ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS WITH CURVED MAJORANTS

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Abstract. Let \( \phi(x) \geq 0 \) for \(-1 \leq x \leq 1\). For a fixed \( x_0 \) in \([-1, 1]\) what can be said for \( \max |p_n(x_0)| \) if \( p_n(x) \) belongs to the class \( P_\phi \) of all polynomials of degree \( n \) satisfying the inequality \( |p_n(x)| \leq \phi(x) \) for \(-1 \leq x \leq 1\)? The case \( \phi(x) = 1 \) was considered by A. A. Markov and S. N. Bernstein. We investigate the problem when \( \phi(x) = (1 - x^2)^{1/2} \). We also study the case \( \phi(x) = |x| \) and the subclass consisting of polynomials typically real in \(|z| < 1\).

The following theorem was proved by A. A. Markov in 1889.

**Theorem A.** If \( p_n(x) \) is a polynomial of degree \( n \), such that \( |p_n(x)| \leq 1 \) for \(-1 \leq x \leq 1\), then

\[
\max_{-1 \leq x \leq 1} |p'_n(x)| \leq n^2.
\]

The original paper [9] of Markov is not readily accessible but an excellent account of this and other related results is presented in [4]. In Theorem A equality is attainable only at \( \pm 1 \) and only for \( p_n(x) = e^{inT_n(x)} \) where

\[
T_n(x) = \cos (n \cos^{-1} x) = 2^{n-1} \prod_{j=1}^{n} \{x - \cos (((j-\frac{1}{2})\pi)/n)\}
\]

\[
= \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}
\]

is the so-called Chebyshev polynomial of the first kind.

For points \( x \) lying in the interval \(|x| < \cos (\frac{1}{2} \pi/n)\) the following theorem of Bernstein [2] gives a better estimate for \( |p'_n(x)| \).

**Theorem B.** Under the conditions of Theorem A

\[
|p'_n(x)| \leq n(1 - x^2)^{-1/2}, \quad -1 < x < 1.
\]

This dominant \( n(1 - x^2)^{-1/2} \) is the best possible dominant only at the points \( x = \cos ((2k+1)\pi/(2n)) \), \( k = 1, 2, \ldots, n-1 \). It is, however, asymptotically equal to the precise bound at every fixed point in the interior of the interval as \( n \) becomes infinite.

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At a conference on Constructive Function Theory held in Varna, Bulgaria, Professor P. Turán proposed the following problem:

Problem. For any \( x_0 \) in \([-1, 1]\) determine \( \max |p'_n(x_0)| \) for all polynomials \( p_n(x) \) of degree \( \leq n \) satisfying the restriction

\[
\max_{-1 \leq x \leq 1} \frac{|p_n(x)|}{\sqrt{1-x^2}} = 1.
\]

He remarked that even the value of \( \max_{-1 \leq x \leq 1} |p'_n(x)| \) did not seem to be known for the class in (3).

For real-valued polynomials the hypothesis says that the graph of \( p_n(x) \) on the interval \(-1 < x < 1\) is contained in the closed unit disk.

Let \( \pi_n \) denote the class of polynomials \( p_n(x) \) of degree \( n \) which satisfy \( |p_n(x)| \leq 1 \) for \(-1 \leq x \leq 1\) it is enough to consider the subclass \( A_n \) whose members are in addition real on the real axis. Let \( p_n^k(z^*) = e^{i\phi}|p_n^k(z^*)| \) and let \( e^{-i\phi}p_n(z) = p_{n,1}(z) + ip_{n,2}(z) \) where \( p_{n,1} \) and \( p_{n,2} \) are elements of \( A_n \). Since \( p_{n,1}^k(z^*) = |p_{n,1}^k(z^*)| \) the maximum of \( |p_{n,1}^k(z^*)| \) is attained, if at all, for some \( p_n \) in \( A_n \). The following theorem of Duffin and Schaeffer [7, p. 240] shows that the functions in \( A_n \) are uniformly bounded on every compact set. Hence [1, p. 216] the maximum of \( |p_{n,1}^k(z^*)| \) is, in fact, attained.

