bigcap_N \mathcal{A}_N(T) \) is the \( \sigma \)-algebra of \( T \)-invariant members of \( \mathcal{B} \).

(b) The class of invertible measure-preserving transformation with \( \mathcal{A}_N = \mathcal{B} \) is closed under (i) factors, (ii) countable direct products, and (iii) inverse limits. In fact (iii) can be strengthened to the property:

if \( \xi_n \not\rightarrow \xi \) and \( T^n \xi_n = \xi_n, T^n \xi = \xi \), then \( \alpha_N(T_{\xi_n}) \not\rightarrow \alpha_N(T) \).

**Proof.** (a) Let \( A \in \bigcap_N \mathcal{A}_N(T) \). Then \( m(TA\Delta A) \leq m(T^{n_1}A\Delta A) + m(A\Delta T^n A) \to 0 \).

(b) (i) is trivial.

(ii) Let \( T_i \) act on \((X_i, \mathcal{B}_i, m_i)\) and let \( T_\infty = \prod_{i=1}^\infty T_i \) acting on \((X, \mathcal{B}, m) = \prod_{i=1}^\infty (X_i, \mathcal{B}_i, m_i)\). Assume \( \mathcal{A}_N(T_i) = \mathcal{B}_i \) for each \( i \). It is easy to show that each measurable rectangle is in \( \mathcal{A}_N(T_\infty) \) and hence \( \mathcal{A}_N(T_\infty) = \mathcal{B} \).
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(iii) It suffices to take $\xi = e$. Let $f \in L^2(\alpha_N(T))$ and put $f_n = E(f/\xi_n)$, where $E(f/\xi_n)$ is the conditional expectation of $f$ relative to the $\sigma$-algebra generated by $\xi_n$. Then $\|f - f_n\|_2 \to 0$ and $\|U_{T_n} f - f_n\|_2 \leq \|U_{T_n} f - f\|_2 \to 0$. Hence $f_n \in L^2(\alpha_N(T_{T_n}))$ and $\alpha_N(T_{T_n}) \to \alpha_N(T)$. Let $\mathcal{W}$ denote the class of invertible measure-preserving transformations of $(X, \mathcal{B}, m)$ with the weak topology [6]. $\mathcal{W}$ is a complete metric space and hence has the Baire property that a countable intersection of open dense sets is dense. From Theorem 1.1 of [7] it follows that the collection of all transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$ contains a dense $G_\delta$ in the space $\mathcal{W}$. Since the weak-mixing transformations form a dense $G_\delta$ in $\mathcal{W}$ ([6]) it follows that the class of all weak-mixing transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$ contains a dense $G_\delta$ in $\mathcal{W}$. It follows from this and the next theorem that the class of weak-mixing transformations which are not strong-mixing contains a dense $G_\delta$ in $\mathcal{W}$.

3. Mixing properties. An example of a property of $\alpha_N(T)$ inherited from $\pi_* (T)$ is

**Theorem 7.** There are no nonconstant mixing functions in $L^2(\mathcal{A}_* (T))$ (i.e. $(U_{T} f, f) \to (f, 1)(1, f)$ for $f \in L^2(\mathcal{A} (T))$ implies $f = \text{constant}$).

**Proof.** If $(U_{T} f, f) \to (f, 1)(1, f)$ and $f \in L^2(\mathcal{A}_* (T))$ then $(f, 1)(1, f) = f$ and $f$ is constant. 

**Corollary 7.1.** $\alpha_N(T)$ has singular spectrum.

**Proof.** If the spectrum is not singular there exists $f \in L^2(\mathcal{A}_* (T))$ with absolutely continuous spectral measure $\sigma_f$. Then $(U_{T} f, f) = \int \lambda^n d\sigma_f (\lambda) \to 0$ by the Riemann-Lebesgue lemma and since $(U_{T} f, f) \to \|f\|_2^2$ we have $f = 0$.

