

THE L^1 - AND C^* -ALGEBRAS OF $[FIA]_B^-$ GROUPS, AND THEIR REPRESENTATIONS

BY

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Abstract. Let G be a locally compact group, and B a subgroup of the (topologized) group $\text{Aut}(G)$ of topological automorphisms of G ; G is an $[FIA]_B^-$ group if B has compact closure in $\text{Aut}(G)$. Abelian and compact groups are $[FIA]_B^-$ groups, with $B=I(G)$; the purpose of this paper is to generalize certain theorems about the group algebras and representations of these familiar groups to the case of general $[FIA]_B^-$ groups. One defines the set \mathfrak{X}_B of B -characters to consist of the nonzero extreme points of the set of continuous positive-definite B -invariant functions ϕ on G with $\phi(1) \leq 1$. \mathfrak{X}_B is naturally identified with the set of pure states on the subalgebra of B -invariant elements of $C^*(G)$. When this subalgebra is commutative, this identification yields generalizations of known duality results connecting the topology of G with that of \hat{G} . When $B=I(G)$, \mathfrak{X}_B can be identified with the structure spaces of $C^*(G)$ and $L^1(G)$, and one obtains thereby information about representations of G and ideals in $L^1(G)$. When G is an $[FIA]_B^-$ group, one has under favorable conditions a simple integral formula and a functional equation for the B -characters. $L^1(G)$ and $C^*(G)$ are "semisimple" in a certain sense (in the two cases $B=(1)$ and $B=I(G)$ this "semisimplicity" reduces to weak and strong semisimplicity, respectively). Finally, the B -characters have certain separation properties, on the level of the group and the group algebras, which extend to $[SIN]_B$ groups (groups which contain a fundamental system of compact B -invariant neighborhoods of the identity). When $B=I(G)$ these properties generalize known results about separation of conjugacy classes by characters in compact groups; for example, when $B=(1)$ they reduce to a form of the Gelfand-Raikov theorem about "sufficiently many" irreducible unitary representations.

Introduction. The now classical theory of abstract harmonic analysis on compact groups and locally compact abelian groups furnishes detailed information on the structure of such groups, on their representations, and on their group algebras. The attempt to extend this theory to other classes of groups leads quickly to the study of groups with certain more general compactness conditions. The class of $[Z]$ groups (locally compact groups G such that G/Z is compact, where Z is the center of G) was studied extensively by S. Grosser and M. Moskowitz in [11], [12], and [13]; they have shown that in this class all the essential characteristics of the

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classical theory are preserved. There are two other classes to which one can generalize much of the classical theory without extensive change. The first, $[FIA]^-$, is the class of locally compact groups G such that $I(G)$ (the group of inner automorphisms) has compact closure in $\text{Aut}(G)$, the topologized (see [2]) group of topological group automorphisms of G . The second class, $[SIN]$, contains $[FIA]^-$; $[SIN]$ is the class of locally compact groups G such that G has "small invariant neighborhoods" (that is, each neighborhood of the identity contains a neighborhood of the identity invariant under $I(G)$). The structure of groups in these classes and others has been studied by Grosser and Moskowitz [10], Hofmann and Mostert [15], Iwasawa [16], and Mostow [21].

The purpose of the present paper is to generalize certain theorems about the group algebras and representations of abelian and compact groups to the classes of $[FIA]^-$ groups and $[SIN]$ groups. Following an idea of Thoma [26] in the study of harmonic analysis on discrete groups and of Grosser and Moskowitz [10] in the study of the group structure, we shall in fact study groups in the more general classes $[FIA]_B^-$ and $[SIN]_B$ (where in the definition one replaces $I(G)$ by the arbitrary subgroup $B \subset \text{Aut}(G)$). The basic object in our analysis is the set \mathfrak{X}_B of B -characters, which we introduce in 2.4 for arbitrary locally compact groups G and arbitrary subgroups $B \subset \text{Aut}(G)$ (when $B = I(G)$ we write simply " \mathfrak{X} "). In the definition of B -character we use and extend the generalized notion of group character (of finite type) introduced and studied extensively by Godement in [9].

When $B = I(G)$ and G is compact or abelian (or in fact any type I group) then \mathfrak{X} reduces to the set of normalized characters (in the usual sense) of the (finite-dimensional) irreducible representations of G . In general $[FIA]_B^-$ groups (and to a lesser extent in $[SIN]_B$ groups) \mathfrak{X}_B plays the role of the set of these classical group characters quite well. For example, when B^- is compact, the B -characters can be identified with the pure states on the C^* -subalgebra of B -invariant elements in $C^*(G)$ (we denote this subalgebra by $Z_B(C^*(G))$). If in particular $B = I(G)$, this subalgebra is precisely the center of $C^*(G)$ (see 1.2) and \mathfrak{X} is therefore identified with the maximal ideal space of this center. If B^- is compact and $Z_B(C^*(G))$ is commutative, the B -characters satisfy a functional equation, and if for example $B \supset I(G)$ then they are characterized by it (see 4.4 and 4.12). In this latter case, moreover, they are characterized also by an integral formula (4.10, and remarks following 5.8). When $G \in [SIN]_B$, the B -characters separate the closures of the relatively compact B -orbits (as well as the relatively compact orbits from the non-relatively compact orbits), and form a separating family for $L^1(G)$ (see 3.3 and 3.6). If $B = I(G)$ and G is compact, these properties yield classical results about separating the conjugacy classes; when $B = (1)$ and G is arbitrary, they reduce to certain forms of the Gelfand-Raikov theorem about "sufficiently many" irreducible unitary representations.

Moreover, when $G \in [FIA]_B^-$ one can use the properties of \mathfrak{X}_B to prove various results about the representations and the group and C^* -algebras of G . For example,

when $G \in [FIA]^-$ then \mathfrak{X} is naturally homeomorphic with the structure spaces of $C^*(G)$, so that $\text{Prim } C^*(G)$ is a locally compact Hausdorff space (see 5.3). If in addition G is separable, and \hat{G} is the space of equivalence classes of irreducible continuous unitary representations of G in the Fell topology [5], then the following are equivalent: G is a type I group; every element of \hat{G} is finite dimensional; \hat{G} is a Hausdorff space (see 5.12). If $G \in [FIA]_B^-$ then one can show (compare 4.7) that $L^1(G)$ and $C^*(G)$ are "semisimple" in a certain sense; in the two cases $B=(1)$ and $B=I(G)$ this "semisimplicity" reduces to weak and strong semisimplicity, respectively. When $G \in [FIA]_B^-$ and $Z_B(C^*(G))$ is commutative, then G is discrete iff \mathfrak{X}_B is compact (see 4.3); if G is compact then \mathfrak{X}_B is discrete, and one would expect that here also the converse is true (these last results generalize part of Satz 3 of Kaniuth [16a]).

The organization of the paper is as follows. In §1 we define the operator # for $[FIA]_B^-$ groups, generalizing the definition introduced by Godement in [6] for $B=I(G)$. # is an idempotent norm-reducing map from $L^1(G)$ (resp. $C^*(G)$) onto $Z_B(L^1(G))$ (resp. $Z_B(C^*(G))$). In §2 we introduce the B -characters and give some of their elementary properties, and in §3 we give the separation properties associated with $\mathfrak{X}_B(G)$ for $G \in [SIN]_B$. In §4 we assume throughout that B^- is compact, and show that \mathfrak{X}_B can be identified with the set of pure states on $Z_B(C^*(G))$. From this fact we derive some of the results mentioned above about integral formulas and about ideals in $L^1(G)$ and $C^*(G)$. These results depend in part upon recent results of A. Hulanicki [15a]. From the description of \mathfrak{X}_B it also follows that if $Z_B(C^*(G))$ is commutative, then \mathfrak{X}_B can be identified with the set of positive-definite B -spherical functions on the semidirect product $G \times_{\eta} B^-$ (holomorph), and we can therefore transfer some results from the theory of spherical functions (compare [8] and [14]) to our case. Finally, in §5 we restrict our attention further to the case $G \in [FIA]_B^-$, with $B \supset I(G)$. We discuss the topological properties of \hat{G} and $\text{Prim } C^*(G)$ mentioned earlier, as well as other results on \mathfrak{X}_B : for example, there is a continuous, open, and proper surjection $\hat{G} \rightarrow \mathfrak{X}_B$, and \mathfrak{X}_B is naturally homeomorphic with \mathfrak{X}/B^- .

The notation in this paper generally follows that of [4], although we make the following special notations and conventions. G will always be a locally compact group, with modular function $\delta: G \rightarrow \mathbf{R}^+$. The convolution product of the L^1 functions f and g will be denoted $f * g$, and the involution will be denoted by \vee where

$$f^{\vee}(x) = \overline{f(x^{-1})}\delta(x^{-1}).$$

For the extension to $C^*(G)$, however, we shall write ab and a^* , respectively. If f is a function on G bounded on some subset $K \subset G$, we shall denote by $\|f\|_K$ the supremum over $x \in K$ of $|f(x)|$. If $a \in C^*(G)$, the norm of a will be denoted $\|a\|_*$. If $\phi \in L^\infty(G)$, we shall denote by $P[\phi]$ the linear functional on $L^1(G)$ given by $P[\phi](f) = \int_G f(x)\phi(x) dx$. If in addition ϕ is continuous and positive-definite, so that $P[\phi]$ can be extended to $C^*(G)$ as a positive functional, we shall again call this

extension $P[\phi]$. Φ will be the set of continuous positive-definite functions on G of norm ≤ 1 , while Φ_0 will denote the set of nonzero extreme points of Φ . If B is a group operating on a set X , and $E \subset X$, we shall denote by $B[E]$ the set

$$\{\beta \cdot x \mid \beta \in B, x \in E\}.$$

In definitions and notations concerning a subgroup B of $\text{Aut}(G)$ (for example, “ B -invariant” or “ B -character”) we shall, except where noted, omit the “ B ” if $B = I(G)$.

We note here two results of Grosser and Moskowitz which we shall have occasion to use. The first is Proposition 2.6 of [10], and the second is Theorem 4.1 of [12].

0.1. Let G be locally compact. Then $G \in [SIN]_B$ iff for every neighborhood U of 1 there exists a nonnegative continuous B -invariant function h_U such that $h_U(1) > 0$ and support $(h_U) \subset U$. These functions may be chosen positive-definite. It should be noted that the family $\{h_U\}$ forms an approximate identity in the spaces $L^p(G)$ ($1 \leq p < \infty$).

0.2. Let G be locally compact, B a subgroup of $\text{Aut}(G)$. Then B has compact closure in $\text{Aut}(G)$ iff $G \in [SIN]_B$ and $G \in [FC]_B^-$ (that is, the B -orbit $B[x]$ of each point $x \in G$ has compact closure).

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1. The operator #.

1.0. Let G be a locally compact group, and let B be a subgroup of $\text{Aut}(G)$. If f is a complex-valued function on G , and $\beta \in B$, we shall put $f^\beta(x) = f(\beta^{-1}x)$ for $x \in G$; we shall write U^β for the linear mapping $f \rightarrow f^\beta$. It is clear that if f is continuous (or continuous with compact support) so is f^β , and $\|f^\beta\|_G = \|f\|_G$ if f is bounded. For fixed $f \in K(G)$ (the continuous complex-valued functions with compact support) the map $\beta \rightarrow U^\beta f$ is a continuous map of $\text{Aut}(G) \rightarrow K(G)$, where the latter has the uniform norm. In fact if $\varepsilon > 0$, let U be an open neighborhood of the identity in G such that $xy^{-1} \in U$ implies $|f(x) - f(y)| < \varepsilon$. If K is the (compact) support of f , let $W = W(K, U)$ be the neighborhood of the identity in $\text{Aut}(G)$ such that $\beta \in W$ iff $\beta(x)x^{-1} \in U$ and $\beta^{-1}(x)x^{-1} \in U$ for all $x \in K$. If $\beta \in W$, then $\|f^\beta - f\|_G < \varepsilon$: for if $x \in K$, then $\beta^{-1}(x)x^{-1} \in U$, so $|f(\beta^{-1}x) - f(x)| < \varepsilon$; if $\beta^{-1}(x) \in K$, then $\beta(\beta^{-1}(x))\beta^{-1}(x^{-1}) \in U$, so $|f(x) - f(\beta^{-1}x)| < \varepsilon$; while if neither x nor $\beta^{-1}(x)$ is an element of K , then $f(\beta^{-1}(x)) = f(x) = 0$. Thus $\beta \rightarrow U^\beta f$ is continuous as $\beta \rightarrow 1$ in $\text{Aut}(G)$, hence $\beta \rightarrow U^\beta$ is a strongly continuous representation of B on $K(G)$.

We shall show that this representation extends to a strongly continuous representation of B on $L^p(G)$ ($1 \leq p < \infty$) and on $C^*(G)$. For this, recall that we have a continuous homomorphism $\Delta: \text{Aut}(G) \rightarrow \mathbf{R}^+$ (the multiplicative group of positive

real numbers) satisfying $\int_G f^\beta(x) = \Delta(\beta) \int_G f(x) dx$ for $f \in K(G)$ and $\beta \in \text{Aut}(G)$ (see [2, p. 75]). If $g \in L^p(G)$, then $\|g^\beta\|_p = \Delta(\beta)^{1/p} \|g\|_p$, so U^β is a bounded operator on L^p . Moreover, approximating by a continuous function with compact support, we see from the continuity of Δ that, for fixed $g \in L^p(G)$, $\beta \rightarrow g^\beta$ is a continuous map of $\text{Aut}(G)$ into $L^p(G)$ (compare [2, p. 78]). Now, for $C^*(G)$, we show first that if $f \in L^1(G)$, then $\|f^\beta\|_* = \Delta(\beta) \|f\|_*$, so that U^β can be extended to a bounded operator on $C^*(G)$. In fact, $\|f^\beta\|_*$ is the supremum over all $\pi \in \hat{G}$ of $\|\pi(f^\beta)\|$. But

$$\pi(f^\beta) = \int_G f^\beta(x) \pi_x dx = \Delta(\beta) \int_G f(x) (\pi \circ \beta)_x dx = \Delta(\beta) \pi \circ \beta(f).$$

As π runs through \hat{G} so does $\pi \circ \beta$, so the supremum mentioned above is precisely $\Delta(\beta) \sup \|\pi(f)\|$ ($\pi \in \hat{G}$), or $\Delta(\beta) \|f\|_*$. Again, approximating by an L^1 function then shows that $\beta \rightarrow U^\beta$ is a strongly continuous representation of B on $C^*(G)$.

We can therefore make the following definition.

DEFINITION. Let G be locally compact, B a subgroup of $\text{Aut}(G)$ and A one of the spaces $K(G)$, $L^1(G)$, or $C^*(G)$. Then $Z_B(A)$ is the set of elements a in A such that $U^\beta a = \Delta(\beta)a$ for each $\beta \in B$. Moreover, if $1 \leq p < \infty$, then $Z_B(L^p(G))$ is the set of elements $f \in L^p(G)$ such that $U^\beta f = \Delta(\beta)^{1/p} f$ for each $\beta \in B$.

It is clear from the above discussion that $Z_B(A)$ is a closed subspace of A , when A is any of the normed spaces $K(G)$, $L^p(G)$ ($1 \leq p < \infty$), or $C^*(G)$. Moreover, the continuity of the representation $\beta \rightarrow U^\beta$ and of the "modular" function Δ show that $Z_B(A) = Z_{B^-}(A)$, and more generally that closed B -stable subsets of A are also B^- -stable.

If A is one of the algebras $K(G)$, $L^1(G)$, or $C^*(G)$, then $Z_B(A)$ is also a $*$ -subalgebra of A . This will follow directly from Lemma 1.1, in which we list several relations which will also be useful to us later. Let $L_x: f \rightarrow f_{x^{-1}}$ (resp. $R_x: f \rightarrow f^x$) be the left (resp. right) regular representation of G (where $f_x(y) = f(xy)$, $f^x(y) = f(yx)$). Then L_x and R_x , regarded as representations of G on $L^1(G)$, can be extended to strongly continuous representations of G on $C^*(G)$ just as the representation U^β was extended. If $a \in C^*(G)$ we shall write $a_{x^{-1}}$ for $L_x a$, a^x for $R_x a$.