Theorem C. Let \( p_n(z) \) be a polynomial of degree \( n \) or less such that in the real interval \((-1, 1) \) \( |p_n(z)| \leq 1 \). Then for \( z \) lying on the ellipse \( \ell_R \) with foci at \(-1, +1 \) and semi-axes \( \frac{1}{2}(R+R^{-1}), \frac{1}{2}(R-R^{-1}) \), we have \( |p_n(z)| \leq \frac{1}{2}(R^n + R^{-n}) \).

For precisely the same reason as above the maximum of \( |p_{n,1}^k(z^*)| \) over the class \( \pi_n \) is attained for a polynomial which is real on the real axis.

We prove

Theorem 1. If \( p_n(x) \) is a polynomial of degree \( n \) such that \( |p_n(x)| \leq (1-x^2)^{1/2} \) for \(-1 < x < 1\), then

\[
\max_{-1 \leq x \leq 1} |p'_n(x)| \leq 2(n-1).
\]

If

\[
U_n(x) = (1-x^2)^{-1/2} \sin \{(n+1) \cos^{-1} x\} = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m)!}{m!(n-2m)!} (2x)^{n-2m}
\]

is the \( n \)th Chebyshev polynomial of the second kind then \( p_n(x) = (1-x^2)U_{n-2}(x) \) satisfies the conditions of Theorem 1 and \( |p'_n(\pm 1)| = 2(n-1) \). Hence the result is best possible.

Theorem 2. Under the conditions of Theorem 1

\[
|p'_n(x)| \leq (x^2(1-x^2)^{-1} + (n-1)^2)^{1/2}, \quad -1 < x < 1.
\]
The example $p_n(x) = e^{i\theta}(1-x^2)U_{n-2}(x)$ shows that in (5) equality can be attained at those points of the interval $-1 < x < 1$ where $(n-1)(1-x^2)^{1/2} \tan \left((n-1)\cos^{-1} x\right) = x$.

It follows from Theorem B that if $p_n(x)$ is a polynomial of degree $n$ such that $|p_n(x)| \leq 1$ for $-1 \leq x \leq 1$ then $n^{-1}(1-x^2)p'_n(x) \in \pi_{n+1}$. Hence we have the following corollary of Theorem 2.

**Corollary 1.** If $p_n(x)$ is a polynomial of degree $n$ such that $|p_n(x)| \leq 1$ for $-1 \leq x \leq 1$, then

$$|p_n(x)| \leq n^{1/2}(1-x^2)^{-1/2} \tan^{-1} x,$$  

When $x = 0$ inequality (5) may be restated as follows:

If $p_n(x) = \sum_{k=0}^{n} a_k x^k \in \pi_n$ then $|a_k| = |p_n(0)| \leq n-1$. This inequality is sharp for odd $n$. Here again the extremal polynomial is $e^{i\theta}(1-x^2)U_{n-2}(x)$.

We also estimate $|a_2|$.

**Theorem 3.** If $p_n(x) = \sum_{k=0}^{n} a_k x^k \in \pi_n$ then

$$|a_2| \leq \frac{(n-1)^2 + 1}{2}.$$  

For even $n$ the bound in (7) is attained when $p_n(x) = e^{i\theta}(1-x^2)U_{n-2}(x)$.

The next theorem is a refinement of Theorem C for polynomials belonging to $\pi_n$.

**Theorem 4.** If $p_n(z) \in \pi_n$ then for $z$ lying on the ellipse $\mathcal{E}_R$ with foci at the points $-1, +1$ and semiaxes $\frac{1}{2}(R+R^{-1}), \frac{1}{2}(R-R^{-1})$, we have

$$|p_n(z)| \leq \frac{|1-z^2|^{1/2}}{2}(R^N+1).$$  

The problem of Turán mentioned earlier is a special case of the following general question subsequently asked by him:

Let $\phi(x) \geq 0$ for $-1 \leq x \leq 1$. For a fixed $x_0$ in $[-1, 1]$ what can be said for $\max |p'_n(x_0)|$ if $p_n(x)$ belongs to the class $P_\phi$ of all polynomials of degree $\leq n$ satisfying the inequality $|p_n(x)| \leq \phi(x)$ for $-1 \leq x \leq 1$.

We shall consider only the simple class $P_{|x|}$ of polynomials $p_n(x)$ of degree $\leq n$ which are dominated by the function $|x|$ on $[-1, 1]$.