**Corollary 7.2.** If $T$ is totally ergodic with quasi-discrete spectrum then $\alpha_N(T) \leq \pi_* (T)$ for all sequences $N$ and there exists a sequence $N$ with $\alpha_N(T) = \pi_* (T)$. Hence $\alpha(T) = \pi_* (T)$.

**Proof.** $L^2(\mathcal{E}) = L^2(\pi_* (T)) \oplus \mathcal{H}$ where $U_T \mathcal{H} = \mathcal{H}$ and $U_T | \mathcal{H}$ has Lebesgue spectrum. Let $f \in L^2(\alpha_N(T))$ and $f = f_1 + f_2, f_1 \in L^2(\pi_* (T)), f_2 \in \mathcal{H}$. Then $\|U_{T} f - f\|_2^2 = \|U_{T} f_1 - f_1\|_2^2 + \|U_{T} f_2 - f_2\|_2^2$ implies $f_1 \in L^2(\alpha_N(T))$ and $f_2 \in L^2(\alpha_N(T))$. By Theorem 7 $f_2 = 0$ and hence $\alpha_N(T) \leq \pi_* (T)$. The rest of the corollary follows from Theorem 4.

Later we shall give examples of weak-mixing transformations with $\mathcal{A}_N(T) = \mathcal{B}$ for some $N$. Consideration of $\mathcal{A}(T)$ gives an interesting connection with mixing. We first give a definition.

**Definition 1.** $T$ is intermixing if whenever $m(A) > 0$ and $m(B) > 0$, $A, B \in \mathcal{B}$, we have $\lim_{n \to \infty} m(T^n A \cap B) > 0$.

Friedman and Ornstein [2] give examples of intermixing transformations which are not strong-mixing.
Theorem 8.

\[ T \text{ strong-mixing} \Rightarrow T \text{ intermixing} \Rightarrow \mathcal{A}(T) = \mathcal{N} \Rightarrow T \text{ weak-mixing}. \]

Proof. \( T \) strong-mixing \( \Rightarrow \) \( T \) intermixing is clear. The example of Friedman and Ornstein mentioned above shows the converse is false. If \( T \) is intermixing and \( 0 < m(A) < 1, A \in \mathcal{B}, \) then \( \lim \inf_{n \to \infty} m(T^nA \cap A^c) > 0 \) and so \( A \notin \mathcal{A}_n(T) \) for any sequence \( N \). Therefore \( \mathcal{A}(T) = \mathcal{N} \). Theorem 4 shows \( \mathcal{A}(T) = \mathcal{N} \Rightarrow T \) weak-mixing, and the converse is false by the examples of §5.

We do not know if \( \mathcal{A}(T) = \mathcal{N} \) implies \( T \) is intermixing. Pinsker [12] has shown that if \( T \xi = \xi \) and \( \pi(T \xi) = \pi \) then \( \pi(T) \) and \( \xi \) are independent partitions. We shall show the corresponding result for the partitions \( \alpha_n(T) \). Two Borel measures (or types of measures) on the unit circle \( K \) will be called singular modulo \( \{1\} \) if their restrictions to \( K - \{1\} \) are singular.

Theorem 9. If \( \mathcal{H} \) is a \( U_T \)-invariant subspace of \( L^2(\mathcal{B}) \) with \( L^2(\mathcal{A}_n(T)) \cap \mathcal{H} = \{0\} \) or the constants, then the maximal spectral types of \( U_T|L^2(\mathcal{A}_n(T)) \) and \( U_T|\mathcal{H} \) are singular modulo \( \{1\} \).

Proof. Let \( \sigma \) be a measure with type equal to the maximal spectral type of \( T_{\alpha_n(T)} \) and \( \mu \) a measure with type the maximal spectral type of \( U_T|\mathcal{H} \). If \( \sigma \) and \( \mu \) are not singular modulo \( \{1\} \) there exists a measure \( \tau \) not concentrated on \( \{1\} \) with \( \tau \leq \sigma \) and \( \tau \leq \mu \). As in the proof of Theorem 3, \( \int |\lambda^n - 1|^2 d\tau \to 0 \). Let \( g \in \mathcal{H} \) have spectral measure \( \tau \). \( g \) is not constant and \( \|U_T^g - g\|_2^2 = \int |\lambda^n - 1|^2 d\tau \to 0 \) so \( g \in L^2(\mathcal{A}_n(T)) \), a contradiction.