1.1. LEMMA. Let $x \in G$, and $\beta \in \text{Aut}(G)$. Then

- (i) $a^\beta b^\beta = \Delta(\beta)(ab)^\beta$ ($a, b \in C^*(G)$).
- (ii) $(a^*)^\beta = (a^\beta)^*$ ($a \in C^*(G)$).
- (iii) $(a_x)^* b_x = a^* b$ ($a, b \in C^*(G)$).

Proof. It suffices to prove these relations when $a, b \in K(G)$, since they extend by continuity to $C^*(G)$. If $g \in K(G)$, then

$$\begin{aligned} f^\beta * g^\beta(t) &= \int_G f^\beta(tx) g^\beta(x^{-1}) dx = \int_G f(\beta^{-1}t \cdot \beta^{-1}x) g(\beta^{-1}x^{-1}) dx \\ &= \Delta(\beta) \int_G f(\beta^{-1}t \cdot x) g(x^{-1}) dx = \Delta(\beta) f * g(\beta^{-1}t) \\ &= \Delta(\beta)(f * g)^\beta(t). \end{aligned}$$

For (ii), we have

$$(f^\vee)(\beta^{-1}t) = \overline{f(\beta^{-1}t^{-1})}\delta(\beta^{-1}t^{-1}),$$

while

$$(f^\beta)^\vee(t) = \overline{f(\beta^{-1}t^{-1})}\delta(t^{-1}).$$

In order to prove (ii), therefore, it suffices to prove that for $t \in G$ and $\beta \in \text{Aut}(G)$, $\delta(\beta t) = \delta(t)$, and this follows easily from a straightforward calculation. The proof of (iii) is similar to that of (i).

In case $B \supset I(G)$, then $Z_B(A)$ is a commutative algebra. In fact, it is obvious from the definition that $Z_B(A) \subset Z(A)$ (the set of $a \in A$ such that $U^\beta a = \Delta(\beta)a$ for each inner automorphism β), and $Z(A)$ is the center of A , as the following proposition shows. This proposition generalizes the known result that for G unimodular, the center of $L^1(G)$ consists of the L^1 functions f with $f(xtx^{-1}) = f(t)$ a.e. in t (the null set depends on x).

1.2. PROPOSITION. *Let G be a locally compact group, and A one of the spaces $K(G), L^1(G)$, or $C^*(G)$. Then the center of A equals $\{a \in A \mid a^\beta = \Delta(\beta)a \text{ for all } \beta \in I(G)\} = Z(A)$.*

Proof. If π is any strongly continuous unitary representation of G , then $\{\pi_x \mid x \in G\}$ and $\{\pi(a) \mid a \in A\}$ have the same commutants in $B(H_\pi)$. Moreover, if $x \in G$, and β is the conjugation $t \rightarrow x^{-1}tx$, then an easy calculation (extended by continuity, if necessary, to $C^*(G)$) shows that $\pi(a^\beta) = \Delta(\beta)\pi_{x^{-1}}\pi(a)\pi_x$. Suppose first that a is in the center of A . Then $\pi(a)$ commutes with $\pi(b)$ for each $b \in A$, so $\pi(a)$ commutes with π_x for each $x \in G$. The above identity shows then that $\pi(a^\beta) = \Delta(\beta)\pi(a)$, and since π is arbitrary, $a^\beta = \Delta(\beta)a$. Thus $a \in Z(A)$. Conversely, if $a^\beta = \Delta(\beta)a$, then by the above identity $\pi(a)$ commutes with π_x for $x \in G$ and with $\pi(b)$ for each $b \in A$. Thus a is in the center of A , since π is arbitrary.

In the future we shall most often consider the case when each $\beta \in B$ preserves Haar measure; that is, $\Delta(\beta) = 1$ for each $\beta \in B$. One sufficient condition for this is $G \in [SIN]_B$ (see [10, 2.4]), and, since $G \in [FIA]_B^-$ iff $G \in [SIN]_B \cap [FC]_B^-$ (by 0.2), $G \in [FIA]_B^-$ is also a sufficient condition. Of course, by the continuity of Δ , if each element of B preserves Haar measure so will each element of B^- . We review explicitly our definitions and results so far in case $\Delta(B) = 1$. First of all, $\|U^\beta f\|_p = \|f\|_p$ for $f \in L^p$, and $\|U^\beta a\|_* = \|a\|_*$ for $a \in C^*(G)$. $Z_B(A)$ is exactly the set of $a \in A$ such that $a^\beta = a$ for each $\beta \in B$; that is, $Z_B(A)$ is the set of B -invariant elements of A . If A is one of the algebras $K(G), L^1(G)$ or $C^*(G)$, then $(ab)^\beta = a^\beta b^\beta$ for $\beta \in B$ and $a, b \in A$. Moreover, if $B = I(G)$, so that G is unimodular, then the center of A is exactly the set of $I(G)$ -invariant elements of A .

If $G \in [FIA]_B^-$, we can define a norm-reducing idempotent mapping $\#$ of A onto $Z_B(A)$. We shall follow a definition for $\#$ introduced by Godement [6] in the case that $B = I(G)$ has compact closure in $\text{Aut}(G)$. The elementary theory of such a $\#$ operator on $L^1(G)$ and $C^*(G)$ for $G \in [FIA]_B^-$ extends and unifies the theory of the

various # operators used by Grosser and Moskowitz in [13] and [11], by Segal in [24], and by Thoma in [26]. The elementary properties of our # operator are the analogues for the group algebras of results discussed by Godement in [9] for certain von Neumann algebras.

Assume then that $G \in [FIA]_{B^-}$, and let $d\beta$ be the normalized Haar measure over the compact group B^- . If f is a continuous function on G and $x \in G$, we shall write

$$f^\#(x) = \int_{B^-} f^\beta(x) d\beta.$$

1.3. PROPOSITION. *Let $G \in [FIA]_{B^-}$ and let f be a continuous function on G ; let $f^\#: G \rightarrow C$ be defined as above. Then*

- (i) *$f^\#$ is a continuous B^- -invariant function. If f is B^- -invariant, then $f=f^\#$. If f is left (resp. right) uniformly continuous, so is $f^\#$. If $f \in K(G)$, then $f^\# \in K(G)$.*
- (ii) *If f is bounded, then $\|f^\#\|_G \leq \|f\|_G$. If $f \in K(G)$, then $\|f^\#\|_1 \leq \|f\|_1$ and $\|f^\#\|_* \leq \|f\|_*$. If F is a compact set in G , then $\|f^\#\|_F \leq \|f\|_{B^-[F]}$.*
- (iii) *If f is positive-definite, so is $f^\#$.*

Proof. It is easy to see that the evaluation map $\text{Aut}(G) \times G \rightarrow G$ is continuous, since the topology on $\text{Aut}(G)$ contains the topology of uniform convergence on compacta (see [2, p. 58]). If F is any compact set, therefore, $B^- [F]$ is also compact, hence

$$\|f^\#\|_F \leq \sup_{x \in F, \beta \in B^-} |f^\beta(x)| = \|f\|_{B^-[F]} < \infty.$$

Applying the above relation to $F=\{x\}$ shows that $f^\#$ is well defined and that if f is bounded then $\|f^\#\|_G \leq \|f\|_G$.

It is clear that if the support F of f is compact, then the support of $f^\#$ is contained in the compact set $B^- [F]$. For the continuity, assume first that f is, say, left uniformly continuous, and let $\epsilon > 0$. Since $G \in [SIN]_B$ we may choose a compact B^- -invariant neighborhood V of 1 in G such that $y^{-1}x \in V$ implies $|f(y) - f(x)| < \epsilon$. Then for each $\beta \in B^-$ we have $\beta^{-1}(y^{-1})\beta^{-1}(x)$ is an element of $\beta^{-1}V=V$, so $y^{-1}x \in V$ implies $|f^\beta(y) - f^\beta(x)| < \epsilon \int d\beta = \epsilon$. Suppose now that f is merely continuous, and let $x \in G$. Let V be a fixed compact B^- -invariant neighborhood of x (for example, the B^- -orbit of a compact neighborhood of x). By Urysohn's lemma, there is a (uniformly) continuous function g with compact support, agreeing with f on V . Since V contains $B^- [y]$ for each $y \in V$, $g^\beta(y) = f^\beta(y)$ for $y \in V$. Since $g^\#$ is uniformly continuous, $f^\#$ is continuous at x .

It is obvious from the definition that if f is B^- -invariant (hence B^- -invariant) then $f=f^\#$, and it is easy to see that if f is positive-definite, so is $f^\#$. If $f \in K(G)$ then we have (since each $\beta \in B^-$ preserves measure)

$$\int_{B^-} \int_G |f(\beta^{-1}x)| dx d\beta = \int_{B^-} \int_G |f(x)| dx d\beta = \|f\|_1.$$

From the Fubini theorem it follows that $\|f^\#\|_1 \leq \|f\|_1$. For the other estimate, let $\pi \in \hat{G}$ and let $u, v \in H_\pi$. Then by the Fubini theorem we have

$$\begin{aligned} |\langle \pi(f^\#)u, v \rangle| &= \left| \int_G f^\#(x) \langle \pi_x u, v \rangle dx \right| = \left| \int_{B^-} \int_G f^\beta(x) \langle \pi_x u, v \rangle dx d\beta \right| \\ &\leq \int_{B^-} |\langle \pi(f^\beta)u, v \rangle| d\beta \leq \sup_{\beta \in B^-} |\langle \pi(f^\beta)u, v \rangle|. \end{aligned}$$

Considering now the supremum first over $u, v \in H_\pi$ with $\|u\| = \|v\| = 1$ and then over $\beta \in B^-$, we get $\|\pi(f^\#)\| \leq \|f\|_*$, so that $\|f^\#\|_* \leq \|f\|_*$.

It follows from the density of $K(G)$ in each of the Banach spaces $L^1(G)$ and $C^*(G)$ that the $\#$ operator of 1.3 can be extended uniquely to a norm-decreasing idempotent operator on $L^1(G)$ and $C^*(G)$; its range, moreover, is precisely the subalgebra of B -invariant elements. This will follow from the next lemma, in which A denotes either the space $K(G)$ with the uniform norm, or one of the Banach algebras $L^1(G)$, $C^*(G)$ in their respective norms. This lemma cannot be subsumed under the results of [9] for $[SIN]$ groups, and is false even for discrete groups with a natural choice of $\#$ operator that reduces to the one given in 1.3 on discrete $[FIA]^-$ groups ($f^\#(x)$ is the average of f over the conjugacy class of x when this class is finite, and 0 otherwise). In fact, as Godement mentions in the introduction to [9], when G is discrete the bounded central functionals σ on $L^1(G)$ do not all have the property that $\sigma(f) = \sigma(f^\#)$ for all $f \in L^1(G)$ (consider $\sigma(f) = \sum_{x \in G} f(x)$), and this implies that $f^\#$ is not always in the L^1 -closed convex hull of $I(G)^-[f]$.

1.4. LEMMA. *Let $G \in [FIA]^-_B$. If $a \in A$, then $a^\#$ is an element of the closed convex hull of the set $B^-[a] = \{a^\beta \mid \beta \in B^-\}$.*

Proof. We note first that, by the density of $K(G)$ in $L^1(G)$ and $C^*(G)$, it suffices to prove the assertion for $A = K(G)$ (then use the triangle inequality). Let $f \in K(G)$ and $\varepsilon > 0$. Let V be an open neighborhood of the identity in B^- such that $\beta \in V$ implies $\|f^\beta - f\|_G < \varepsilon$ (by 1.0). Cover B^- by finitely many open sets $\beta_i V$ ($i = 1, \dots, n$). Let $B_1 = \beta_1 V$, and by induction, let $B_k = \beta_k V - (B_1 \cup \dots \cup B_{k-1})$. Then B_1, \dots, B_n is a family of disjoint Borel sets covering B^- , and, for each i , $B_i \subset \beta_i V$. If $\lambda_i =$ measure of B_i , then $\lambda_i \geq 0$ and $\sum \lambda_i = 1$. Then for $x \in G$, $|f^\#(x) - \sum \lambda_i f^{\beta_i}(x)| < \varepsilon$, so that $\|f^\# - \sum \lambda_i f^{\beta_i}\|_G < \varepsilon$.

1.5. PROPOSITION. *Let $G \in [FIA]^-_B$, and let A be either $L^1(G)$ or $C^*(G)$. The operation $\#$ on A is a norm-decreasing idempotent onto $Z_B(A)$. The following identities hold.*

- (i) *If $a, b \in A$ and $b \in Z_B(A)$, then $(ab)^\# = a^\#b$, $(ba)^\# = ba^\#$.*
- (ii) *If $a \in A$ and $\beta \in B^-$, then $(a^\beta)^\# = a^\#$.*
- (iii) *If in addition $B \triangleright I(G)$, and $a, b \in A$, then $(ab)^\# = (ba)^\#$.*

Proof. We already know that for $f \in K(G)$, $f^\#$ is B -invariant, so that by continuity $a^\#$ is B -invariant for any $a \in A$. If $a \in Z_B(A)$, then the convex hull of $B^-[a]$ reduces

to $\{a\}$, so $a^\# = a$ by 1.4. To verify the identities in A it suffices to do so in $K(G)$. (i) follows immediately from the first identity in 1.1, while (ii) follows from invariance of Haar measure on B^- . For (iii), recall that in this case G is unimodular, and let $f, g \in K(G)$. Applying the Fubini theorem, we get

$$(g * f)^\#(x) = \int_{B^-} \int_G g(t^{-1})f(t \cdot \beta^{-1}x) dt d\beta = \int_G g(t^{-1}) \int_{B^-} f(t \cdot \beta^{-1}x) d\beta dt.$$

Let $\gamma(y) = t^{-1}yt$ for each $y \in G$; then $\gamma \in I(G) \subset B$, so translating by γ^{-1} we have in the inner integral

$$\int_{B^-} f(t \cdot \beta^{-1}x) d\beta = \int_{B^-} f(t \cdot \gamma\beta^{-1}x) d\beta = \int_{B^-} f(\beta^{-1}x \cdot t) d\beta.$$

Consequently we have, for the entire integral,

$$(g * f)^\#(x) = \int_{B^-} \int_G f(\beta^{-1}x \cdot t)g(t^{-1}) dt d\beta = (f * g)^\#(x).$$

1.6. COROLLARY. *If $G \in [FIA]_B^-$, then $Z_B(K(G))$ is dense in $Z_B(L^1(G))$ and in $Z_B(C^*(G))$, in their respective norms. In particular, if $G \in [FIA]^-$, then the center of $L^1(G)$ is $\|\cdot\|_*$ -dense in the center of $C^*(G)$.*

Proof. Let A be either $L^1(G)$ or $C^*(G)$. Let $a \in Z_B(A)$, and let (f_n) be a sequence of continuous functions with compact support such that $\|f_n - a\| \rightarrow 0$. Then by 1.5, $\|f_n^\# - a^\#\| = \|f_n^\# - a\|$ approaches 0, and $f_n^\# \in Z_B(K(G))$.

Given the existence of the $\#$ operator, the next corollary can be proved as Segal proved the analogous statement for groups of the form $A \times K$ (A abelian, K compact) in [24, Theorem 1.7]. This corollary is a special case of the more general result 4.7.

1.7. COROLLARY. *If $G \in [FIA]^-$, then $L^1(G)$ is strongly semisimple; that is, the intersection of all regular maximal two-sided ideals is (0) .*

We gather here some more or less obvious facts which will be useful to us later. The first assertion of 1.9 follows from the Fubini theorem and 1.8.

1.8. LEMMA. *Let G be a locally compact group, let ϕ be a bounded continuous function on G , and let $\alpha \in \text{Aut}(G)$ be measure-preserving. Then $P[\phi^\alpha](h) = P[\phi](h^{\alpha^{-1}})$ for $h \in L^1(G)$. If in addition ϕ is positive-definite (so that $P[\phi]$ extends to a positive linear functional on $C^*(G)$), then the same statement is true for $h \in C^*(G)$.*

1.9. PROPOSITION. *Let $G \in [FIA]_B^-$. If $h \in L^1(G)$ and ϕ is a bounded continuous function on G , then*

$$P[\phi^\#](h) = \int_{B^-} P[\phi^\beta](h) d\beta = P[\phi](h^\#).$$

If therefore ϕ is also B -invariant, then $P[\phi](h) = P[\phi](h^\#)$ for all $h \in L^1(G)$. Consequently, if ϕ, ψ are bounded continuous B -invariant functions such that $P[\phi], P[\psi]$

agree on $Z_B(L^1(G))$, then $\phi = \psi$. If ϕ, ψ are in addition positive-definite, then all the above statements hold with $C^*(G)$ replacing $L^1(G)$ everywhere.