If $p_n(z)$ is a polynomial of degree $n$ such that $|p_n(x)| \leq |x|$ for $-1 \leq x \leq 1$ then $p_n(z) = z g_{n-1}(z)$ where $g_{n-1}(z)$ is a polynomial of degree $n-1$. Since, clearly, $|g_{n-1}(z)| \leq 1$ for $-1 \leq z \leq 1$, Theorem A gives

$$|p'_n(x)| \leq |x| |g'_n(x)| + |g_{n-1}(x)| \leq (n-1)^2 + 1$$  

for $-1 \leq x \leq 1$. Thus we have

**Theorem 5.** If $p_n(z)$ is a polynomial of degree $n$ such that $|p_n(x)| \leq |x|$ for $-1 \leq x \leq 1$ then

$$\max_{-1 \leq x \leq 1} |p'_n(x)| \leq (n-1)^2 + 1.$$
The example $p_n(z) = zT_{n-1}(z)$, where $T_{n-1}(z)$ is the Chebyshev polynomial of the first kind of degree $n-1$, shows that (9) is sharp.

We also prove

**Theorem 6.** If $p_n(x)$ is a polynomial of degree $\leq n$ such that $|p_n(x)| \leq |x|$ for $-1 \leq x \leq 1$ then for a fixed $x_0$ in $(-1, 1)$ we have

\[(10) \quad |p_n'(x_0)| \leq (n-1)^2 x_0^2 (1-x_0^2)^{-1} + 1)^{1/2}.\]

The example $p_n(z) = e^{iy}T_{n-1}(z)$ shows that in (10) equality is attained at those points of the interval $-1 < x < 1$ where $(1-x^2)^{1/2} \tan \{(n-1)\cos^{-1} x\} = (n-1)x$.

From a geometric point of view, those members of the class $P_{1x1}$ which are typically real [10] in $|z| < 1$ constitute an interesting subclass. If we restrict ourselves to this subclass we can replace (9) by a considerably stronger inequality.

**Theorem 7.** Let $p_n(z)$ be a polynomial of degree $n$ such that $|p_n(x)| \leq |x|$ for $-1 \leq x \leq 1$. If $p_n(z)$ is typically real in $|z| < 1$ then

\[(11) \quad |p_n'(x)| \leq (n+1)/2\]

for $-1 \leq x \leq 1$.

A function $g(z)$ analytic in $|z| < 1$ is typically real in $|z| < 1$ if and only if [10, p. 112] $g(z)$ is real for real $z$ and $\text{Re} \{(1-z^2)/z)g(z)\} \neq 0$ in $|z| < 1$. Hence if $n$ is odd the polynomial

\[p_n(z) = 2(z+z^3+\cdots+z^n)/(n+1)\]

is typically real in $|z| < 1$. It also belongs to $P_{1x1}$, and since $|p_n'(1)| = (n+1)/2$ the bound in (11) cannot in general be improved.

**Lemmas.** We shall now collect some results which we shall use in the proofs of the above theorems.

If $p_n(x)$ is a polynomial of degree $n$ then $p_n(\cos \theta)$ is a trigonometric polynomial of degree $n$. Since $d\theta = -(1-x^2)^{-1/2} dx$, inequality (2) states that $|d/d\theta)p_n(\cos \theta)| \leq n$. Hence Theorem B is a consequence of the following result known as Bernstein’s theorem for trigonometric polynomials.

**Theorem D.** If $t(\theta)$ is a trigonometric polynomial of degree $n$ and $|t(\theta)| \leq 1$, then $|t'(\theta)| \leq n$.

It has been remarked by Boas [4, p. 169] that Markov’s theorem (Theorem A) would also follow from Bernstein’s theorem for trigonometric polynomials if it could be shown that $|p_n'(x)|$ attains its maximum at $\pm 1$ if $p_n(x)$ is extremal.