The next corollary is the analogue of the result of Pinsker mentioned above.

Corollary 9.1. Suppose \( T \xi = \xi \) and \( \alpha_n(T \xi) = \pi \). Then \( \xi \) and \( \alpha_n(T) \) are independent partitions.

Proof. By Theorem 9, \( T \xi \) and \( T_{\alpha_n(T)} \) have singular types mod \( \{1\} \). Let \( f \in L^2(\alpha_n(T)) \) and \( g \in L^2(\xi) \) both have integral zero. Then \( f \) and \( g \) have singular spectral types and hence are orthogonal [13, p. 124].

Corollary 9.2. If \( \alpha_n(T) = \pi \) then \( T \) is disjoint from all strong-mixing transformations. (For the definition of disjointness see [3].)


This corollary is a strengthening of Theorem 7. The converse to Corollary 9.2 is false since the transformation of the 2-torus \( T(z, w) = (e^{2\pi i a z}, zw) \), where \( a \) is irrational, is disjoint from all strong-mixing transformations [3] and yet \( \alpha(T) \neq \pi \) by Corollary 7.2 since \( T \) is totally ergodic with quasi-discrete spectrum.

4. Group extensions. We now investigate how the partitions \( \alpha_n(T) \) behave under group extensions.
Theorem 10. Let $G$ be a compact abelian metric group acting as a group of measure-preserving transformations of $(X, \mathcal{B}, m)$ such that $gT = Tg$. Let $\xi(G)$ denote the partition of $X$ into orbits of $G$. If $\alpha_n(TG) = \nu$ then $T_{\alpha_n(T)}$ is conjugate to a rotation on a factor group of $G$. (The triviality of this factor group means $\alpha_n(T) = \nu$ and this will occur if $T$ is weak-mixing.)

Proof. $gT = Tg$ implies $g\alpha_n(T) = \alpha_n(T)$ and so $G$ acts on $X/\alpha_n(T)$. We first show that $G$ acts ergodically on $X/\alpha_n(T)$. Let $\xi(G, N)$ denote the partition of $X$ determined by the partition of the space $X/\alpha_n(T)$ into orbits of $G$. Then $\xi(G, N) \leq \xi(G)$ and so $\alpha_n(T_{\xi(G, N)}) = \nu$. But $\xi(G, N) \leq \alpha_n(T)$ and therefore $\xi(G, N) = \nu$. That $T_{\alpha_n(T)}$ is conjugate to a rotation on a factor group of $G$ follows from Lemma 3 of [11].

Results of this nature have been proved about $\pi(T)$ by Parry [11] and Thomas [17].

Corollary 10.1. Suppose $T$ is totally ergodic and $G$ is a finite group acting as measure-preserving transformations of $(X, \mathcal{B}, m)$ so that $gT = Tg$ for each $g \in G$. If $\alpha_n(TG) = \nu$ then $\alpha_n(T) = \nu$.

Proof. By Theorem 10, $X/\alpha_n(T)$ is a finite space and the total ergodicity of $T$ implies it is one point. Hence $\alpha_n(T) = \nu$.