Let \mathcal{C}_B denote the set of continuous bounded B -invariant functions on G , and let \mathcal{P}_B denote the set of continuous positive-definite B -invariant functions on G . If E is any normed linear space, we shall denote its dual by E' .

1.10. COROLLARY. Let $G \in [FIA]_B^-$. The map $\phi \rightarrow P[\phi]|_{Z_B(L^1(G))}$ is an isometry of \mathcal{C}_B onto its image in $Z_B(L^1(G))'$, and a homeomorphism for the weak topologies $\sigma(L^\infty, L^1), \sigma(Z_B(L^1(G))', Z_B(L^1(G)))$. Similarly, the map $\phi \rightarrow P[\phi]|_{Z_B(C^*(G))}$ is an isometry and a homeomorphism of \mathcal{P}_B onto its image in $Z_B(C^*(G))'$.

2. **B -characters.**

2.0. Let G be a locally compact group. If ϕ is any continuous positive-definite function on G , and A is either $L^1(G)$ or $C^*(G)$, we shall denote by $M(\phi)$ the closed left ideal $\{a \in A \mid P[\phi](a^*a) = 0\}$. It is well known (see for example [4, 13.4.5]) that if $A/M(\phi)$ is completed in the ϕ -norm $\langle [a], [b] \rangle = P[\phi](b^*a)$ to the Hilbert space H_ϕ , one obtains a strongly continuous unitary representation ρ of G on H_ϕ which extends the representation $\rho_x[a] = [a_x^{-1}]$. There is a unique vector $u \in H_\phi$ such that $P[\phi](a) = \langle [a], u \rangle$ for each $a \in A$, and $[a] = \rho(a)u$ for each $a \in A$. Thus u is a cyclic vector for ρ , and for each $x \in G, \phi(x) = \langle \rho_x u, u \rangle$. Moreover, $P[\phi](a) = \langle \rho(a)u, u \rangle$ for each $a \in A$; finally, $a \in M(\phi)$ iff $[a] = 0$, that is, $a \in M(\phi)$ iff $\rho(a)u = 0$.

2.1. Suppose now that B is a subgroup of $\text{Aut}(G)$, and that ϕ is also B -invariant. From 1.1 it follows that if $a, b \in A$, then $P[\phi]((b^\beta)^*a^\beta) = \Delta(\beta)^2 P[\phi](b^*a)$. This shows first that $M(\phi)$ is B -stable. Moreover, the strongly continuous representation $\beta \rightarrow U^\beta$ of B on A (see 1.0) then induces a strongly continuous representation of B on H_ϕ (we shall also denote this representation by $\beta \rightarrow U^\beta$) having the property that $U^\beta/\Delta(\beta)$ is a unitary operator. The vector u obtained above has the property that $U^\beta u = \Delta(\beta)u$ for each $\beta \in B$. For if $\beta \in B$ and $a \in A$ is arbitrary then by B -invariance of ϕ we have

$$\begin{aligned} \langle [a], U^\beta u \rangle &= \Delta(\beta)^2 \langle U^{\beta^{-1}}[a], u \rangle = \Delta(\beta)^2 \langle [a^{\beta^{-1}}], u \rangle \\ &= \Delta(\beta)^2 P[\phi](a^{\beta^{-1}}) = \Delta(\beta) P[\phi](a) = \Delta(\beta) \langle [a], u \rangle = \langle [a], \Delta(\beta)u \rangle. \end{aligned}$$

We shall denote by $H^\#$ the set of elements $v \in H = H_\phi$ such that $U^\beta v = \Delta(\beta)v$ for each $\beta \in B$. It is clear that $H^\#$ is a closed subspace of H , and $\dim H^\# \geq 1$ since $u \in H^\#$. If B is a group of measure-preserving automorphisms, then $H^\#$ is the set of B -invariant elements of H ; by analogy to the case $\Delta(B) = 1$, we shall always call $H^\#$ the B -fixed subspace of H . We denote by L the von Neumann algebra in $B(H_\phi)$ generated by $\{\rho_x \mid x \in G\} \cup \{U^\beta \mid \beta \in B\}$, and by $L^\#$ the commutant of L . It is easy to see that $\dim L^\# = 1$ iff the only closed subspaces of H simultaneously stable under ρ and U are (0) and H_ϕ .

One obtains, of course, two different Hilbert spaces by taking $A = L^1(G)$, $A = C^*(G)$, but we remark that the canonical inclusion of $L^1(G)$ in $C^*(G)$ induces a unitary isomorphism for the two spaces, intertwining with both ρ and U .

2.2. If $B \supset I(G)$ then $M(\phi)$ has another interpretation; in fact $M(\phi) = \ker \rho$. For if $\rho(a) = 0$ then $\rho(a)u = 0$, and therefore $a \in M(\phi)$. Conversely, if $a \in M(\phi)$ then $a^\beta \in M(\phi)$ for each $\beta \in B$, and in particular for each inner automorphism $\beta(t) = x^{-1}tx$. Thus $\rho(a^\beta)u = \Delta(\beta)\rho_x^{-1}\rho(a)\rho_x u = 0$ for each $x \in G$ (see the identity in 1.2), and since u is a ρ -cyclic vector, $\rho(a) = 0$.

If $B = I(G)$ in the above, and G is unimodular, one obtains the "double unitary representations" defined and extensively studied by Godement in [9]. Some of Godement's major results can be extended, with minor changes in the preliminary definitions and proofs, to the nonunimodular case as well. (Take " V_x " to be the operator such that $V_x[g] = \delta(x)[g^x]$, " V_f " to be the operator $\int f(x)V_x^{-1} dx$.) For example, $L^\#$ is precisely the center of the von Neumann algebra generated by $\{\rho_x \mid x \in G\}$, so that $L^\#$ is one-dimensional iff ρ is a factor representation. Moreover, since H contains a ρ -cyclic invariant ("central") vector u , the factor induced by ρ is a finite factor [9, Théorème 5]. Our preliminary results in this section extend some of the techniques and results of [9].

2.3. LEMMA. *Let G be a locally compact group, and B a subgroup of $\text{Aut}(G)$. Then*

- (i) *If $x \in G$ and $\beta \in B$, then $U^\beta \rho_x U^{\beta^{-1}} = \rho_{\beta(x)}$.*
- (ii) *If $a \in A$ and $\beta \in B$, then $U^\beta \rho(a) U^{\beta^{-1}} = \Delta(\beta^{-1})\rho(a^\beta)$.*

Proof. We shall prove (ii) as an example. It suffices to consider $a = f \in L^1(G)$, and since the relation is preserved under the reduction $A \rightarrow A/M(\phi)$, it suffices to verify the corresponding relation when both sides are applied to an arbitrary $g \in L^1(G)$. But then by 1.1 we have (with some abuse of notation)

$$U^\beta \rho(f) U^{\beta^{-1}} g = U^\beta (\rho(f) g^{\beta^{-1}}) = (f * g^{\beta^{-1}})^\beta = \Delta(\beta)^{-1} f^\beta * g = \Delta(\beta^{-1}) \rho(f^\beta) g.$$

2.4. Let G be locally compact, and B a subgroup of $\text{Aut}(G)$. Let $\Phi_B = \Phi_B(G)$ denote the set of continuous positive-definite B -invariant functions ϕ on G with $\phi(1) \leq 1$ (here we make an exception to the convention mentioned in the introduction; we shall always denote by $\Phi = \Phi(G)$ the set of continuous positive-definite functions ϕ on G with $\phi(1) \leq 1$). It is easy to see that Φ_B is a convex and weak*-compact subset of the convex compact set Φ . By the Kreĭn-Milman theorem, therefore, Φ_B is the closed convex hull of the set of its extreme points. We shall denote by $\mathfrak{X}_B = \mathfrak{X}_B(G)$ the set of nonzero extreme points, and call the elements of \mathfrak{X}_B the B -characters. The set \mathfrak{X} of (Godement) characters (in the terminology of [9] and [4], the characters of finite type) is thus the set of nonzero extreme points of the set of continuous *central* positive-definite functions ϕ on G with $\phi(1) \leq 1$.

If we introduce the standard partial ordering in Φ_B (" $\phi \leq \psi$ " means " $\psi - \phi \in \Phi_B$ ") and the notion of "elementary" relative to this ordering, then the usual proof shows that, for $\phi \in \Phi_B$, ϕ is an element of \mathfrak{X}_B iff ϕ is elementary in Φ_B and $\phi(1) = 1$. Using this fact, it is easy to modify the standard Gelfand-Raikov arguments to prove the following proposition.

2.5. PROPOSITION. *Let G be locally compact, and B a subgroup of $\text{Aut}(G)$. Let $\phi \in \Phi_B$ with $\phi(1)=1$. Then $\phi \in \mathfrak{X}_B$ iff $\dim L^\# = 1$ (that is, iff the only subspaces invariant under the representations ρ and U of 2.0 and 2.1 are (0) and H_ϕ).*

Proof. The only difficulty in modifying the standard proof comes in showing that if $\psi \in \Phi_B$ and $\psi \leq \phi$, then there is a positive selfadjoint operator $T \in B(H)$ of norm ≤ 1 such that $T \in L^\#$ and $\psi(x) = \langle T\rho_x u, u \rangle$. In the usual way one can find $T \in B(H)$ with $\|T\| \leq 1$, T positive and selfadjoint, such that for arbitrary $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \in \mathbb{C}, x_1, \dots, x_m, y_1, \dots, y_n \in G$, we have

$$\left\langle T \sum_{i=1}^m \lambda_i \rho_{x_i} u, \sum_{j=1}^n \mu_j \rho_{y_j} u \right\rangle = \sum_{i,j} \lambda_i \bar{\mu}_j \psi(y_j^{-1} x_i).$$

Hence if $\beta \in B$ is arbitrary, and $x, y \in G$, then by 2.3 and 2.1 we have

$$\begin{aligned} \langle U^{\beta^{-1}} T U^\beta \rho_x u, \rho_y u \rangle &= \Delta(\beta)^{-2} \langle T U^\beta \rho_x u, U^\beta \rho_y u \rangle = \Delta(\beta)^{-2} \langle T \rho_{\beta(x)} U^\beta u, \rho_{\beta(y)} U^\beta u \rangle \\ &= \langle T \rho_{\beta(x)} u, \rho_{\beta(y)} u \rangle \psi(\beta(y^{-1} x)) = \psi(y^{-1} x) = \langle T \rho_x u, \rho_y u \rangle, \end{aligned}$$

since ψ is B -invariant and $U^\beta u = \Delta(\beta)u$. The linearity properties of the inner product then show that $U^{\beta^{-1}} T U^\beta = T$ on a dense subspace of H (the space generated by $\{\rho_x u \mid x \in G\}$) and therefore on all of H . Since in addition the usual calculation shows that $\rho_x T = T \rho_x$ for all $x \in G$, we have $T \in L^\#$.

If $B=I(G)$, then the above proposition shows (in view of the remarks in 2.2) that $\phi \in \mathfrak{X}$ iff the representation ρ induced by ϕ is a finite factor representation. In particular, suppose G is a type I group, and let $\phi \in \mathfrak{X}$. Then the von Neumann algebra generated by $\{\rho_x \mid x \in G\}$ is a type I finite factor, hence is finite dimensional (see for example J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier-Villars, Paris, 1957, pp. 97, 121). Since H_ρ contains a cyclic vector, H_ρ is also finite dimensional.

Let $\mathfrak{X}_{\text{fin}}$ be the set of $\phi \in \mathfrak{X}$ such that the representation ρ induced by ϕ is finite dimensional. Then we have just shown that for any type I group, $\mathfrak{X}(G) = \mathfrak{X}_{\text{fin}}(G)$. Now $\mathfrak{X}_{\text{fin}}$ is easy to calculate in many instances. In fact, if π is a continuous finite-dimensional irreducible unitary representation of G (briefly, $\pi \in \hat{G}_{\text{fin}}$) with character χ_π and dimension d_π , then χ_π/d_π is an element of $\mathfrak{X}_{\text{fin}}$, and the map $\pi \rightarrow \chi_\pi/d_\pi$ induces a bijection $sp: \hat{G}_{\text{fin}} \rightarrow \mathfrak{X}_{\text{fin}}$. (This fact was proved in [9, pp. 72-75] for unimodular groups; with minor modifications, the proof is valid in the general case also.) Therefore, if G is any type I group, then \mathfrak{X} is the set of normalized characters (in the usual sense) of the elements of \hat{G}_{fin} . In particular, this holds for any group all of whose continuous irreducible unitary representations are finite dimensional (so that $\hat{G} = \hat{G}_{\text{fin}}$), since any such group is evidently a CCR group, and therefore type I [4, 5.5.2].

2.6. PROPOSITION. *Let G be locally compact, B a subgroup of $\text{Aut}(G)$, and $\phi \in \Phi_B$ with $\phi(1)=1$. Let A be either $L^1(G)$ or $C^*(G)$. If $\dim H^\# = 1$, then $\dim L^\# = 1$ (that is, $\phi \in \mathfrak{X}_B$), and $P[\phi]|Z_B(A)$ is multiplicative.*

Proof. Suppose $\dim H^\# = 1$, and let $T \in L^\#$. Then $Tu \in H^\#$, since for any $\beta \in B$, $U^\beta Tu = TU^\beta u = \Delta(\beta)Tu$. Therefore $Tu = \lambda(T)u$ for some $\lambda(T) \in \mathbb{C}$. Since $T\rho_x = \rho_x T$ for all $x \in G$, we have $T\rho_x u = \rho_x Tu = \lambda(T)\rho_x u$, so by linearity and continuity T agrees with $\lambda(T)I$ on H (since u is a ρ -cyclic vector). Therefore $\dim L^\# = 1$. If $a \in Z_B(A)$, then $U^\beta a = \Delta(\beta)a$, so $[a] \in H^\#$; but by 2.0, $[a] = \rho(a)u$, so $\rho(a)u = \lambda(a)u$ for some $\lambda(a) \in \mathbb{C}$. Since ρ is a homomorphism, λ is a multiplicative linear functional on $Z_B(A)$. But $P[\phi]|_{Z_B(A)}$ agrees with λ , since for $a \in Z_B(A)$ we have

$$P[\phi](a) = \langle \rho(a)u, u \rangle = \langle \lambda(a)u, u \rangle = \lambda(a)\phi(1) = \lambda(a).$$

Suppose now that $B \supset I(G)$, and let $\phi \in \Phi_B$. Then ϕ is a central function, so that one can define as in [9] the bounded elements in H_ϕ : these are the elements $w \in H$ such that there is an operator $V_w \in B(H)$ satisfying $V_w[a] = \rho(a)w$ for all $a \in A$. If H^b denotes the vector subspace of bounded elements in H , then H^b is dense in H , and it is easy to modify the arguments in [9] to show that: (i) $H^b \cap H^\#$ is dense in $H^\#$; and (ii) $w \rightarrow V_w$ is a faithful linear map of $H^b \cap H^\#$ into $L^\#$. In particular, if $\dim L^\# = 1$, then $\dim (H^b \cap H^\#) \leq 1$, so $\dim H^\# \leq 1$; but $0 \neq u \in H^\#$, so $\dim H^\# = 1$. In view of 2.5 and 2.6 we therefore have the following proposition (a partial converse to 2.6).

2.7. PROPOSITION. *Let G be locally compact, and B a subgroup of $\text{Aut}(G)$ with $B \supset I(G)$. If $\phi \in \Phi_B$ with $\phi(1) = 1$, then the following are equivalent.*

- (i) $\phi \in \mathfrak{X}_B$.
- (ii) $\dim L^\# = 1$.
- (iii) $\dim H^\# = 1$.

One cannot, however, obtain a complete converse to 2.6, even for $B = I(G)$, as the following example shows.