We observe that the following result which is a refined version of Bernstein’s theorem for trigonometric polynomials and which was independently proved by Szegö [12, p. 69], Boas [3, p. 287], van der Corput and Schaeke [6, p. 321] is more appropriate for the study of polynomials on the unit interval and gives Markov’s theorem as an immediate corollary.
Lemma 1. Let \( t(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta} \) be a real trigonometric polynomial of degree \( n \). If \( |t(\theta)| \leq 1 \) then
\[
2(r(\theta))^2 + (r'(\theta))^2 \leq n^2.
\]

This result plays a central role in our paper. First of all we use it to prove:

Lemma 2. If \( p_{n-1}(x) \) is a real valued polynomial of degree \( n-1 \) such that
\[
(1-x^2)^{1/2} |p_{n-1}(x)| \leq 1 \quad \text{for} \quad -1 < x < 1
\]
then
\[
|p_{n-1}(x)| \leq n \quad \text{for} \quad -1 \leq x \leq 1.
\]

Our hypothesis implies that \( (\sin \theta)p_{n-1}(\cos \theta) \) is a real trigonometric polynomial of degree \( n \) whose absolute value does not exceed 1. Hence according to Lemma 1
\[
n^2 \sin^2 \theta (p_{n-1}(\cos \theta))^2 + (\cos \theta)(d/d\theta)p_{n-1}^2(\cos \theta))^2 \leq n^2
\]
for real \( \theta \). At a point where \( |p_{n-1}(\cos \theta)| \) attains its maximum value, \( (d/d\theta)p_{n-1}(\cos \theta) \) must vanish. Consequently, at such a point \( \theta_0 \),
\[
n^2(\sin^2 \theta_0)(p_{n-1}(\cos \theta_0))^2 + (\cos^2 \theta_0)(p_{n-1}(\cos \theta_0))^2 \leq n^2
\]
or
\[
(n^2-1)(\sin^2 \theta_0)(p_{n-1}(\cos \theta_0))^2 + (p_{n-1}(\cos \theta_0))^2 \leq n^2.
\]
Therefore \( |p_{n-1}(\cos \theta_0)| \leq n \) which gives the desired result.

For the sake of completeness we include a proof of Lemma 1. In this way we will also be giving a complete and independent proof of Markov’s theorem (since it follows from Theorem B in conjunction with Lemma 2). It may be noted that our proof of Lemma 1 depends only on the maximum modulus principle and the Gauss-Lucas theorem [8, p. 84].

Proof of Lemma 1. Let \( p(z) \) be a polynomial of degree \( m \) such that \( |p(z)| \leq M \) for \( |z| \leq 1 \). Then for \( |\lambda| > 1 \) the polynomial \( P(z) = p(z) - \lambda M \) does not vanish in \( |z| \leq 1 \). Let
\[
Q(z) = z^n P(1/z) = z^n p(1/z) - \lambda M z^m = q(z) - \lambda M z^m.
\]
Since the function \( Q(z)/P(z) \) is holomorphic in \( |z| \leq 1 \) and \( |Q(z)| = |P(z)| \) for \( |z| = 1 \) it follows from the maximum modulus principle that \( |Q(z)| \leq 1 \) for \( |z| \leq 1 \). Replacing \( z \) by \( 1/z \) we conclude that \( |P(z)| \leq |Q(z)| \) for \( |z| \geq 1 \). Thus for \( |\mu| > 1 \) all the zeros of the polynomial \( P(z) - \mu Q(z) \) lie in \( |z| < 1 \) and so do the zeros of \( P(z) - \mu Q(z) \) by Gauss-Lucas theorem. Consequently,
\[
|p'(z)| = |P'(z)| \leq |Q'(z)| = |q'(z) - \lambda M z^{m-1}| \quad \text{for} \quad |z| \geq 1.
\]
According to our hypothesis \( |p(z)| \leq M \) for \( |z| \leq 1 \). Since
\[
p(z) = z^n q(1/z),
\]
we have
\[
|z^n q(1/z)| \leq M \quad \text{for} \quad |z| \leq 1,
\]
i.e., \(|q(z)| \leq M|z|^m\) for \(|z| \geq 1\). Hence for \(|A| > 1\) all the zeros of the polynomial \(q(z) - \Lambda Mz^m\) lie in \(|z| < 1\) and by the Gauss-Lucas theorem so do the zeros of \(q'(z) - \Lambda Mmz^{m-1}\). This implies that \(|q'(z)| \leq Mm|z|^{m-1}\) for \(|z| \geq 1\). Given a point \(z\) in the circular domain \(|z| \geq 1\), this inequality permits us to choose \(\arg \lambda\) in (14) such that \(|q'(z) - \Lambda Mmz^{m-1}| = |\lambda| Mm|z|^{m-1} - |q'(z)|\). We readily obtain

\[
p'(z) + |q'(z)| \leq Mm|z|^{m-1}\quad \text{for } |z| \geq 1.
\]