Let $G$ be a compact connected abelian metric group which acts freely as a group of homeomorphisms on a compact metric space $X$. Let $T: X \to X$ be a homeomorphism with $gT = Tg$ for every $g \in G$. Suppose $T$ and $G$ preserve a measure $m$ defined on the completion of the Borel subsets of $X$. $T$ induces a homeomorphism $T_G: X/G \to X/G$ of the orbit space and every lift of $T_G$ to $X$ is of the form $x \to \phi(x)T(x)$ where $\phi \in C_0(X, G) = \{\phi: X \to G | \phi$ is continuous and $\phi(gx) = \phi(x) \forall g \in G, x \in X\}$. $C_0(X, G)$ becomes a complete metric space when endowed with the metric $D(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x))$ where $d$ is an invariant metric for $G$. $T_G$ preserves the measure on $X/G$ determined by $m$ and the maps $x \to \phi(x)T(x)$ preserve the measure $m$. An (unpublished) result of Jones and Parry announced in [4] states that if $T_G$ is weak-mixing the set of $\phi$ making $x \to \phi(x)Tx$ weak-mixing contains a dense $G_6$ in $C_0(X, G)$. From this, Theorem 8 and Theorem 10 we conclude

Corollary 10.2. (i) If $\alpha_n(T_G) = \nu$ and $T_G$ is weak-mixing the set of $\phi \in C_0(X, G)$ having the property that $x \to \phi(x)Tx$ has $\alpha_n = \nu$ contains a dense $G_6$ in $C_0(X, G)$.

(ii) If $\alpha(T_G) = \nu$ the set of $\phi \in C_0(X, G)$ having the property that $x \to \phi(x)T(x)$ has $\alpha = \nu$ contains a dense $G_6$ in $C_0(X, G)$.

We now consider the problem of extending a transformation with $\alpha_n = \varepsilon$ to obtain one with the same property. We shall consider only extensions by $Z^2 = \{1, -1\}$. The measure on $Z^2$ is always taken to be the measure giving weight $\frac{1}{2}$ to each point.

Theorem 11. Let $(Y, \mathcal{G}, \mu)$ be a Lebesgue space and let $X = Y \times Z^2$. Define $T: X \to X$ by $T(y, e) = (S_y, \phi(y)e)$ where $S: Y \to Y$ is measure-preserving and
\( \phi : Y \to Z^2 \) is measurable. If \( \mathcal{A}_g(S) = \mathcal{C} \) then
\[
\mathcal{A}_g(T) = \mathcal{B} \iff \mu\{y | \phi(S^{n-1}y)\phi(S^{n-2}y) \cdots \phi(y) = -1\} \to 0.
\]

**Proof.** Suppose \( \mathcal{A}_g(T) = \mathcal{B} \) and take \( f(y, \varepsilon) = \varepsilon \). Then
\[
\int |\phi(S^{n-1}y)\cdots \phi(y) - 1|^2 \, d\mu(y) = \| U_n^f f - f \|_2^2 \to 0 \text{ as } i \to \infty
\]
and hence \( \mu\{y | \phi(S^{n-1}y)\phi(S^{n-2}y) \cdots \phi(y) = -1\} \) \( \to 0 \).
Conversely if this condition holds, the above function \( f \) belongs to \( L^2(\mathcal{A}_g(T)) \) and hence \( \mathcal{A}_g(T) = \mathcal{B} \).

We use this in the following theorem the proof of which comes from ideas in [7].

**Theorem 12.** Let \( T : K \times Z^2 \to K \times Z^2 \) be defined by \( T(z, \varepsilon) = (e^{2\pi i a z}, \phi(z) \varepsilon) \) where
\[
\phi(z) = \begin{cases} 
-1 & \text{if } \arg z \leq \gamma 2\pi \\
1 & \text{if } \arg z > \gamma 2\pi
\end{cases} \quad (0 < \gamma < 1).
\]
Then \( \mathcal{A}_g(T) = \mathcal{B} \) if there exist integers \( p \) and even integers \( r \), with \( (p, n) = 1 \), \( |a - p_i|/n \) = \( o(1/n^2) \) and \( |\gamma - r_i/n| = o(1/n) \).