2.8. EXAMPLE. Let G be the (discrete) free group on two generators. Then each conjugacy class in G is infinite, so that the center of $L^p(G)$ ($1 \leq p < \infty$) reduces to $\mathbb{C}\delta$, where δ is the identity in $L^1(G)$. If ϕ is any bounded function on G with $\phi(1) = 1$, then $P[\phi]$ is trivially multiplicative when restricted to the center of L^1 . Now the remarks following 2.5 show that any normalized irreducible finite-dimensional character of G is an element of \mathfrak{X} , so that one can certainly choose two distinct elements of \mathfrak{X} , say θ and ψ . If $\phi = (\theta + \psi)/2$, then ϕ is central and positive-definite, and $\phi(1) = 1$, but ϕ is not an element of \mathfrak{X} .

We conclude this section with a proposition about the relationship between B -characters of groups and B -characters of subgroups; this proposition generalizes Lemma 14 of [25]. If G is a group, N a subgroup of G , and f a mapping with domain G , we denote by f_N the restriction of f to N . If B is a group of automorphisms of G and N is B -invariant, then β_N is an automorphism of N for any $\beta \in B$; we shall denote by B_N the group of restrictions β_N .

2.9. PROPOSITION. *Let G be locally compact, and B a subgroup of $\text{Aut}(G)$. Let $N \subset G$ be an open B -invariant subgroup. If $\psi \in \mathfrak{X}_{B_N}(N)$, then there exists a $\chi \in \mathfrak{X}_B(G)$*

such that $\psi = \chi_N$. Conversely, if in addition $B \supset I(G)$, then χ_N is a B_N -character of N whenever χ is a B -character of G , and $\chi \rightarrow \chi_N$ maps $\mathfrak{X}_B(G)$ continuously onto $\mathfrak{X}_{B_N}(N)$ (in the topologies of uniform convergence on compacta).

Proof. Let $\psi \in \mathfrak{X}_{B_N}(N)$, and let ψ^\sim be the continuous function on G agreeing with ψ on N and vanishing elsewhere. Then ψ^\sim is a continuous B -invariant function on G , and an easy calculation shows that it is positive-definite. Let K be the set of $\phi \in \Phi_B(G)$ such that $\phi_N = \psi$. Then K is nonempty (since $\psi^\sim \in K$) and clearly convex and weak*-compact, hence contains extreme points. We shall show that any extreme point χ of K is actually extreme in $\Phi_B(G)$, thus showing that there exists a $\chi \in \mathfrak{X}_B$ such that $\chi_N = \psi$. For suppose $\chi = \lambda\phi + \mu\theta$ with $\phi, \theta \in \Phi_B(G)$ and λ, μ positive non-zero numbers with $\lambda + \mu = 1$. Then $\psi = \chi_N = \lambda\phi_N + \mu\theta_N$, and ϕ_N, θ_N are elements of $\Phi_{B_N}(N)$. Since ψ is extreme in $\Phi_{B_N}(N)$, we must have $\phi_N = \theta_N = \psi$, so that ϕ and θ are elements of K . But χ is extreme in K , so $\phi = \theta = \chi$, and therefore $\chi \in \mathfrak{X}_B(G)$.

Now, since N is open, Haar measure on N can be taken as the restriction of Haar measure on G . This implies that the modular function $\delta: G \rightarrow \mathbf{R}^+$ when restricted to N coincides with the modular function on N , and also, since N is B -invariant, that the modulus $\Delta(\beta)$ of $\beta \in B \subset \text{Aut}(G)$ equals the modulus of $\beta_N \in \text{Aut}(N)$. If now $f \in K(N)$, then the function f^\sim defined as above is an element of $K(G)$ (since N is open), and by the above, the map $f \rightarrow f^\sim$ extends to an isometric *-algebra homomorphism of $L^1(N) \rightarrow L^1(G)$. Moreover, if ϕ is any bounded continuous function on G , and $f \in L^1(N)$, then $P[\phi_N](f) = P[\phi](f^\sim)$. Suppose now that $\phi \in \Phi_B(G)$; let $M(\phi) = \{f \in L^1(G) \mid P[\phi](f^\vee * f) = 0\}$, and

$$N(\phi) = \{f \in L^1(N) \mid P[\phi_N](f^\vee * f) = 0\}.$$

Then $f \rightarrow f^\sim$ induces a linear isometry of $L^1(N)/N(\phi) \rightarrow L^1(G)/M(\phi)$ in the respective prehilbert norms, and this extends to an isometry θ of H_{ϕ_N} into H_ϕ . Moreover, θ maps the B_N -fixed subspace of H_{ϕ_N} into the B -fixed subspace of H_ϕ . For if $f \in L^1(N)$ is arbitrary, then

$$\theta U^{\beta_N}[f] = \theta[f^{\beta_N}] = [(f^{\beta_N})^\sim] = [(f^\sim)^\beta] = U^\beta \theta[f]$$

so that $\theta U^{\beta_N} = U^\beta \theta$ on H_{ϕ_N} . Therefore if v is in the B_N -fixed subspace of H_{ϕ_N} we have

$$U^\beta \theta(v) = \theta U^{\beta_N}(v) = \theta(\Delta(\beta_N)v) = \Delta(\beta)\theta(v)$$

so $\theta(v)$ is in the B -fixed subspace of H_ϕ .

Assume now that $B \supset I(G)$. Then if $\chi \in \mathfrak{X}_B(G)$, we shall show that $\chi_N \in \mathfrak{X}_{B_N}(N)$. For the B -fixed subspace of H_χ is one-dimensional, by 2.7, so that B_N -fixed subspace of H_{χ_N} , which is always nonzero, must by the above considerations also be one-dimensional. Consequently, by 2.7 we have $\chi_N \in \mathfrak{X}_{B_N}(N)$. The first half of the proof shows then that the map $\chi \rightarrow \chi_N$ of $\mathfrak{X}_B(G) \rightarrow \mathfrak{X}_{B_N}(N)$ is surjective. The continuity of this map is obvious.

3. B -characters and separation properties. Our first lemma can be proved exactly as in [4, 13.6.4], using the Kreĭn-Milman theorem applied to Φ_B .

3.1. LEMMA. *Let G be locally compact, and let B be a subgroup of $\text{Aut}(G)$. Let $\phi \in \Phi_B$ with $\phi(1)=1$ (see 2.4). Then ϕ is the uniform limit on compacta of G of convex combinations $\sum_{i=1}^n \mu_i \chi_i$, where $\chi_i \in \mathfrak{X}_B$ for each i , and $\mu_i \geq 0$, $\sum_{i=1}^n \mu_i = 1$.*

The following theorem generalizes (3.3) of [13], and as in [13] will be used to prove the main theorem about separation properties.

3.2. THEOREM. *Let $G \in [FIA]_{\bar{B}}$, let F be a compact set in G and f a continuous B -invariant function. For each $\varepsilon > 0$ there exist characters $\chi_1, \dots, \chi_n \in \mathfrak{X}_B$ and complex numbers $\lambda_1, \dots, \lambda_n$ such that $\|f - \sum_{i=1}^n \lambda_i \chi_i\|_F < \varepsilon$.*

Proof. Given $\varepsilon > 0$, there are complex numbers c_1, \dots, c_r and $\phi_1, \dots, \phi_r \in \Phi_0$ (see introduction) such that $\|f - \sum_{j=1}^r c_j \phi_j\|_F < \varepsilon$ (see [7, p. 47]). We may assume that F is B -invariant, for if not it may be replaced by the larger compact set $B^-[F]$. By 1.3, application of the $\#$ operator yields $\|f - \sum_j c_j \phi_j^\#\|_F < \varepsilon$, since $f^\# = f$. Now $\phi_j(1) = 1$ for each j , hence $\phi_j^\#(1) = 1$; consequently we may apply the lemma to each $\phi_j^\#$ and obtain the required result.

We remark that in case $B \supset I(G)$ the above proof can be somewhat simplified. We shall see in 5.1 that in this case $\phi \in \Phi_0$ implies $\phi^\# \in \mathfrak{X}_B$, so use of the lemma is unnecessary.

The next theorem has been proved for $[Z]$ groups by Grosser and Moskowitz in [13, (3.4)], for discrete $[FIA]_{\bar{B}}$ groups with $B \supset I(G)$ by Thoma in [26, Korollar 4], and for $[FIA]^-$ groups by Kaniuth [16a, Lemma]. In their proofs Thoma and Kaniuth use Bochner-Plancherel theorems which seem to depend on the commutativity of the algebra of B -invariant elements considered (see also [8]). Since we are not assuming that $Z_B(K(G))$ is commutative, such a Bochner theorem seems unavailable; we have, however, various applications of the Kreĭn-Milman theorem at hand (compare 2.9 and 3.1), and our proof will use these.

3.3. THEOREM. *Let $G \in [SIN]_B$, and $x, y \in G$. If $B[x]^-$ is compact and $y \notin B[x]^-$, then there exists $\phi \in \mathfrak{X}_B(G)$ such that $\phi(x) \neq \phi(y)$.*

Proof. Following a method of Thoma in [25], we shall consider the set F of elements in G whose B -orbits have compact closures. F is a B -invariant subgroup of G , and F is open since G contains a compact B -invariant neighborhood of 1. Moreover, if B_F is the group of restrictions to F of the automorphisms in B , it is clear that $F \in [SIN]_{B_F} \cap [FC]_{\bar{B}_F}$. By 0.2, B_F^- is compact; that is, $F \in [FIA]_{\bar{B}_F}$.

Suppose first that $y \in F$. Then $B_F[y]^- = B_F^-[y]$ is disjoint from $B[x]^- = B_F^-[x]$, and therefore by Urysohn's lemma there is an $f \in K(F)$ such that f restricted to $B_F^-[x]$ identically equals 1, and f vanishes on $B_F^-[y]$. Then $f^\#(x) = 1$ and $f^\#(y) = 0$ (where $\#$ is taken with respect to B_F^-). Since, by 3.2, f can be uniformly approximated on $B_F[x]^- \cup B_F[y]^-$ by linear combinations of B_F -characters, there is a

$\chi \in \mathfrak{X}_{B^F}(F)$ with $\chi(x) \neq \chi(y)$. If $\phi \in \mathfrak{X}_B(G)$ is chosen (by 2.9) such that $\phi_F = \chi$, then $\phi(x) \neq \phi(y)$.

Suppose now that $y \notin F$. Let V be a fixed compact symmetric B -invariant neighborhood of the identity, and let $K = V \cup B[x]^- V \cup VB[x^{-1}]^-$. If f is the characteristic function of K , then an easy calculation shows that $f * f$ is a continuous B -invariant positive-definite function on G . Moreover, $f * f(t) = \text{Haar measure of } K \cap t^{-1}K$, so $f * f(x) > 0$ (since $V \subset K \cap x^{-1}K$) while $f * f(y) = 0$ (since $K \subset F$). Lemma 3.1 now shows that there is a B -character ϕ on G with $\phi(x) \neq \phi(y)$.

As a corollary we get a proposition which was first proved for $B = I(G)$ and G unimodular by Godement [9, p. 54]. For $B = (1)$ it is essentially the Gelfand-Raikov theorem about "sufficiently many" irreducible unitary representations for G .

3.4. COROLLARY. *Let $G \in [SIN]_B$, $x \in G$, $x \neq 1$. Then there exists $\phi \in \mathfrak{X}_B(G)$ with $\phi(x) \neq 1$.*

In Theorem 3.3, the hypothesis that $B[x]^-$ is compact cannot be dropped, as the following example shows.

3.5. EXAMPLE. Let G be the discretely topologized group $GL_2(K)$, where K is an infinite discrete field. Kirillov has shown [18] that $\chi \in \mathfrak{X}(G)$ iff χ has one of the following two forms: (i) χ is identically zero off $Z(G)$, the center of G , and χ restricted to the center is a character of $Z(G)$, or (ii) $\chi(t) = \pi(\det t)$ for any $t \in G$, where π is a multiplicative character of K . Now, let s and t be two nonscalar matrices with the same determinant but different traces. Then s and t are not in $Z(G)$, so all characters of type (i) vanish at s and t . Similarly, all characters of type (ii) agree on s and t . It is clear on the other hand, that the classes of s and t are infinite.

We now turn our attention to $L^1(G)$ and $C^*(G)$. Recall that, for $G \in [SIN]_B$, each $\beta \in B$ preserves Haar measure, so that $Z_B(A)$ is just the set of B -invariant elements of A when $A = L^p(G)$ or $C^*(G)$ (see discussion following 1.2). When $B = (1)$ the following theorem yields a form of the Gelfand-Raikov theorem, asserting that $L^1(G)$ possesses "sufficiently many" irreducible $*$ -representations.

3.6. THEOREM. *Let $G \in [SIN]_B$, and let $f \in L^1(G)$.*

(i) *If $P[\phi] (f^\vee * f) = 0$ for all $\phi \in \mathfrak{X}_B$, then $f = 0$.*

(ii) *If f is B -invariant and $P[\phi](f) = 0$ for all ϕ in \mathfrak{X}_B , then $f = 0$.*

If G is amenable, or B^- is compact, then similar statements hold for $a \in C^(G)$.*

Proof. Let T denote the regular representation of G on $L^2(G)$ (as well as the extension of this representation to $L^1(G)$ and $C^*(G)$). We notice first that if $a \in C^*(G)$ and $T(a)g = 0$ for all $g \in Z_B(L^2(G))$, then $T(a) = 0$. For if $h \in K(G)$ is arbitrary and $g \in Z_B(K(G))$, then $T(a)(g * h) = (T(a)g) * h = 0$ (the first relation holds when a is an L^1 -function, and extends by continuity to $C^*(G)$). Since, by 0.1, (KG) contains a B -invariant approximate identity for $L^2(G)$, $T(a)h = 0$ for all $h \in K(G)$, hence $T(a) = 0$. If a is actually an L^1 -function, or if G is amenable, this implies that $a = 0$. (For the

equivalence of amenability with the fact that T extends to $C^*(G)$ with trivial kernel, see for example, F. P. Greenleaf, *Invariant means on locally compact groups and their applications*, Van Nostrand, Princeton, N. J., 1969.)

The Kreĭn-Milman theorem shows that we may assume in (i) and (ii) that ϕ is any continuous, positive-definite B -invariant function. In particular, we may assume that ϕ is of the form $\phi(x)=(T_x g, g)$, where $g \in Z_B(L^2(G))$, since an easy calculation shows that such a ϕ is continuous, positive-definite, and B -invariant. If $a \in C^*(G)$ and $P[\phi](a^*a)=(T(a)g, T(a)g)=0$ for all $g \in Z_B(L^2(G))$, then by our first remarks $T(a)=0$. Similarly, if a is B -invariant and $P[\phi](a)=(T(a)g, g)=0$ for all such g , then $T(a)=0$ (we may apply the polarization identity since 1.1(i) shows that $T(a)$ maps $Z_B(L^2(G))$ into itself). This proves (i) and (ii) for $f \in L^1(G)$, and also for $a \in C^*(G)$ when G is amenable.

Suppose on the other hand that B^- is compact, and let T be a faithful $*$ -representation of $C^*(G)$ on a Hilbert space H . Let $v \in H$ be arbitrary, and let $\phi(x)=(T_x v, v)$ (where $x \rightarrow T_x$ is the corresponding continuous unitary representation of G —compare [4, 13.9.3]). Then $\phi^\#$ is continuous, positive-definite, and B -invariant. Now, let $a \in C^*(G)$ and suppose $P[\phi^\#](a^*a)=0$; then, by 1.9, $P[\phi^\#](a^*a)=\int P[\phi^\beta](a^*a) d\beta$ (integral over B^-), and since by 1.0 the integrand is a positive continuous function of β , we must have in particular $P[\phi](a^*a)=(T(a)v, T(a)v)=0$. Since v is arbitrary, $a=0$. Similarly, suppose that $a \in Z_B(C^*(G))$ and $P[\phi^\#](a)=0$; then $P[\phi](a)=P[\phi](a^\#)=P[\phi^\#](a)=0$, by 1.9, so that $(T(a)v, v)=0$. It follows as before that $a=0$.

We remark that for $[FIA]_B^-$ groups, $B \supset I(G)$, the second half of the proof is actually unnecessary. In fact, Leptin has shown [19] that if $G \in [FC]^-$ then G is amenable.

3.7. COROLLARY. *Let $G \in [FIA]_B^-$, and $a \in C^*(G)$. If $(a^*a)^\# = 0$, then $a = 0$.*

Proof. If $(a^*a)^\# = 0$, then, by 1.9, $P[\phi](a^*a) = P[\phi]((a^*a)^\#) = 0$ for all $\phi \in \mathfrak{X}_B$. Therefore $a = 0$.