In particular, we have

\[
|\left(\frac{d}{d\theta}\right)p(e^{i\theta})| + | -imp(e^{i\theta}) + \left(\frac{d}{d\theta}\right)p(e^{i\theta})| \leq Mm.
\]

If \(t(\theta) = \sum_{n=-\infty}^{\infty} a_ne^{ik\theta}\) is a trigonometric polynomial of degree \(n\) and \(|t(\theta)| \leq 1\) then \(e^{i\theta}t(\theta) = p(e^{i\theta})\) where \(p(z)\) is a polynomial of degree \(2n\) such that \(|p(z)| \leq 1\) for \(|z| \leq 1\). From (16) we get \(|int(\theta) + t'(\theta)| + | -int(\theta) + t'(\theta)| \leq 2n\). Hence \(n^2(t(\theta))^2 + (t'(\theta))^2 \leq n^2\) if the trigonometric polynomial is real.

For the proof of Theorem 7 we shall need the following result due to de Bruijn [5, Theorem 4].

**Lemma 3.** Let \(C\) be a circular domain in the \(z\)-plane, and \(S\) an arbitrary point set in the \(w\)-plane. If the polynomial \(P(z)\) of degree \(n\) satisfies \(P(z) = w \in S\) for any \(z \in C\), then we have, for any \(z \in C\) and any \(\xi \in C\),

\[
\frac{(\xi/n)P'(z) + P(z) - zP'(z)/n}{P(z)} \in S.
\]

**Proofs of the theorems.**

**Proof of Theorem 1.** As remarked earlier there is no loss of generality in assuming \(p_n(x)\) to be real-valued. Since \(p_n(x)\) vanishes at the points \(-1, +1\) we have \(p_n(x) = (1-x^2)q_{n-2}(x)\) where \(q_{n-2}(x)\) is a polynomial of degree \(n-2\). We set \((1-x^2)^{1/2}q_{n-2}(x) = f(x)\) and write \(p_n(x)\) as the product of \((1-x^2)^{1/2}\) and \(f(x)\). Thus

\[
\mid p_n(x) \mid = \mid -x(1-x^2)^{-1/2}f(x) + (1-x^2)^{1/2}f'(x) \mid
\]

\[
\leq \mid x \mid \mid (1-x^2)^{-1/2}f(x) \mid + \mid (1-x^2)^{1/2}f'(x) \mid.
\]

We observe that \(f(\cos \theta)\) is a trigonometric polynomial of degree \(n-1\) whose absolute value does not exceed 1. Hence according to Theorem D \(\mid (d/d\theta)f(\cos \theta) \mid \leq n-1\) for real \(\theta\). Hence according to Theorem D \(\mid (d/d\theta)f(\cos \theta) \mid \leq n-1\) for real \(\theta\). Since

\[
(-\sin \theta) \frac{d}{d(\cos \theta)} f(\cos \theta) = \frac{d}{d\theta} f(\cos \theta)
\]

we get

\[
\mid (1-x^2)^{1/2}f'(x) \mid \leq n-1\quad \text{for } -1 \leq x \leq 1.
\]

Now let us note that \((1-x^2)^{-1/2}f(x)\) is a polynomial of degree \(n-2\) such that \(\mid (1-x^2)^{-1/2}f(x) \mid = \mid f(x) \mid \leq 1\) for \(-1 \leq x \leq 1\). Hence by Lemma 2

\[
\mid (1-x^2)^{-1/2}f(x) \mid \leq n-1\quad \text{for } -1 \leq x \leq 1.
\]
Using (18) and (19) in (17) we get the desired estimate for \( \max_{-1 \leq x \leq 1} |p'_n(x)| \).