**Proof.** For the proof we shall consider the circle group \( K \) as the additive group \([0, 1)\) with addition modulo 1. Set
\[
\phi_i(x) = \phi(x)\phi(a + x) \cdots \phi((n_i - 1)a + x)
\]
and
\[
\phi^*_i(x) = \phi(x)\phi(p_i/n_i + x) \cdots \phi(((n_i - 1)/n_i)p_i + x);
\]
\[
\{x | \phi_i(x) = -1\} \subseteq \{x | \phi^*_i(x) = -1\} \cup \{x | \phi_i(x) \neq \phi^*_i(x)\}.
\]
Let \( \mu \) denote Lebesgue measure on \([0, 1)\).
\[
\mu\{|x | \phi_i(x) \neq \phi^*_i(x)\}) \leq \sum_{j=0}^{n_i-1} \mu\{|x | \phi(ja + x) \neq \phi(jp_i/n_i + x)\}
\]
\[
\leq \sum_{j=0}^{n_i-1} |ja - jp_i/n_i| \leq (n_i(n_i + 1)/2) o(1/n^2) \to 0.
\]
If \( n_i\gamma - r_i \geq 0 \) then
\[
\phi^*_i(x) = ( -1)^{i+1} \text{ if } \{n_i x\} \leq n_i \gamma - r_i,
\]
\[
= ( -1)^i \text{ if } \{n_i x\} > n_i \gamma - r_i.
\]
If \( n_i\gamma - r_i < 0 \) then
\[
\phi^*_i(x) = ( -1)^{i} \text{ if } \{n_i x\} \leq 1 + n_i \gamma - r_i,
\]
\[
= ( -1)^{i-1} \text{ if } \{n_i x\} > 1 + n_i \gamma - r_i.
\]
Therefore
\[
\mu\{|x | \phi_i^*(x) = -1\}) = \mu\{|x | \{n_i x\} \leq n_i \gamma - r_i \} \cup \{x | \{n_i x\} > 1 + n_i \gamma - r_i\}
\]
\[
\leq 2 |n_i \gamma - r_i| \to 0 \text{ as } i \to \infty.
\]
Hence \( \mu\{|x | \phi_i(x) = -1\}) \to 0 \text{ as } i \to \infty \) and \( \mathcal{A}_g(T) = \mathcal{B} \) by Theorem 11.
5. Further examples. In this section we shall consider some weak-mixing transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$. $T$ will be the shift generated by a stationary Gaussian process. Let $X = \bigcap_{n=0}^{\infty} R$, $\mathcal{C}$ the product $\sigma$-algebra generated by the Borel subsets of $R$ and let $p_j$ denote the $j$th coordinate function. Hence if $x = \{x_n\}$ then $p_j(x) = x_j$. One assigns a probability measure to $(X, \mathcal{C})$ by requiring that $\{p_j\}$ be a stationary Gaussian process with covariance sequence $R(n)$ where $R(n) = \int_K \lambda^n d\mu(\lambda)$ and $\mu$ is a finite measure on the unit circle $K$ symmetric with respect to the real axis. $\mu$ is called the covariance measure of the process. Let $\mathcal{B}$ denote the completion of $\mathcal{C}$ and let the measure on $\mathcal{B}$ be $\mu$. $T$ is then defined by $p_{n+1}(x) = p_n(Tx)$, and is an invertible measure-preserving transformation of $(X, \mathcal{B}, \mu)$. Hence every symmetric finite Borel measure on $K$ is the covariance measure of a stationary Gaussian process.

**Theorem 13.** Let $T$ be the shift on a stationary Gaussian process with covariance measure $\mu$. Then $\mathcal{A}_N(T) = \mathcal{B}$ if $\int |\lambda^n - 1|^2 d\mu(\lambda) \to 0$.

**Proof.** Suppose $\mathcal{A}_N(T) = \mathcal{B}$, then $\int |\lambda^n - 1|^2 d\mu(\lambda) = \| U_{\lambda^N} p_1 - p_1 \|_2^2 \to 0$. Conversely $\int |\lambda^n - 1|^2 d\mu(\lambda) \to 0$ implies $p_1 \in L^2(\mathcal{A}_N(T))$ and hence $p_k \in L^2(\mathcal{A}_N(T))$ for each $k$ and hence $\mathcal{A}_N(T) = L^2(\mathcal{B})$.