Theorem 3.6 has certain measure-theoretic consequences.

3.8. PROPOSITION. *Let $G \in [SIN]_B$, and let $x \in G$ have the following properties: $B[x]$ is measurable with measure $m > 0$, and $B[x]^-$ has finite measure m_1 . Then $m = m_1$.*

Proof. Let θ, θ_1 be the characteristic functions of $B[x], B[x]^-$ respectively. Then by hypothesis $\theta, \theta_1 \in Z_B(L^1(G))$. Moreover, for each $\phi \in \mathfrak{X}_B$,

$$m^{-1}P[\phi](\theta) = (m_1)^{-1}P[\phi](\theta_1) = \phi(x)$$

so that by 3.6 we must have $m^{-1}\theta = (m_1)^{-1}\theta_1$ except on a set T of measure 0. Since $m > 0$, $B[x]$ cannot be contained in T , so there exists a point y of $B[x]$ such that $m^{-1}\theta(y) = (m_1)^{-1}\theta_1(y)$. But $\theta(y) = \theta_1(y) = 1$, so $m = m_1$.

3.9. COROLLARY. Let $G \in [SIN]_B$, and let $x, y \in G$ be such that

- (i) $B[x] \cap B[y] = \emptyset$.
- (ii) $B[x], B[y]$ are measurable with measures $m_x, m_y > 0$, and $B[x]^-$ has finite measure.

Then $B[x]^- \cap B[y]^- = \emptyset$.

Before going to the proof, we remark that for $G \in [SIN]_B$ the relation “ $t \in B[x]^-$ ” is an equivalence relation. For clearly the relation is reflexive and transitive, so it suffices to show that it is symmetric. Let $t \in B[x]^-$, let W be a neighborhood of x , and let V be a B -invariant neighborhood of 1 in G such that $Vx \subset W$. Choose β in B such that $\beta x \in V^{-1}t$ (since $t \in B[x]^-$). Then $x \in \beta^{-1}(V^{-1})\beta^{-1}t = V^{-1}\beta^{-1}t$, so $\beta^{-1}t \in Vx \subset W$, and $x \in B[t]^-$.

Proof of Corollary 3.9. By the above remarks either $B[x]^-$ equals $B[y]^-$ or their intersection is empty; we shall show that the former case cannot occur. For if $B[x]^- = B[y]^-$, then the disjoint union $B[x] \cup B[y]$ is contained in $B[x]^-$. But 3.8 shows that the measure of $B[x]^- = m_x$, so $m_x + m_y \leq m_x$, which contradicts $m_y > 0$.

An argument similar to that in 3.8 allows us also to supplement Theorem 3.3.

3.10. THEOREM. Let $G \in [SIN]_B$, and let $x, y \in G$. If $B[x]^-$, $B[y]^-$ are disjoint and have positive finite measures, then there is a $\phi \in \mathfrak{X}_B$ such that $\phi(x) \neq \phi(y)$.

Proof. Suppose $\phi(x) = \phi(y)$ for all $\phi \in \mathfrak{X}_B$; we shall derive a contradiction. Let θ_x and m_x (resp. θ_y and m_y) denote the characteristic function and measure, respectively, of $B[x]^-$ (resp. $B[y]^-$). Since $\phi(x) = P[\phi](m_x^{-1}\theta_x)$ and $\phi(y) = P[\phi](m_y^{-1}\theta_y)$ for each $\phi \in \mathfrak{X}_B$, by our assumption and 3.6 we have $m_x^{-1}\theta_x = m_y^{-1}\theta_y$ except on a set T of measure 0. Since $m_x > 0$, $B[x]^-$ cannot be contained in T , so there exists $s \in B[x]^-$ such that $m_x^{-1} = m_x^{-1}\theta_x(s) = m_y^{-1}\theta_y(s)$. Now θ_y takes on only the values 0 and 1, so that $\theta_y(s) = 1$ and $s \in B[y]^- \cap B[x]^-$, contradicting the hypothesis.

4. **B -characters and maximal ideals.** In this section we show that the B -characters of an $[FIA]_B^-$ group can be identified with the set of pure states of the subalgebra $Z_B(C^*(G))$ of B -invariant elements in $C^*(G)$. This will allow us to use the powerful theory of C^* -algebras to derive generalizations of some classical theorems about compact and locally compact abelian groups. Before stating our results, however, we recall some facts and establish some notation. If G is locally compact and B is a subgroup of $\text{Aut}(G)$, then \mathfrak{X}_B has the topology of uniform convergence on compacta of G . By a theorem of Yoshizawa [4, 13.5.2] this topology coincides with the weak*-topology on \mathfrak{X}_B regarded as a subset of the dual of $L^1(G)$ (via the map $\phi \rightarrow P[\phi]$), and therefore also with the weak*-topology on \mathfrak{X}_B regarded as a subset of the dual of $C^*(G)$ [4, 2.7.5]. If ϕ is a continuous positive-definite function on G , we shall in this section often use $p[\phi]$ to denote the restriction to $Z_B(C^*(G))$ of the positive linear functional $P[\phi]$.

We shall assume throughout this section that $G \in [FIA]_B^-$. In this case we have observed in 1.10 that the map $\phi \rightarrow p[\phi]$ of Φ_B into the dual of $Z_B(C^*(G))$ is an isometry and a homeomorphism onto its image.

4.1. THEOREM. *Let $G \in [FIA]_B^-$. The map $\phi \rightarrow p[\phi]$ is an isometry and a weak*-homeomorphism of Φ_B onto the set of positive linear functionals on $Z_B(C^*(G))$ of norm ≤ 1 . Moreover, under this map \mathfrak{X}_B is identified with the set of pure states of $Z_B(C^*(G))$.*

Proof. To show the first statement we must show that the map is surjective. Let p be a positive functional of norm ≤ 1 on $Z_B(C^*(G))$. Now $Z_B(C^*(G))$ contains an approximate identity (see 0.1), so p has a positive extension to the algebra obtained by adjoining an identity to $Z_B(C^*(G))$ [4, 2.1.5], and thence to the algebra obtained by adjoining an identity to $C^*(G)$ [4, 2.10.1]. By [4, 2.7.5] this extension actually comes from a positive linear functional on $L^1(G)$, hence there exists a continuous positive-definite function ψ such that $p = P[\psi]|_{Z_B(C^*(G))}$. If we let $\phi = \psi^\#$, then also $p = P[\phi]|_{Z_B(C^*(G))}$ (by 1.9), and thus it follows that p is the restriction to $Z_B(C^*(G))$ of $P[\phi]$ for some $\phi \in \Phi_B$.

Now, let $\phi \in \Phi_B$, and suppose $p[\phi]$ is a pure state; we shall show that ϕ is extreme in Φ_B . For if $\phi = c\phi_1 + d\phi_2$, where $\phi_i \in \Phi_B$ ($i=1, 2$), and c, d are positive nonzero numbers with $c+d=1$, then $p[\phi] = cp[\phi_1] + dp[\phi_2]$, and since $p[\phi]$ is extreme, $p[\phi] = p[\phi_1] = p[\phi_2]$. The injectivity of our map then shows that $\phi = \phi_i$ ($i=1, 2$), and since ϕ is nonzero, $\phi \in \mathfrak{X}_B$. Conversely, let $\phi \in \mathfrak{X}_B$, and suppose that $p[\phi]$ is a convex combination of positive functionals on $Z_B(C^*(G))$ (of norm ≤ 1). Since the map is surjective we can assume that $p[\phi] = cp[\phi_1] + dp[\phi_2]$, with c, d, ϕ_i as before; by the injectivity, we have $\phi = c\phi_1 + d\phi_2$. But ϕ is extreme, so $\phi = \phi_i$ ($i=1, 2$), and therefore $p[\phi]$ is extreme.

Our first corollary follows immediately from the above and the elementary theory of commutative C^* -algebras (see [23, 4.6.12]).

4.2. COROLLARY. *Let $G \in [FIA]_B^-$, with $Z_B(C^*(G))$ commutative. If $\phi \in \Phi_B$, then $\phi \in \mathfrak{X}_B$ iff $p[\phi]$ is multiplicative. Therefore, \mathfrak{X}_B is a locally compact Hausdorff space, homeomorphic with the maximal ideal space of $Z_B(C^*(G))$, and $\mathfrak{X}_B \cup (0)$ (in the weak*-topology) is its one-point compactification.*

We note that, in view of 1.6, $Z_B(C^*(G))$ is commutative iff the algebra of B -invariant functions in $L^1(G)$ (or in $K(G)$) is commutative. Of course, 1.2 shows that a sufficient condition for this is that $B \supset I(G)$; this condition is however not necessary, as simple examples show (compare 4.10 and 4.4).

The next corollary generalizes part of Satz 3 of Kaniuth [16a]; our method of proof is quite different from his.

4.3. COROLLARY. *Let $G \in [FIA]_B^-$, with $Z_B(C^*(G))$ commutative.*

- (i) *G is discrete iff \mathfrak{X}_B is compact.*
- (ii) *If G is compact then \mathfrak{X}_B is discrete.*

Proof. If G is discrete then the characteristic function of the set $\{1\}$ is an identity for $Z_B(C^*(G))$, so \mathfrak{X}_B is compact. Conversely, suppose that \mathfrak{X}_B is compact. We show first that $C^*(G)$ has an identity. For $Z_B(C^*(G))$ is a commutative C^* -algebra with compact maximal ideal space, hence has an identity e (e is of course $*$ -invariant). If now $a \in C^*(G)$, then by 1.5

$$[(ae - a)^*(ae - a)]^\# = e(a^*a)^\#e - (a^*a)^\#e - e(a^*a)^\# + (a^*a)^\#;$$

since the right-hand side vanishes we may apply 3.7 to conclude that $ae = a$, so e is an identity for $C^*(G)$. Now, pick f in $K(G)$ such that $\|e - f\|_* < 1$; then f is invertible with inverse $b \in C^*(G)$. Let T denote the left regular representation of G on $L^2(G)$ (as well as its extension to $C^*(G)$), and let $h = T(b)f$. We show next that $h * g = g$ for any g such that $g = \chi_V$ is the characteristic function of a compact neighborhood V in G . For on the one hand, $T(b)(f * g) = (T(b)f) * g$ (this equality holds for fixed g when b is an arbitrary element of $L^1(G)$, and extends by continuity to $C^*(G)$); thus $h * g = T(b)(f * g) = T(b)T(f)g = T(e)g$. On the other hand, $T(e)$ is the identity operator on $L^2(G)$, since on the dense subspace generated by functions of the form $r * s$ ($r, s \in K(G)$) we have

$$T(e)(r * s) = T(e)T(r)s = T(r)s = r * s.$$

Now suppose that G is not discrete, so that the measures of open sets in G are not bounded away from 0. Then (by absolute continuity of the measure $h_{\chi_W} dx$, for W a compact neighborhood of 1) pick a compact neighborhood U of 1 in G with $\int_U |h(x)| dx < 1$; let V be a symmetric neighborhood of 1 such that $V^2 \subset U$, and let $g = \chi_V$. Then for almost all x in V , we have

$$1 = g(x) = h * g(x) = \left| \int_{xV} h(y) dy \right| \leq \int_U |h(y)| dy < 1,$$

a contradiction. Therefore G is discrete.

Suppose now that G is compact. It is easy to see that, for $f \in K(G)$, $T(f)$ is a compact operator in $L^2(G)$. Since the ideal of compact operators is uniformly closed in $B(L^2(G))$, $T(a)$ is compact for each $a \in C^*(G)$. But T is faithful on $C^*(G)$ [4, 18.3.3], so $Z_B(C^*(G))$ is a commutative C^* -algebra of compact operators on $L^2(G)$. It then follows from the theory of dual C^* -algebras (see I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255, and [4, 4.7.20]) that the maximal ideal space \mathfrak{X}_B of $Z_B(C^*(G))$ is discrete.

We remark that Kaniuth has also proved that for $G \in [FIA]^-$, if $\mathfrak{X}(G)$ is discrete then G is compact; his proof uses only the fact that the constant function 1 is open in $\mathfrak{X}(G)$. Now when $B \supset I(G)$ is relatively compact, it will follow from 5.8 that there is a continuous map from $\mathfrak{X}(G)$ to $\mathfrak{X}_B(G)$ such that the inverse image of 1 is 1. Therefore if 1 is open in $\mathfrak{X}_B(G)$ it is open in $\mathfrak{X}(G)$, and we therefore have a slight extension of Kaniuth's result: *if $G \in [FIA]^-$, $B \supset I(G)$, and $\mathfrak{X}_B(G)$ is discrete, then G is compact. One would expect that the hypothesis " $B \supset I(G)$ " can be replaced by " $Z_B(C^*(G))$ is commutative."*

From 4.2 it follows that when $G \in [FIA]_B^-$ with $Z_B(K(G))$ commutative, then the B -characters on G can be identified with the positive-definite spherical functions on a certain semidirect product. In fact, let $H = G \times_{\eta} B^-$ (holomorph). Any continuous B -invariant function f on G can be regarded as a continuous function f' on H , bi-invariant with respect to the compact group B^- ; merely let $f'(x, \beta) = f(x)$ for $(x, \beta) \in H$. If f has compact support on G , then f' has compact support on H . Conversely, if f' is any continuous bi-invariant function on H , and f is defined by the above formula, then f is continuous and B -invariant on G ; and if f' has compact support, then f has compact support. It is clear that the mapping $f \rightarrow f'$ is a bijective linear mapping of $Z_B(K(G))$ onto the algebra $Z'_B(K(H))$ of continuous bi-invariant functions on H with compact support; and a straightforward calculation shows that the mapping is also an algebra homomorphism. Now the spherical functions on H are by definition (see [8]) the (continuous) bi-invariant functions ϕ' on H such that $P[\phi']|_{Z'_B(K(H))}$ is multiplicative. If we take the B -invariant functions ϕ on G corresponding as above to the spherical functions on H , it is clear that these are exactly the continuous B -invariant functions ϕ on G such that $P[\phi]|_{Z_B(K(G))}$ is multiplicative. In view of 4.2, therefore, the positive-definite spherical functions on H correspond exactly to the B -characters on G . We may therefore apply the well-known theory of spherical functions (see [8], as well as I. M. Gelfand, *Spherical functions on symmetric Riemannian spaces*, Dokl. Akad. Nauk SSSR **70** (1950), 5–8; Amer. Math. Soc. Transl. (2) **37** (1964), 39–44) to obtain the following supplement to 4.2 (see also 2.5 and 2.6).

4.4. PROPOSITION. *Let $G \in [FIA]_B^-$, with $Z_B(K(G))$ commutative. Let $\phi \in \Phi_B$ with $\phi(1) = 1$. With the notation of 2.0 and 2.1, the following are equivalent: (i) $\phi \in \mathfrak{X}_B$; (ii) $\dim L^\# = 1$; (iii) $\dim H^\# = 1$; (iv) $P[\phi]|_{Z_B(K(G))}$ is multiplicative; (v) ϕ satisfies the B -character formula*

$$(*) \quad \phi(x)\phi(y) = \int_{B^-} \phi(\beta x \cdot y) \, d\beta \quad (x, y \in G).$$

Moreover, if $G \in [FIA]_B^-$, then $Z_B(K(G))$ is commutative iff for each $\phi \in \mathfrak{X}_B$, $\dim H^\# = 1$.

We remark that conditions (iv) and (v) are equivalent for any continuous bounded B -invariant function ϕ on G .

To each B -character we shall associate in a natural way a “maximal” ideal of $C^*(G)$ (see 4.8; the situation of $L^1(G)$ is discussed in 4.11). In order to do this, we digress first to discuss certain properties of B -stable ideals in $L^1(G)$ and $C^*(G)$. Assume then that $G \in [FIA]_B^-$, and let A denote either $L^1(G)$ or $C^*(G)$. If M is a closed left B -stable ideal of A , let $M^\#$ be the set of elements $a^\#$ for $a \in M$; it is easy to see from 1.4 and 1.5 that $M^\# = M \cap Z_B(A)$, and is therefore a closed left ideal in $Z_B(A)$. On the other hand, if m is a closed left ideal in $Z_B(A)$, let I_m be the set $\{a \in A \mid (ba)^\# \in m \text{ for all } b \in A\}$; it follows directly from 1.1 and 1.5 that I_m is a closed left B -stable ideal in A . We shall consider also two sets of “maximal”

ideals. By $\mathcal{M}_B(A)$ we shall mean the set of ideals maximal among the left regular B -stable ideals of A . $\mathcal{M}(Z_B(A))$ will denote the set of regular maximal left ideals of $Z_B(A)$.