**Remark.** Our proof of Theorem 1 makes particular use of the fact that the polynomial \( p_n(z) \) under consideration vanishes at the points \(-1, +1\). However, the bound for \( \max_{-1 \leq x \leq 1} |p'_n(x)| \) is not very much improved if we only add this requirement to the hypothesis in Markov’s theorem. By considering the polynomial \( p_n(x) = \cos n \cos^{-1} x \cos (\pi/2n) \) we see that \( \max_{-1 \leq x \leq 1} |p'_n(x)| \) can be as large as \( n \cot (\pi/2n) \) if \( p_n(\pm 1) = 0 \) and \( \max_{-1 \leq x \leq 1} |p_n(x)| = 1 \). A theorem of Schur [11, pp. 284-285] says that \( \max_{-1 \leq x \leq 1} |p'_n(x)| \leq n \cot (\pi/2n) \) for every polynomial of degree \( \leq n \) satisfying the inequality \( |p_n(x)| \leq 1 \) for \(-1 \leq x \leq 1\) and vanishing at the points \(-1, +1\).

**Proof of Theorem 2.** Without loss of generality we may assume \( p_n(x) \) to be real-valued. Again setting \( f(x) = (1 - x^2)^{-1/2} p_n(x) \) we see that \( f(\cos \theta) \) is a real trigonometric polynomial of degree \( n - 1 \) whose absolute value does not exceed 1. Hence from Lemma 1

\[
(n-1)^2 f^2(x) + (1-x^2)(f'(x))^2 \leq (n-1)^2 \quad \text{for} \quad -1 \leq x \leq 1.
\]

Using this inequality in (17) we conclude that for \(-1 < x < 1\)

\[
|p'_n(x)| \leq |x|(1-x^2)^{-1/2}|f(x)| + (n-1)(1-|f(x)|^2)^{1/2} \leq \max_{0 \leq x \leq 1} \{|x|(1-x^2)^{-1/2} + (n-1)(1-y^2)^{1/2}\}.
\]

For a given \( x \) in \((-1, 1)\) the maximum of the expression \( |x|(1-x^2)^{-1/2} + (n-1)(1-y^2)^{1/2} \) is \( \{x^2(1-x^2)^{-1} + (n-1)^2\}^{1/2} \) which is attained when

\[
y = |x|((n-1)^2(1-x^2) + x^2)^{-1/2}.
\]

**Proof of Theorem 3.** We have

\[
|a_k| = \frac{1}{2} |p'_n(0)| = \frac{1}{2} |f'(0) - f(0)| \leq \frac{1}{2} \{|f'(0)| + |f(0)|\}
\]

where \( f(x) = (1 - x^2)^{-1/2} p_n(x) \). Now if \( F(\theta) = f(\cos \theta) \) then \( |f''(0)| = |F''(\pi/2)| \) and hence by Theorem D \( |f''(0)| \leq (n-1)^2 \). Since \( |f(0)| \leq 1 \) we get the desired result.

**Proof of Theorem 4.** This result is proved in exactly the same way as Theorem C was proved by Duffin and Schaeffer [7, p. 240]. We need only to observe that \( f(\cos z) = (\csc z) p_n(\cos z) \) is an entire function of exponential type \( n - 1 \).

If \( p_n(z) = (1 - z^2) U_{n-2}(z) \), then (8) becomes an equality at the points

\[
z = (R + R^{-1})/2 \cos \phi_k \pm i((R - R^{-1})/2) \sin \phi_k
\]

where \( \phi_k = (2k + (-1)^k)/2(n-1)\pi, \ k = 0, 1, 2, \ldots \).

**Proof of Theorem 6.** It is enough to prove the theorem for polynomials which assume real values on the real axis. We have \( p_n(x) = x g_{n-1}(x) \) where \( g_{n-1}(x) \) is a polynomial of degree \( n-1 \) which assumes real values for real \( x \) and \( |g_{n-1}(x)| \leq 1 \) for \(-1 \leq x \leq 1\). Thus \( g_{n-1}(\cos \theta) \) is a real trigonometric polynomial of degree \( n-1 \) such that \( |g_{n-1}(\cos \theta)| \leq 1 \). Hence from Lemma 1 we get

\[
(n-1)^2 (g_{n-1}(x))^2 + (1-x^2)(g'_{n-1}(x))^2 \leq (n-1)^2 \quad \text{for} \quad -1 \leq x \leq 1.
\]
We use this inequality in \( |p'_n(x)| \leq |g_{n-1}(x)| + |x| |g'_{n-1}(x)| \) to complete the proof of the theorem in precisely the same way as for Theorem 2.