**Theorem 14.** Let $\mu$ be a continuous symmetric finite measure concentrated on $D \cup D^{-1}$ where $D$ is a Kronecker subset of $K$. Let $T$ be the shift on the Gaussian process determined by $\mu$. Then $T$ is weak-mixing and $\mathcal{A}_N(T) = \mathcal{B}$ for some sequence $N$ (and hence is not strong-mixing or intermixing).

**Proof.** The conclusion about mixing is in [10]. We have

$$\int_{D \cup D^{-1}} |\lambda^n - 1|^2 d\mu(\lambda) \leq \int_D |\lambda^n - 1|^2 d\mu(\lambda) + \int_{D^{-1}} |\lambda^n - 1|^2 d\mu(\lambda) = 2 \int_D |\lambda^n - 1|^2 d\mu(\lambda).$$

Let $\epsilon_i \to 0$ and for each $i$ choose $n_i \in Z$ with $\sup_{x \in D} |1 - z^{n_i}| < \epsilon_i$. This is possible since $D$ is a Kronecker set. Then $\int_D |\lambda^n - 1|^2 d\mu(\lambda) < \epsilon_i^2 \mu(D) \to 0$ as $i \to \infty$. Therefore $\mathcal{A}_N(T) = \mathcal{B}$ by Theorem 13 if $N = \{n_i\}$.

We also note the following

**Theorem 15.** If $S$ is an invertible measure-preserving transformation with $\alpha_S(S) = \epsilon$ and $S$ does not have discrete spectrum there exists a weak-mixing shift $T$ of a stationary Gaussian process with $\alpha_S(T) = \epsilon$.

**Proof.** Let $\mu_S$ denote a measure having type equal to the maximal spectral type of $S$. $\mu_S$ can be chosen symmetric with respect to the real axis. Let $\mu$ be its continuous part which is nontrivial (by the assumption that $S$ does not have discrete spectrum) and is symmetric. By the proof of Theorem 3, since $\mu \ll \mu_S$ we have $\int_K |\lambda^n - 1| d\mu(\lambda) \to 0$. So letting $T$ be the shift defined on the Gaussian process with covariance
measure μ we obtain a transformation with αₙ(T) = e by Theorem 13, and T is
weak-mixing since μ is a continuous measure [9].

Other examples of weak-mixing transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$ are
constructed in [7] by taking transformations induced from rotations of the unit
circle.

6. Problems. We now discuss whether properties of ergodic transformations
with discrete spectrum carry over to ergodic transformations with αₙ(T) = e for
some $N$. Ergodic transformations with discrete spectrum have simple spectrum
and this may account for the fact that some properties do not carry over. We first
note that if $T$ is ergodic and αₙ(T) = e then $T$ need not have simple spectrum, for
we could choose $T$ weak-mixing and then $T \times T$ does not have simple spectrum but
is ergodic and αₙ(T × T) = e (Theorem 6).

If $T$ is ergodic with discrete spectrum then $T$ is coalescent, i.e. if $S$ is measure-

preserving and $ST = TS$ then $S$ is invertible. All transformations with simple

spectrum have this property but it does not hold for all ergodic $T$ with αₙ(T) = e.

Let $T$ acting as $(X, \mathcal{B}, m)$ be weak-mixing and αₙ(T) = e then $T_\omega = \prod_{i=1}^\infty T$ acting

on $Y = \prod_{i=1}^\infty X$ is ergodic with αₙ(T_ω) = e but commutes with the 1-sided shift with

state space $X$.

The main result of [1] is that if $T$ admits an approximation by periodic automor-

phisms (in the sense of Chacon and Schwartzbauer) and if $S$ is an invertible measure-

preserving transformation commuting with $T$ there exists a sequence $\{j_n\}$ of integers

such that $m(T^{j_n} A \triangle S A) \to 0$ for every $A \in \mathcal{B}$. This property is, of course, true for an

ergodic $T$ with discrete spectrum since every measure-preserving transformation

commuting with an ergodic rotation of a compact abelian group is itself a rotation.