The first part of the following theorem relating $\mathcal{M}_B(A)$ and $\mathcal{M}(Z_B(A))$ was proved in the context of von Neumann algebras by Godement [9]; our proof is essentially patterned on his.

4.5. THEOREM. *Let $G \in [FIA]_B^-$, and let A be either $L^1(G)$ or $C^*(G)$. The map $\nu: M \rightarrow M^\#$ is a bijection of $\mathcal{M}_B(A)$ onto $\mathcal{M}(Z_B(A))$, with inverse $m \rightarrow I_m$. Moreover, for any subset F of $\mathcal{M}_B(A)$, $\text{hull}(\text{kernel } \nu(F)) = \nu(\text{hull}(\text{kernel } F))$ (the hulls are taken in $\mathcal{M}(Z_B(A))$ and $\mathcal{M}_B(A)$, respectively).*

Proof. We note first that if M is any closed left B -stable ideal in A , and M is regular with right identity e modulo M , then $M^\#$ is regular with right identity $e^\#$: for if $a \in Z_B(A)$, then by 1.5 we have $a - ae^\# = a^\# - (ae)^\# = (a - ae)^\#$, an element of $M^\#$. If M is a proper closed left B -stable ideal, then $M^\#$ is a proper ideal in $Z_B(A)$: for if not, then $M \supset Z_B(A)$, so that M contains an approximate identity for L^1 (see 0.1), and since M is closed, $M = A$, contradiction.

On the other hand, let m be any closed left ideal in $Z_B(A)$. We show first that $I_m^\# = m$. For let (a_ν) be an approximate identity for A , and let $a \in I_m$. Then $a_\nu a \rightarrow a$, so $(a_\nu a)^\# \rightarrow a^\#$. Since $(a_\nu a)^\# \in m$ and m is closed, $a^\# \in m$. Therefore $(I_m)^\# \subset m$. Conversely, if $a \in m$ and $b \in A$, then $(ba)^\# = b^\# a \in m$, so $a \in I_m$. Since $(I_m)^\# = I_m \cap Z_B(A)$, $a \in (I_m)^\#$, and therefore $(I_m)^\# = m$. If $m \neq Z_B(A)$, then $I_m \neq A$, for otherwise $m = (I_m)^\# = Z_B(A)$. If in addition m is regular, with right identity e modulo m , then e is a right identity modulo I_m . For let $a, b \in A$; we shall show that $[b(a - ae)]^\# \in m$, and since b is arbitrary, $a - ae \in I_m$. But by 1.5 and B -invariance of e , we have

$$[b(a - ae)]^\# = (ba)^\# - (bae)^\# = (ba)^\# - (ba)^\# e \in m.$$

Finally, the two operations are connected also as follows. If M is a closed left B -stable ideal in A and $m = M^\#$, then $M \subset I_m$: for if $a \in M$ and $b \in A$, then

$$(ba)^\# \in M^\# = m.$$

Suppose now that M is an element of $\mathcal{M}_B(A)$, and let $m = M^\#$. Then $M \subset I_m$, so by maximality $M = I_m$. Moreover, by Zorn's lemma there exists $n \in \mathcal{M}(Z_B(A))$ such that $m \subset n$. Since clearly $I_m \subset I_n$, and since $I_n \neq A$, we have $I_n = I_m = M$. Therefore $n = (I_n)^\# = M^\# = m$. Consequently, if $M \in \mathcal{M}_B(A)$ then $M^\# = m \in \mathcal{M}(Z_B(A))$ and $M = I_m$.

Conversely, suppose $n \in \mathcal{M}(Z_B(A))$. Then by Zorn's lemma there exists $N \in \mathcal{M}_B(A)$ with $I_n \subset N$. Then $n = (I_n)^\# \subset N^\# \neq Z_B(A)$, so by maximality $N^\# = n$ and $N = I_n$. Therefore $I_n \in \mathcal{M}_B(A)$, and $(I_n)^\# = n$. This completes the proof of the first statement.

Now let F be any subset of $\mathcal{M}_B(A)$. Then

$$\begin{aligned} (*) \quad \text{kernel } (\nu(F)) &= \bigcap_{M \in F} (M \cap Z_B(A)) = \left(\bigcap_{M \in F} M \right) \cap Z_B(A) \\ &= (\text{kernel } F) \cap Z_B(A) = (\text{kernel } F)^\#. \end{aligned}$$

Moreover, let N be any closed two-sided B -stable ideal of A . We shall show that for $m \in \mathcal{M}(Z_B(A))$, $m \in \text{hull}(N^\#)$ iff $I_m \in \text{hull}(N)$; since $\nu(I_m) = m$ this will show that $\text{hull}(N^\#) = \nu(\text{hull}(N))$. But if $I_m \supset N$ then letting $n = N^\#$ we have $n \subset (I_m)^\# = m$; conversely, if $n \subset m$, then $I_n \subset I_m$, and since $N \subset I_n$, this shows that $N \subset I_m$. In particular, setting $N = \text{kernel } F$ we have by (*) that $\text{hull}(\text{kernel } \nu(F)) = \nu(\text{hull}(\text{kernel } F))$.

When $B \supset I(G)$, Theorem 4.5 has special meaning. In this case $Z_B(A) \subset Z(A)$ is a commutative algebra, so $\mathcal{M}(Z_B(A))$ is just the maximal ideal space of $Z_B(A)$. Moreover, it is easy to see, using 1.5(iii), that if m is in $\mathcal{M}(Z_B(A))$, then I_m is a regular two-sided ideal. It then follows from 4.5 that $\mathcal{M}_B(A)$ is exactly the set of regular maximal two-sided B -stable ideals of A . In particular, if $B = I(G)$ then $\mathcal{M}_B(A) = \mathcal{M}(A)$ is exactly the set of regular maximal two-sided ideals of A : for it is easy to modify the argument in [23, p. 321] to show that a closed left (right) ideal of A is invariant under the left (right) regular representation of G on A (see 1.1), so that a closed two-sided ideal of A is invariant under $I(G)$. When $B \supset I(G)$, therefore, we can define a hull-kernel topology for both $\mathcal{M}(Z_B(A))$ and $\mathcal{M}_B(A)$: for the former this is well known since $Z_B(A)$ is commutative, and for the latter one can apply the argument in [24, Theorem 1]. Theorem 4.5 now asserts that the map ν is a homeomorphism.

4.6. COROLLARY. *Let $G \in [FIA]_{\bar{B}}$, with $B \supset I(G)$. Then the map ν of 4.5 is a homeomorphism for the respective hull-kernel topologies. Moreover, the weak*-topology on the maximal ideal space of $Z_B(C^*(G))$ coincides with the hull-kernel topology, so that $\mathcal{M}_B(C^*(G))$ is a locally compact Hausdorff space.*

Proof. For the second statement, we observe that the algebra $Z_B(C^*(G))$ is a commutative C^* -algebra, hence is isometrically $*$ -isomorphic to the algebra of continuous functions vanishing at infinity on the locally compact Hausdorff space $\mathcal{M}(Z_B(C^*(G)))$ (in the weak*-topology) [23, 4.2.2]. But any such algebra has the property that the weak*-topology on the maximal ideal space coincides with the hull-kernel topology (see [23, 3.7.1 and 3.7.2]).

We shall see later that for $G \in [FIA]^-$ the set of primitive ideals of $C^*(G)$ coincides with the set of regular maximal two-sided ideals (5.3), and consequently is a locally compact Hausdorff space. Moreover, 4.5 implies then that $C^*(G)$ is central in the sense of Kaplansky [17]; using Theorem 9.1 of [17] one then has an alternative proof that $\text{Prim } C^*(G)$ is locally compact and T_2 .

In this connection, we remark that even the injectivity assertion of 4.5 is false when G is assumed only to be in $[SIN]$. For example, let G be the discretely topologized free group on two generators. Then $Z(L^1(G))$ is one-dimensional (see 2.8), hence has a unique ideal (0). On the other hand, Example 3.4 of [11] shows that the intersection of all the maximal two-sided ideals in $L^1(G)$ of finite codimension is (0), and since $L^1(G)$ is not finite dimensional, there are infinitely many such ideals.

Let G be locally compact and B a subgroup of $\text{Aut}(G)$. If A is $L^1(G)$ or $C^*(G)$, we shall say that A is strongly B -semisimple if the intersection of all regular maximal

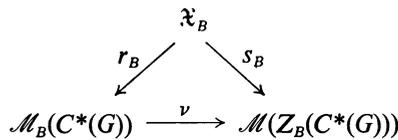
left B -stable ideals is (0) . As we have seen, if $G \in [FIA]_B^-$ and $B \supset I(G)$, then the regular maximal left B -stable ideals of A are exactly the regular maximal two-sided B -stable ideals. In particular, if $G \in [FIA]^-$ (so $B = I(G)$), then to say A is strongly B -semisimple is just to say that A is strongly semisimple in the usual sense (see, for example, [23, p. 59]). The next result therefore generalizes 1.7. On the other hand, it also contains the result that $L^1(G)$ and $C^*(G)$ are (weakly) semisimple (set $B = (1)$).

4.7. THEOREM. *Let $G \in [FIA]_B^-$, and let A be either $L^1(G)$ or $C^*(G)$. Then A is strongly B -semisimple.*

Proof. Let R denote the intersection of all the regular maximal left B -stable ideals in A . Then 4.5 shows that $R \cap Z_B(A)$ is just the intersection of all regular maximal left ideals of $Z_B(A)$, that is, the (weak) radical of $Z_B(A)$. Now, $Z_B(A)$ can be identified with a $*$ -subalgebra of bounded operators on some Hilbert space (for $A = L^1(G)$ this follows from the injectivity of the regular representation on $L^2(G)$; for $A = C^*(G)$ this is [4, 2.6.1]). By a result of Segal [24, p. 77n.], the (weak) radical of $Z_B(A)$ is therefore (0) , so $R \cap Z_B(A) = (0)$. Now if $a \in R$, then $(a^*a)^\# \in R \cap Z_B(A)$, so $a = 0$. Therefore $R = (0)$.

We can now establish the connection between B -characters and “maximal” ideals. If ϕ is any continuous positive-definite function, and $A = L^1(G)$ or $C^*(G)$, then $M(\phi)$ denotes the set $\{a \in A \mid P[\phi](a^*a) = 0\}$, and we let $m(\phi)$ denote $M(\phi) \cap Z_B(A)$. We let $r_B(\phi) = M(\phi)$, $s_B(\phi) = m(\phi)$.

4.8. PROPOSITION. *Let $G \in [FIA]_B^-$, $\phi \in \Phi_B$, $\phi(1) = 1$. Then $m(\phi)$ is in $\mathcal{M}(Z_B(C^*(G))) \Leftrightarrow \phi \in \mathfrak{X}_B \Leftrightarrow M(\phi)$ is in $\mathcal{M}_B(C^*(G))$. The maps in the diagram below are bijections (ν is the map of 4.5) and the diagram is commutative. Moreover, if $Z_B(C^*(G))$ is commutative, then s_B is a homeomorphism, and if in addition $B \supset I(G)$, then r_B is also a homeomorphism.*



Proof. Suppose first that $\phi \in \Phi_B$ with $\phi(1) = 1$, and let $p[\phi]$ be the corresponding functional on $Z_B(C^*(G))$. By Theorem 4.1, $\phi \in \mathfrak{X}_B$ iff $p[\phi]$ is a pure state on $Z_B(C^*(G))$. But by [4, 2.9.5], $p[\phi]$ is a pure state iff $m(\phi)$ is a regular maximal left ideal in $Z_B(C^*(G))$. In this case, moreover, $M(\phi)$ is a regular maximal left B -stable ideal in $C^*(G)$: for if $m = m(\phi)$, $M = M(\phi)$, and $a \in I_m$, then $(a^*a)^\# \in m$, and $P[\phi](a^*a) = 0$; thus $I_m \subset M$, and since, by 4.5, I_m is maximal, so is M . Conversely, if $M(\phi)$ is in $\mathcal{M}_B(C^*(G))$, then, by 4.5, $m(\phi)$ is in $\mathcal{M}(Z_B(C^*(G)))$. This proves the first statement, and the commutativity of the diagram now follows from the

definitions. Now, by 4.1, the map $\phi \rightarrow p[\phi]$ is a bijection of \mathfrak{X}_B onto the set of pure states of $Z_B(C^*(G))$, and it follows from [4, 2.9.5] again that $p[\phi] \rightarrow m(\phi)$ is a bijection of the set of pure states onto $\mathcal{M}(Z_B(C^*(G)))$. Thus s_B is a bijection, and since ν is a bijection (4.5), so is r_B . Finally, when $Z_B(C^*(G))$ is commutative, s_B is a homeomorphism by 4.2 (we may take either the hull-kernel or the weak*-topology on $\mathcal{M}(Z_B(C^*(G)))$), by 4.6). If in addition $B \supset I(G)$, then 4.6 shows that ν is a homeomorphism, so the same is true of r_B .

4.9. PROPOSITION. *Let $G \in [FIA]_B^-$, and let C be a subgroup of B . If $\chi \in \mathfrak{X}_B$ then there is a $\phi \in \mathfrak{X}_C$ such that $\phi^\# = \chi$. Moreover, if $\phi \in \Phi_C$ with $\phi(1) = 1$, then $\phi^\# = \chi$ iff $M(\phi)$ contains $M(\chi)$; in this case $M(\chi)$ is the largest B -stable subset of $M(\phi)$.*

Proof. We show first that if $\phi \in \Phi_C$, then $M(\phi) \supset M(\phi^\#)$, and in fact $M(\phi^\#)$ is the largest B -stable subset of $M(\phi)$. For let $a \in C^*(G)$, and suppose that $P[\phi^\#](a^*a) = 0$; then by 1.9 $\int_B P[\phi^\beta](a^*a) d\beta = 0$, and since by 1.0 the integrand is a positive continuous function of β , we have $M(\phi^\#) = \bigcap M(\phi^\beta)$ ($\beta \in B^-$), hence $M(\phi^\#) \subset M(\phi)$. Moreover, if M is any B -stable subset of $M(\phi)$, then $M \subset M(\phi^\beta)$ for each β , so $M \subset M(\phi^\#)$. Consequently, if $\chi \in \mathfrak{X}_B$ and $M(\chi) \subset M(\phi)$, then $M(\chi) \subset M(\phi^\#)$. But by 4.8 $M(\chi)$ is maximal among the B -stable ideals, so $M(\chi) = M(\phi^\#)$; this implies by 4.8 that $\phi^\#$ is actually in \mathfrak{X}_B ($\phi(1) = 1$), and by the injectivity of r_B that $\chi = \phi^\#$. Finally, to prove the first statement, let $\chi \in \mathfrak{X}_B$, and observe that $M(\chi)$ is a regular left C -stable ideal of $C^*(G)$. By Zorn's lemma and 4.8 $M(\chi)$ is contained in $M(\phi)$ for some $\phi \in \mathfrak{X}_C$, so that $\phi^\# = \chi$.