**Proof of Theorem 7.** According to hypothesis \( p_n(z) \) assumes real values in \( |z| < 1 \) if and only if \( z \) is real. Hence \( p'_n(x) \neq 0 \) for \( -1 < x < 1 \) and \( p_n(x) \) is a monotonic function on the interval \( -1 \leq x \leq 1 \). Without loss of generality we may suppose \( p_n(x) \) to be increasing on \([-1, 1]\). Let \( x_0 \) be a given point of the open interval \((0, 1)\). The polynomial \( P(z) = p_n(x_0 z) \) is typically real in \( |z| < 1/x_0 \) and hence in \( |z| \leq 1 \). Also \( |P(x)| \leq x_0 \) for \( -1 \leq x \leq 1 \). Since the only zero of \( P(z) \) in \( |z| < |x_0|^{-1} \) is a simple zero at the origin, \( Q(z) = z^n P(z^{-1}) \) is a polynomial of degree \( n-1 \) having all its zeros in \( |z| \leq x_0 \). Hence according to Walsh’s generalization of Laguerre’s theorem [13, Lemma 1, p. 13] \( Q'(1)/Q(1) = (n-1)/(1-w) \) where \( |w| \leq x_0 \). Consequently \( |Q'(1)| \geq ((n-1)/(1+x_0)|Q(1)| \), i.e.

\[
|p_n(x_0) - x_0 p'_n(x_0)| \geq \frac{n-1}{1+x_0} |p_n(x_0)| = \frac{n-1}{1+x_0} p_n(x_0).
\]

But if \( z_1, z_2, \ldots, z_{n-1} \) are the zeros of \( z^{-1} p_n(z) \) then

\[
\frac{x_0 p'_n(x_0)}{p_n(x_0)} = \text{Re} \left( \frac{x_0 p'_n(x_0)}{p_n(x_0)} \right) = 1 + \sum_{j=1}^{n-1} \text{Re} \left( \frac{x_0}{x_0 - z_j} \right)
\]

where \( \text{Re} \left( x_0/(x_0 - z_j) \right) \leq \frac{1}{2} \), \( 1 \leq j \leq n-1 \), since \( |z_j| \geq 1 \). Thus

\[
n p_n(x_0) - x_0 p'_n(x_0) \geq \frac{1}{2} (n-1) p_n(x_0) \geq 0,
\]

and (22) can be written as

\[
n p_n(x_0) - x_0 p'_n(x_0) \geq \frac{n-1}{1+x_0} p_n(x_0).
\]

This implies that the point \( p_n(x_0) - n^{-1} x_0 p'_n(x_0) \) lies on the interval

\[
[(1-n^{-1})(1+x_0)^{-1} p_n(x_0), p_n(x_0)].
\]

Now we note that the image \( S \) of the circular domain \( |z| \leq x_0 \) under the mapping \( w = p_n(z) \) lies in the plane cut along the positive real axis from \( p_n(x_0) \) to infinity. According to Lemma 3 the disk

\[
|w - \{ p_n(x_0) - n^{-1} x_0 p'_n(x_0) \}| \leq n^{-1} x_0 p'_n(x_0)
\]

lies in \( S \). This is possible only if

\[
n^{-1} x_0 p'_n(x_0) \leq p_n(x_0) (1 - (1-n^{-1})(1+x_0)^{-1})
\]

Since \( p_n(x_0) \leq x_0 \) we get

\[
p'_n(x_0) \leq n - (n-1)(1+x_0)^{-1}.
\]

By continuity \( p'_n(0) \leq 1 \) and \( p'_n(1) \leq (n+1)/2 \). Hence

\[
\max_{0 \leq x \leq 1} p'_n(x) \leq (n+1)/2.
\]
Applying this result to \(-p_n(-x)\) we get

\[
\max_{-1 \leq x \leq 0} p'_n(x) \leq (n+1)/2.
\]

The desired result follows from (24) and (25).

Inequality (23) gives an estimate for \(|p_n(x_0)|\) at a fixed point \(x_0\) in \([-1, 1]\) but the bound does not appear to be sharp except at \(-1, 0, +1\).

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References