It is not true in general for ergodic transformations with αₙ = e as the following

example shows. Let $T$ acting on $(X, \mathcal{B}, m)$ be weak-mixing with αₙ(T) = e. Put

$T_\omega = \prod_{i=1}^\infty T$, $X_\omega = \prod_{i=1}^\infty X$, $\mathcal{B}_\omega = \prod_{i=1}^\infty \mathcal{B}$, $m_\omega = \prod_{i=1}^\infty m$, $S = \prod_{i=1}^\infty T^i$. $T_\omega$ and $S$

both act on $X_\omega$, αₙ(T_ω) = e and $T_\omega S = S T_\omega$. However there is no sequence $\{j_n\}$ with

$m_\omega(T^{j_n} A \triangle S A) \to 0$ for all $A \in \mathcal{B}_\omega$. However if $T$ admits an approximation by

periodic automorphisms in Chacon and Schwartzbauer’s sense then $T$ has simple

spectrum and so we could pose the following problem that we have been unable

to solve. If $T$ has simple spectrum and αₙ(T) = e for some $N$ and if $S$ is an invertible

measure-preserving transformation with $ST = TS$ then does there exist a sequence

$\{j_n\}$ of integers with $m(T^{j_n} A \triangle S A) \to 0$ for every $A \in \mathcal{B}$?

Another property enjoyed by an ergodic transformation $T$ with discrete spectrum

is that if $S$ is measure-preserving and $ST = TS$ then $\mathcal{A}_M(S) = \mathcal{B}$ for some sequence

$M$. It is possible that if $T$ has simple spectrum and $\mathcal{A}_N(T) = \mathcal{B}$ for some sequence $N$

then each measure-preserving transformation $S$ commuting with $T$ has $\mathcal{A}_M(S) = \mathcal{B}$

for some sequence $M$. This is false if the condition of simplicity of the spectrum

of $T$ is replaced by ergodicity since we could take $T$ to be the 2-sided direct product

of a weak-mixing transformation with $\mathcal{A}_N = \mathcal{B}$ and then $T$ commutes with the

2-sided shift $S$ which is invertible and $\mathcal{A}_M(S) = \mathcal{N}$ for every sequence $M$ (Theorem 8).
Another property of ergodic transformations with discrete spectrum is that \( \{ B \in \mathcal{B} \mid (B, X \setminus B) \) is a generator\} is dense in the metric space \( \mathcal{B} \) (mod 0) with the symmetric difference metric [16]. This is also true for totally ergodic transformations with quasi-discrete spectrum [5]. We have been unable to decide whether it is true for ergodic \( T \) with \( \mathcal{A}_n(T) = \mathcal{B} \) for some \( N \).

7. Noninvertible transformations. Suppose now that \( T \) is a noninvertible measure-preserving transformation of a Lebesgue space \((X, \mathcal{B}, m)\). If we define \( \mathcal{A}_n(T) = \{ A \in \mathcal{B} \mid m(T^{-n}A \Delta A) \to 0 \} \) for a sequence \( N = \{ n_i \}_{i=1}^{\infty} \) of nonnegative integers and let \( \alpha_n(T) \) denote the corresponding partition, then \( T^{-1} \alpha_n(T) \subseteq \alpha_N(T) \) and one can show, as in the proof of Theorem 5, that \( \alpha_n(T) \subseteq \pi(T) \) for each sequence \( N \). Since \( T^{n(T)} \) is an invertible measure-preserving transformation with zero entropy we have \( T^{-1} \alpha_n(T) = \alpha_n(T) \) mod 0 for each sequence \( N \). Hence to study the algebras \( \alpha_n(T) \) for a noninvertible \( T \) it suffices to study \( \alpha_n(T^{n(T)}) \) for the invertible transformation \( T^{n(T)} \).

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