4.10. REMARK. The previous proposition affords an integral formula for the B -characters of G in terms of the C -characters, when C is a subgroup of B . In particular, if $\chi \in \mathfrak{X}_B$ then there is an irreducible representation $\pi \in \hat{G}$ and $\phi \in \Phi_0$ associated with π such that $\chi = \int_B \phi^\beta d\beta$ (set $C = (1)$). However, the converse is not in general true; that is, $\phi \in \mathfrak{X}_C$ does not in general imply that $\phi^\# \in \mathfrak{X}_B$, even when $Z_B(C^*(G))$ is commutative. For example, let G be the symmetric group on three letters, generated by a transposition x and a 3-cycle y , with $xyx = y^2$. Let $B \subset \text{Aut}(G)$ be the subgroup consisting of the identity and conjugation by x . Then B does not contain $I(G)$. Now if $t \in G$ (resp. $T \subset G$) let δ_t (resp. δ_T) denote the characteristic function of $\{t\}$ (resp. T). Then it is easy to see that if $Y = \{y, y^2\}$, then $Z_B(A)$ has as a basis the family of characteristic functions $(\delta_1, \delta_x, \delta_Y, \delta_{xY})$, and also that $Z_B(A)$ is commutative. Let π be the irreducible matrix representation given by

$$\pi_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi_y = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Let $u = (1, 0)$, and $\phi(t) = (\pi_t u, u)$. Then $\phi(1) = 1$, $\phi(x) = 0$; and since $\{1\}$ and $\{x\}$ are already B -orbits, $\phi^\#(1) = 1$, $\phi^\#(x) = 0$. Consequently,

$$P[\phi^\#](\delta_x^2) = P[\phi^\#](\delta_{x^2}) = P[\phi^\#](\delta_1) = \phi^\#(1) = 1$$

while $[P[\phi^\#](\delta_x)]^2 = [\phi^\#(x)]^2 = 0$. Therefore $P[\phi^\#]|_{Z_B(A)}$ is not multiplicative, so by

4.2 $\phi^\# \notin \mathfrak{X}_B$. Note also that if $\psi(t) = (\pi_t v, v)$ where $v = (\sqrt{2})^{-1}(1, 1)$, then $\psi^\#(x) = 1$ so $\psi^\# \neq \phi^\#$. See, however, 5.1 for the case $B \supset I(G)$.

We shall now discuss briefly the corresponding ideal structure in $L^1(G)$. Here we assume throughout that $Z_B(K(G))$ is commutative.

4.11. PROPOSITION. *Let $G \in [FIA]_B^-$, with $Z_B(K(G))$ commutative, and let $\phi \in \Phi_B$ with $\phi(1) = 1$. Then $\phi \in \mathfrak{X}_B$ iff $M(\phi)$ is in $\mathcal{M}_B(L^1(G))$ (resp. $m(\phi)$ is in $\mathcal{M}(Z_B(L^1(G)))$). Moreover, the maps r_B and s_B below are injective, and the diagram is commutative (v is the map of 4.5).*

$$\begin{array}{ccc}
 & \mathfrak{X}_B & \\
 r_B \swarrow & & \searrow s_B \\
 \mathcal{M}_B(L^1(G)) & \xleftarrow{v} & \mathcal{M}(Z_B(L^1(G)))
 \end{array}$$

Proof. Let $\phi \in \Phi_B$ with $\phi(1) = 1$. The proof of 4.8 shows that $m(\phi)$ is in $\mathcal{M}(Z_B(L^1(G)))$ if and only if $M(\phi)$ is in $\mathcal{M}_B(L^1(G))$. If $\phi \in \mathfrak{X}_B$ then $p[\phi]$ is multiplicative; thus $m(\phi) = \ker p[\phi]$ is a regular maximal ideal in $Z_B(L^1(G))$, since for $f \in Z_B(L^1(G))$, $p[\phi](f^\vee * f) = |p[\phi](f)|^2$. Conversely, if $m(\phi)$ is maximal, hence the kernel of a multiplicative functional q , then $\|q\| = 1$, since $Z_B(L^1(G))$ contains an approximate identity (0.1). Moreover, $\ker p[\phi] \supset m(\phi)$, since for any $f \in L^1(G)$, $|P[\phi](f)|^2 \leq P[\phi](f^\vee * f)$ [4, 2.1.5]; since $m(\phi)$ has codimension 1, $\ker p[\phi] = m(\phi) = \ker q$, so $p[\phi] = cq$ for some $c \in \mathbb{C}$. But $\|p[\phi]\| = 1$, so $p[\phi] = q$ is multiplicative and $\phi \in \mathfrak{X}_B$ (4.4). The commutativity of the diagram follows from the definitions; finally, the injectivity of s_B (hence also, of r_B) follows from 1.9.

It seems to be unknown in general whether with the hypotheses of 4.11, the maps r_B and s_B are always surjective (where the ideals considered lie in $L^1(G)$, of course, since for $C^*(G)$ the answer is given by 4.8). In view of the remarks preceding 4.4 this is equivalent to the statement that all the continuous bounded B^- -spherical functions on the semidirect product $G \times_n B^-$ (holomorph) are positive-definite, or again, equivalent to the statement that each bounded continuous solution $\phi \neq 0$ to the B -character formula (*) of 4.4 is a B -character. A. Hulanicki [15a] has recently given sufficient conditions for this to be so. Hulanicki gives his results in a framework more general than ours, and we shall recast them for our purposes. If G is a locally compact group, with Haar measure $|\cdot|$, G is said to satisfy condition (C) if, for each compact subset A of G and $c > 1$, $|A^n| = o(c^n)$ as $n \rightarrow \infty$, where $A^n = \{a_1 \cdots a_n \mid a_j \in A\}$. Hulanicki observes that condition (C) is satisfied by $[FC]^-$ groups.

4.12. THEOREM (A. HULANICKI). *Let $G \in [FIA]_B^-$, and suppose G satisfies condition (C). If $\phi \neq 0$ is a bounded continuous B -invariant function and $P[\phi]|_{Z_B(L^1(G))}$ is multiplicative (that is, if ϕ satisfies the B -character formula (*) of 4.4), then $\phi \in \mathfrak{X}_B$. If in addition $Z_B(L^1(G))$ is commutative, then the maps r_B and s_B of 4.11 are*

bijections, and s_B is a homeomorphism for the respective weak*-topologies. In particular, the above statements hold for $G \in [FIA]_B^-, B \supset I(G)$.

Finally, we state a proposition which relates the question discussed above to conditions which have been discussed by other authors (see [3] and [26]). For the definition of terms in (i) and (ii) see [23, pp. 233, 216].

4.13. PROPOSITION. *Let $G \in [FIA]_B^-$ with $Z_B(K(G))$ commutative. Then the maps r_B and s_B of 4.11 are bijective iff both (i) and (ii) are satisfied.*

(i) $Z_B(L^1(G))$ is symmetric.

(ii) Every (representable) positive linear functional on $Z_B(L^1(G))$ extends to a (bounded)⁽²⁾ positive linear functional on $L^1(G)$.

Proof. Suppose first that (i) and (ii) hold, and let p be a multiplicative linear functional on $Z_B(L^1(G))$. Then p is positive and representable, hence is the restriction to $Z_B(L^1(G))$ of a function $P[\phi]$, for some continuous positive-definite function ϕ . By applying the # operator we may assume that ϕ is also B -invariant; then clearly $\|\phi\|_G = \|p\|$, and since $Z_B(L^1(G))$ contains an approximate identity, $\|p\| = 1$. By 4.4 it follows that $\phi \in \mathfrak{X}_B$, and therefore that r_B and s_B are surjective. Suppose now conversely that s_B is a bijection, hence by 1.10 a homeomorphism. Then (i) holds, since each multiplicative linear functional on $Z_B(L^1(G))$ is of the form $p[\phi]$ ($\phi \in \mathfrak{X}_B$), hence positive. Moreover, if q is any representable positive linear functional on $Z_B(L^1(G))$, then there is a regular Borel measure μ on $\mathcal{M}(Z_B(L^1(G)))$ such that $q = \int p \, d\mu(p)$ (see, for example, [23, p. 230]). Since s_B is a homeomorphism we may assume that μ is defined on \mathfrak{X}_B ; then $Q = \int_{\mathfrak{X}_B} P[\phi] \, d\mu(\phi)$ is the required extension of q to $L^1(G)$.

5. The dual, the structure space, and \mathfrak{X}_B .

5.0. In this section we shall consider the case $G \in [FIA]_B^-, B \supset I(G)$. In particular, therefore, $G \in [SIN]$, so the left and right uniformities coincide, and G is unimodular. We shall be interested mainly in $C^*(G)$, so we shall henceforth denote $C^*(G)$ by A . We recall [4, 3.1.1] that the structure space $\text{Prim } A$ of A is the topological space of primitive ideals in the hull-kernel topology. $\text{Prim } A$ is precisely the set of kernels $\ker \pi$, where π is an irreducible continuous unitary representation of G (extended to A) [4, 2.9.7 and 13.9.3]. \hat{G} , the space of equivalence classes of these representations, is given the inverse image of the topology on $\text{Prim } A$, by the surjective map $k: \pi \rightarrow \ker \pi$. It follows directly from the definition that k is a continuous map which is open and proper. (We recall that a continuous map $f: X \rightarrow Y$ is proper if f is closed and $f^{-1}(y)$ is compact (not necessarily Hausdorff) for each $y \in Y$ —see N. Bourbaki, *Topologie générale* (3rd ed. rev., Hermann, Paris, 1961), Ch. I, §10.2, Théorème 1.)

(²) Any positive linear functional on a Banach algebra with approximate identity is bounded and representable [27]. Since $Z_B(L^1(G))$ contains an approximate identity (see 0.1) the words “bounded” and “representable” in the lemma are in fact unnecessary.

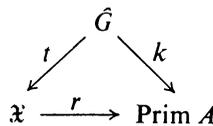
5.1. LEMMA. Let $G \in [FIA]_{\bar{B}}$, with $B \supset I(G)$. If π is a continuous irreducible unitary representation of G , then the operator-valued integral $\int_{B^-} (\pi \circ \beta)_x d\beta$ is a scalar multiple $\pi^\#(x)I$ of the identity, and $\pi^\#$ is in \mathfrak{X}_B . Moreover, if ϕ is any (elementary) normalized positive-definite function associated with π , then $\phi^\# = \pi^\# \in \mathfrak{X}_B$. Therefore $\pi^\#$ depends only on the equivalence class of π in \hat{G} , and the map $t_B: \hat{G} \rightarrow \mathfrak{X}_B$ given by $t_B(\pi) = \pi^\#$ is well defined.

Proof. Let T_x denote the operator defined above, let $v \in H_\pi$ be of norm 1, and let $\phi(x) = (\pi_x v, v)$. If $y \in G$ then conjugation by y is an element of B , so it follows directly from the left invariance of Haar measure on B^- that $T_x \pi_y = \pi_y T_x$. Therefore T_x is a scalar multiple $\pi^\#(x)I$. Moreover, $\pi^\#(x) = (T_x v, v) = \phi^\#(x)$, so that $\phi^\#$ depends only on the class of π . To show that $\phi^\#$ is in \mathfrak{X}_B , we show that $P[\phi^\#]$ is multiplicative on $Z_B(A)$ (compare 4.2). But if $a \in Z_B(A) \subset Z(A)$, then $\pi(a)$ commutes with $\pi(b)$ for each $b \in A$ (since $Z(A)$ is the center of A , by 1.2). Therefore $\pi(a)$ is a scalar multiple $c(a)I$ of the identity, for some complex number $c(a)$. Therefore $P[\phi](a) = (\pi(a)v, v) = c(a)(v, v) = c(a)$, so $P[\phi](a)I = \pi(a)$. By 1.9 we have also that, for $a \in Z_B(A)$, $P[\phi^\#](a)I = \pi(a)$. Since π is multiplicative, $\phi^\# \in \mathfrak{X}_B$.

It follows from the above lemma that if ϕ and ψ are any two elementary normalized positive-definite functions associated with the same irreducible representation, then $\phi^\# = \psi^\# \in \mathfrak{X}_B$. We have already seen a counterexample to this when $B \not\supset I(G)$ (4.10).

We can now determine the relationship of the characters to the structure space $\text{Prim } A$. Recall that we have (4.8) a homeomorphism $r: \mathfrak{X} \rightarrow \mathcal{M}(A)$, $r(\phi) = M(\phi)$, where $\mathcal{M}(A)$ is the set of regular maximal two-sided ideals of A . Now it is easy to see (compare [23, 2.2.9]) that each such ideal is actually a primitive ideal of A . Consequently we shall regard r as an injective map of $\mathfrak{X} \rightarrow \text{Prim } A$.

5.2. THEOREM. Let $G \in [FIA]^-$. The map $t: \hat{G} \rightarrow \mathfrak{X}$ is a continuous, open, and proper surjection. The map $r: \mathfrak{X} \rightarrow \text{Prim } A$ is a homeomorphism onto $\text{Prim } A$, and the diagram below is commutative.



Proof. We have already seen that r is an injection and that t is a surjection⁽³⁾ (let $C=(1)$ in 4.9). If we show that the diagram is commutative, this will show that r is a bijection (since k is a surjection). Let $\pi \in \hat{G}$, and let $\chi = \pi^\#$; we must show that $\ker \pi = M(\chi)$. If v is an arbitrary unit vector in H_π , and $\phi(x) = (\pi_x v, v)$, then the

⁽³⁾ The fact that t is a surjection was first observed in the case of separable $[FIA]^-$ groups by J. Liukkonen. His proof used the theory of direct integral decompositions and continuity properties of the operator.

lemma shows that $\phi^\# = \chi$, so, by 4.9, $M(\chi) \subset M(\phi)$. Thus if $a \in M(\chi)$, then $P[\phi](a^*a) = \|\pi(a)v\|^2 = 0$ for all v , so $\pi(a) = 0$. Therefore $M(\chi) \subset \ker \pi$, and since, by 4.8, $M(\chi)$ is maximal, we have $M(\chi) = \ker \pi$. Thus the diagram is commutative and r is a bijection, hence $\text{Prim } A = \mathcal{M}(A)$; therefore, by 4.8, r is in fact a homeomorphism. The topological properties of t then follow from those of k mentioned in 5.0.

The first statement of the next corollary follows from the proof of 5.2, together with 4.2 (or 4.6). The second statement follows from the definition of the topology on \hat{G} (see [4, 3.1.6]).

5.3. COROLLARY. *Let $G \in [FIA]^-$. The strong structure space $\mathcal{M}(A)$ coincides with the structure space $\text{Prim } A$, and is a locally compact Hausdorff space. Moreover, the following are equivalent: (i) \hat{G} is a T_0 -space; (ii) $t: \pi \rightarrow \pi^\#$ is a homeomorphism of $\hat{G} \rightarrow \mathfrak{X}$; (iii) \hat{G} is a locally compact Hausdorff space.*

The above corollary is of course false for arbitrary $G \in [SIN]$. For example, if Z_2 is the cyclic group of order 2, let $G = \mathbf{R} \times_n Z_2$, where the nontrivial element of Z_2 acts on \mathbf{R} by sending $x \rightarrow -x$. Then $G \in [SIN]$, since $\mathbf{R} \in [SIN]_{Z_2}$; also, it is easy to see from the Mackey theory [20] that each element of \hat{G} is finite dimensional. It follows, of course, that \hat{G} is a T_1 -space (inequivalent irreducible finite-dimensional representations must have different kernels when extended to $C^*(G)$, and each such kernel is a maximal two-sided ideal). On the other hand, \hat{G} is not Hausdorff by [1, 10-1]. The same remark could be made about the discrete semi-direct product $\mathbf{Z} \times_n Z_2$, where \mathbf{Z} is the group of integers.

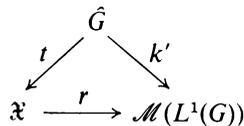
Before turning to the study of \mathfrak{X}_B , we state some propositions about the ideals in $L^1(G)$. The first generalizes Theorem (6.18) of [11] dealing with the maximal ideals in the group algebra of a $[Z]$ group.

5.4. PROPOSITION. *Let $G \in [FIA]^-$. For each $\pi \in \hat{G}$ the L^1 -kernel*

$$\ker \pi = \{f \in L^1(G) \mid \pi(f) = 0\}$$

is a regular maximal two-sided ideal of $L^1(G)$, and the map $\pi \rightarrow \ker \pi$ is a surjection k' of $\hat{G} \rightarrow \mathcal{M}(L^1(G))$. Moreover if $\pi, \sigma \in \hat{G}$, then $k'(\pi) = k'(\sigma)$ iff $\pi^\# = \sigma^\#$.

Proof. If $\pi \in \hat{G}$ then, by 5.2, $\ker \pi = M(\pi^\#)$, so the same relation holds upon intersection with $L^1(G)$. By 4.11, $M(\pi^\#)$ is then a regular maximal two-sided ideal in $L^1(G)$, hence also $\ker \pi$. This shows, moreover, that the diagram below is commutative:



and since t and r are surjective (5.2 and 4.12), so is k' . On the other hand, since r is injective (4.11), the last statement holds also.

5.5. COROLLARY. *Let $G \in [FIA]^-$, and let $\pi \in \hat{G}$. Then the closure in $C^*(G)$ of the L^1 -kernel of π is just the C^* -kernel of π .*

Proof. Let N be the closure in $C^*(G)$ of the L^1 -kernel of π . Then by [4, 2.9.7] N is the intersection of the primitive ideals $\ker \sigma$ ($\sigma \in \hat{G}$) in $C^*(G)$ containing it. If $\ker \sigma \supset N$, then by the above proposition the L^1 -kernel of σ must equal the L^1 -kernel of π . Consequently $\pi^\# = \sigma^\#$, so by 5.2 the C^* -ideals $\ker \pi$ and $\ker \sigma$ agree. Therefore $N = \ker \pi$.

We now turn to the study of \mathfrak{X}_B . We need first a lemma applying to arbitrary locally compact groups G . If B is any subgroup of $\text{Aut}(G)$, then B operates on \mathfrak{X} ; that is, $\phi \rightarrow \phi^\beta$ is a permutation of \mathfrak{X} for each $\beta \in B$, and $\phi^{\beta_1\beta_2} = (\phi^{\beta_2})^{\beta_1}$ for all $\beta_1, \beta_2 \in B$.

5.6. LEMMA. *Let G be locally compact, and let B be a subgroup of $\text{Aut}(G)$. Then B operates continuously on \mathfrak{X} .*

Proof. We must show that the map $(\beta, \phi) \rightarrow \phi^\beta$ is a jointly continuous map of $B \times \mathfrak{X} \rightarrow \mathfrak{X}$ (where we may assume that \mathfrak{X} has the weak topology). Let (α, ψ) be a fixed point in $B \times \mathfrak{X}$, and let $f \in L^1(G)$. We shall show that if $(\beta, \phi) \rightarrow (\alpha, \psi)$, then $P[\phi^\beta](f) \rightarrow P[\psi^\alpha](f)$. Now $P[\phi^\beta](f) = P[\phi](f^{\beta^{-1}}) \Delta(\beta)$, where Δ is the modular function defined in 1.0; since Δ is continuous, it suffices to show that $P[\phi](f^{\beta^{-1}}) \rightarrow P[\psi](f^{\alpha^{-1}})$. Now

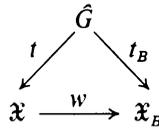
$$|P[\phi](f^{\beta^{-1}}) - P[\psi](f^{\alpha^{-1}})| \leq |P[\phi](f^{\beta^{-1}}) - P[\phi](f^{\alpha^{-1}})| + |P[\phi](f^{\alpha^{-1}}) - P[\psi](f^{\alpha^{-1}})|.$$

Since $\|\phi\| = 1$, the first term is bounded by $\|f^{\beta^{-1}} - f^{\alpha^{-1}}\|_1$, and when $\beta \rightarrow \alpha$, $\beta^{-1} \rightarrow \alpha^{-1}$, so the strong continuity of $\beta \rightarrow f^\beta$ (see 1.0) shows that this term approaches zero. On the other hand as $\phi \rightarrow \psi$ the second term approaches zero independently of β , and this completes the proof.

5.7. PROPOSITION. *Let $G \in [FIA]^-_B$, with $B \supset I(G)$. If $\phi \in \mathfrak{X}$ then $w(\phi) = \phi^\#$ is an element of \mathfrak{X}_B . Moreover, if $\psi \in \mathfrak{X}$, then $\phi^\# = \psi^\#$ iff $\psi \in B^-[\phi] = \{\phi^\beta \mid \beta \in B^-\}$.*

Proof. Let $\phi, \psi \in \mathfrak{X}$. In order to show that $\phi^\# \in \mathfrak{X}_B$, it suffices by 4.2 to show that the restriction of $P[\phi^\#]$ to $Z_B(A)$ is multiplicative. But $P[\phi]$ and $P[\phi^\#]$ agree on $Z_B(A)$, and, by 4.2, $P[\phi]$ is multiplicative on $Z(A) \supset Z_B(A)$. Therefore $\phi^\# \in \mathfrak{X}_B$. Of course, if $\psi \in B^-[\phi]$, then $\psi^\# = \phi^\#$. Conversely, suppose that $\chi = \phi^\# = \psi^\#$. It follows from 1.9 that $M(\chi) = \bigcap M(\phi^\beta)$ ($\beta \in B^-$), and $M(\psi) \supset M(\chi)$. By the definition of the hull-kernel topology on $\text{Prim } A$, $M(\psi)$ is in the closure of the set $\{M(\phi^\beta) \mid \beta \in B^-\}$, so that, by 5.2, ψ is in the closure of $B^-[\phi]$. Since B^- is compact and \mathfrak{X} is Hausdorff, 5.6 shows that $B^-[\phi]$ is closed, so $\psi \in B^-[\phi]$.

5.8. THEOREM. *Let $G \in [FIA]^-_B$, with $B \supset I(G)$. Let w be the map defined above, and t, t_B be as in 5.1. Then the diagram below is commutative, and all the maps are continuous, open, and proper surjections. Moreover, w induces a homeomorphism of $\mathfrak{X}/B^- \rightarrow \mathfrak{X}_B$.*



Proof. Let $\pi \in \hat{G}$, let $\phi = t(\pi)$ and $\chi = t_B(\pi)$. Then for $a \in Z(A)$, $P[\phi](a)I = \pi(a)$, and for $a \in Z_B(A) \subset Z(A)$, $P[\chi](a)I = \pi(a)$ (see proof of 5.1). Consequently $P[\phi]$ and $P[\chi]$ agree on $Z_B(A)$, so, by 1.9, $\phi^\# = \chi$, and the diagram is commutative (of course, this can also be derived directly, using the Fubini theorem). We show next that the extended map of compact spaces $w': \mathfrak{X} \cup (0) \rightarrow \mathfrak{X}_B \cup (0)$ given by $\phi \rightarrow \phi^\#$ ($0 \rightarrow 0$) is continuous ($\mathfrak{X} \cup (0)$ and $\mathfrak{X}_B \cup (0)$ are compact by 4.2). For if $\phi_\nu, \phi \in \mathfrak{X} \cup (0)$ and $\phi_\nu \rightarrow \phi$ in the weak topology $\sigma(A', A)$, then for any $a \in A$ we have by 1.9

$$P[\phi_\nu^\#](a) = P[\phi_\nu](a^\#) \rightarrow P[\phi](a^\#) = P[\phi^\#](a).$$

Thus $\phi_\nu^\# \rightarrow \phi^\#$ in the weak topology $\sigma(A', A)$ and w' is continuous. If now F is any closed set in \mathfrak{X} , then $F \cup (0)$ is a closed (hence compact) set in $\mathfrak{X} \cup (0)$. The image $w'(F \cup (0))$ is compact, hence closed in $\mathfrak{X}_B \cup (0)$. But $w'(F \cup (0)) = w(F) \cup (0)$, so that $w(F) = [w(F) \cup (0)] \cap \mathfrak{X}_B$ is a closed set in \mathfrak{X}_B . Therefore w is a continuous and closed surjection. If $\chi \in \mathfrak{X}_B$ then $w^{-1}(\chi)$ is a compact set by 5.7, so w is in fact a proper mapping. It follows from 5.7 and the above, moreover, that w induces a homeomorphism $w^*: \mathfrak{X}/B^- \rightarrow \mathfrak{X}_B$. Now the canonical map of $\mathfrak{X} \rightarrow \mathfrak{X}/B^-$ is open, since if $E \subset \mathfrak{X}$ is open, then so is $B^-[E] = \bigcup \beta(E)$ ($\beta \in B^-$). Therefore $w: \mathfrak{X} \rightarrow \mathfrak{X}_B$, being the composition of an open map and a homeomorphism, is also an open map. The commutativity of the diagram now shows that t_B is also a continuous, open, and proper surjection.

As an application of 5.8, let us consider the group $G = \mathbf{R}^n$, and let B be the orthogonal group O_n . Then G satisfies the hypotheses of 4.12, so the maximal ideal space of the algebra $Z_B(L^1(G))$ of radial functions is given by \mathfrak{X}_B (of course, it is easy to verify 4.12 when G is abelian directly from 4.13, without using Hulanicki's results, since here $L^1(G)$ is well known to be symmetric). It is clear that for an abelian group $\mathfrak{X} = \hat{G}$ (see also the remarks following 2.5), and from the above theorem it follows that here \mathfrak{X}_B is homeomorphic with $\mathbf{R}^n/O_n = \mathbf{R}_+$. Thus the maximal ideal space of the algebra of radial functions is given by the half-line $[0, \infty)$. Theorem 5.8 therefore yields another proof of a result of Reiter [22, Lemma 3].

We remark also that 5.8 implies that a continuous function ϕ on G is a B -character, for $B \supset I(G)$, iff ϕ has an integral representation $\int_B \psi^\beta d\beta$ for some elementary normalized positive-definite function ψ (resp. some $\psi \in \mathfrak{X}$). We shall see in 5.11 that if $\pi \in \hat{G}$ is finite dimensional, then $t(\pi) = \chi_\pi/d_\pi$ ($\chi_\pi(x)$ is the trace of π_x). In this case, $\phi = \pi^\#$ iff $\phi = d_\pi^{-1} \int_B \chi_\pi^\beta d\beta$.

We can use 5.8 and 5.2 to determine the inverse image by the map $t_B: \hat{G} \rightarrow \mathfrak{X}_B$ of a B -character $\chi \in \mathfrak{X}_B$.

5.9. PROPOSITION. Let $G \in [FIA]_B^-$, with $B \supset I(G)$. If $\pi, \sigma \in \hat{G}$, then $\pi^\# = \sigma^\#$ iff σ is in the closure of the set $\{\pi \circ \beta \mid \beta \in B^-\} = B^-[\pi]$.

Proof. The sufficiency is clear, since t_B is a continuous map and Haar measure on B^- is invariant. Conversely, suppose $t_B(\pi) = t_B(\sigma)$, and let $t(\pi) = \phi$, $t(\sigma) = \psi$. Then by 5.8 $w(\phi) = w(\psi)$, so there exists $\beta \in B^-$ with $\psi = \phi^\beta$. The calculations in 1.0 and 1.1 show that $(\ker \pi)^\beta = \ker(\pi \circ \beta^{-1})$ and $M(\phi^\beta) = M(\phi)^\beta$; since the diagram in 5.2 is commutative, this shows that $\ker(\sigma) = M(\psi) = M(\phi)^\beta = \ker(\pi \circ \beta^{-1})$. By the definition of the topology on \hat{G} , this shows that $\{\sigma\}$ and $\{\pi \circ \beta^{-1}\}$ have the same closure, so σ is certainly in the closure of $B^-[\pi]$.

We can also use 5.8 to get the following supplement to Theorem 5.2.

5.10. PROPOSITION. Let $G \in [FIA]_B^-$, with $B \supset I(G)$.

(i) If $P \in \text{Prim } A$, then P contains a unique ideal M in $\mathcal{M}_B(A)$. In fact, if $P = \ker \pi$ with $\pi \in \hat{G}$, and $\chi = \pi^\# \in \mathfrak{X}_B$, then the unique maximal two-sided B -stable ideal of A in P is $M(\chi)$.

(ii) Let q be the map implicitly defined in (i). If t_B and r_B are as defined in 5.1 and 4.8, respectively, then the diagram below is commutative. All the maps are continuous, open, and proper surjections (r_B is a homeomorphism by 4.8).

$$\begin{array}{ccc}
 \hat{G} & \xrightarrow{k} & \text{Prim } A \\
 t_B \downarrow & & \downarrow q \\
 \mathfrak{X}_B & \xrightarrow{r_B} & \mathcal{M}_B(A)
 \end{array}$$

Proof. Let P be a primitive ideal, so that, by 5.2, $P = M(\phi)$ for some $\phi \in \mathfrak{X}$ (in fact, $\phi = r^{-1}(P)$). By 5.7, $\phi^\# \in \mathfrak{X}_B$; hence, by 4.9, P contains the regular maximal two-sided B -stable ideal $M(\phi^\#)$, and $M(\phi^\#)$ is the largest B -stable subset of P . This proves (i). Now (ii) follows from 5.8, since (i) shows that the lower triangle in the enlarged diagram below is commutative.

$$\begin{array}{ccc}
 \hat{G} & \xrightarrow{\quad} & \text{Prim } A \\
 t_B \downarrow & \begin{array}{l} \searrow t \\ \swarrow w \end{array} & \mathfrak{X} \begin{array}{l} \nearrow r^{-1} \\ \searrow q \end{array} \\
 \mathfrak{X}_B & \xrightarrow{r_B} & \mathcal{M}_B(A)
 \end{array}$$

We conclude with some applications of the above theory to the case when \hat{G} consists entirely of finite-dimensional representations. We recall (see the remarks following 2.5) that for any locally compact group G , the map $sp: \pi \rightarrow \chi_\pi/d_\pi$ is a bijection of $\hat{G}_{\text{fin}} \rightarrow \mathfrak{X}_{\text{fin}}$ (where \hat{G}_{fin} is the set of finite-dimensional $\pi \in \hat{G}$, $\mathfrak{X}_{\text{fin}}$ is the set of $\phi \in \mathfrak{X}$ such that the cyclic representation ρ induced by ϕ is finite dimensional).

5.11. PROPOSITION. Let $G \in [FIA]^-$, and let $\pi \in \hat{G}_{\text{fin}}$. Then $\pi^\# = d_\pi^{-1}\chi_\pi$ (see 5.1).

Proof. Let (e_1, \dots, e_d) be an orthonormal basis for H_π , and let $\phi_i(x) = (\pi_x e_i, e_i)$. Then $d^{-1}\chi_\pi = \sum_{i=1}^d d^{-1}\phi_i$; but χ_π is central, so applying $\#$ to both sides gives $d^{-1}\chi_\pi = \sum_{i=1}^d d^{-1}\phi_i^\#$. But by 5.1, $\phi_i^\# = \pi^\#$ for each i , so $d^{-1}\chi_\pi = \pi^\#$.

5.12. THEOREM⁽⁴⁾. Let $G \in [FIA]^-$. The following conditions are equivalent: (i) $\hat{G} = \hat{G}_{\text{fin}}$; (ii) G is type I; (iii) $\mathfrak{X} = \mathfrak{X}_{\text{fin}}$. These conditions imply the following three equivalent conditions, and if G is separable, then all six conditions are equivalent: (iv) \hat{G} is a T_0 -space; (v) $t: \pi \rightarrow \pi^\#$ is a homeomorphism of $\hat{G} \rightarrow \mathfrak{X}$; (vi) \hat{G} is a locally compact Hausdorff space.

Proof. We have observed already, in the remarks following 2.5, that (i) \Rightarrow (ii) \Rightarrow (iii) for any G . Suppose now that $\mathfrak{X} = \mathfrak{X}_{\text{fin}}$, and let $\pi \in \hat{G}$. Then π is cyclic, certainly, hence H_π contains a dense subspace whose dimension is at most $\dim \pi(A) = \text{codim ker } \pi$. On the other hand, by 5.2, $\text{ker } \pi = M(\chi)$, where $\chi = \pi^\# \in \mathfrak{X} = \mathfrak{X}_{\text{fin}}$. By construction (see 2.0), $\chi \in \mathfrak{X}_{\text{fin}}$ implies that $M(\chi)$ has finite codimension in A , so π is finite dimensional. This shows that (iii) \Rightarrow (i), so all three are equivalent. Moreover, if they are satisfied then by 5.11 the continuous closed surjection $t: \hat{G} \rightarrow \mathfrak{X}$ of 5.2 reduces to the bijective map sp above, so that t is a homeomorphism. Thus (iv), (v), and (vi), which are equivalent by 5.3, are satisfied. If in addition G is separable, then these last conditions imply that G is a type I group [4, 9.1], so all six conditions are equivalent.

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⁽⁴⁾ The question of whether separable type I $[FIA]^-$ groups have only finite-dimensional representations was brought to my attention by J. Liukkonen. On the basis of Lemma 5.1 (of the author), he reached an affirmative answer to this question at approximately the same time that I did. Also, he first showed that when G is a separable $[FIA]^-$ group such that $\hat{G} = \hat{G}_{\text{fin}}$, then \hat{G} is a Hausdorff space. His proof does not depend on the more general fact (Corollary 5.3 of the author) that for any $[FIA]^-$ group, $\text{Prim } C^*(G)$ is Hausdorff.

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