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# A THEOREM OF COMPLETENESS FOR FAMILIES OF COMPACT ANALYTIC SPACES

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**Abstract.** A sufficient condition is given for a family of compact analytic spaces to be complete. This condition generalizes to analytic spaces the Theorem of Completeness of Kodaira and Spencer [6]. It contains, as a special case, the rigidity theorem proved by Schuster in [11].

**Introduction.** If $X$ is a family of deformations of a compact complex manifold, $X_0$, parametrized by a germ of complex manifold, $(S, 0)$, Kodaira and Spencer have proved in [6] that $X$ contains all small deformations of $X_0$ (i.e. is (locally) complete) provided that the Kodaira-Spencer map $\rho_0: T_0(S) \to H^1(X_0, \Theta_0)$ is surjective. This condition is equivalent to the statement that $X$ contains all deformations of $X_0$ parametrized by the analytic space, $D$, whose underlying topological space is a point, and whose structure sheaf is the ring of dual numbers. The paper [6] appeared before Kuranishi's Completeness Theorem [7], [8]. A very simple proof can be given of the Kodaira-Spencer Theorem if we use Kuranishi's result. We give this proof in §1.

If $X_0$ has singularities, it is quite plausible that an analog of the Kuranishi space exists; but, at the moment, no proof exists. A formal analog of the Kuranishi space can be shown to exist, however. In this paper we will prove a generalization of some results of Artin [1] which will allow us to prove the completeness theorem using this formal analog.

The Theorem of Completeness is useful in answering questions of whether a certain property of $X_0$ is shared by all small deformations of $X_0$—for it allows one to recognize that a family of spaces having the special property is complete. We have used the theorem in this way in our paper [13]. Some applications are given in §4.

I am indebted to Michael Artin for a sketch of the proof of Theorem 2.2.

1. **Quasi-representable functors.** By a family of compact complex analytic spaces, we mean a pair of analytic spaces $X$, $S$ together with a flat, proper morphism $\pi: X \to S$. If $X_0$ is a given compact analytic space, a family of deformations of $X_0$ is a family, $(X, \pi, S)$, of compact spaces together with a distinguished point $s_0 \in S$

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and a fixed isomorphism $i\colon X_0 \to X_{s_0}$ ($X_{s_0}$ is the fibre of $\pi$ over $s_0$). An isomorphism of families of deformations is required to be compatible with the given isomorphisms $i$. Since all considerations are local with respect to $S$, we identify a family with its restriction to any open neighborhood of the distinguished point $s_0$. Thus, following Grothendieck [4, 16–06], we define a contravariant functor, $\mathscr{F}$, from the category of germs of analytic spaces to the category of sets by

$$\mathscr{F}(S, 0) = \text{set of isomorphism classes of germs of families of deformations of } X_0 \text{ parametrized by } (S, 0).$$

If $(S', 0) \to (S, 0)$ is a morphism, the map $\mathscr{F}(S) \to \mathscr{F}(S')$ is given by the fibre product. We call $\mathscr{F}$ the deformation functor of $X_0$.

An analytic algebra is a local $\boldsymbol{C}$-algebra which is the quotient of a ring $\boldsymbol{C}\{x_1, \ldots, x_n\}$ of convergent power series. The category $\mathscr{A}$ of analytic algebras (the morphisms being $\boldsymbol{C}$-algebra homomorphisms) is dual to the category of germs of analytic spaces [4, 13-02]. Thus, alternatively, $\mathscr{F}$ may be considered as a covariant functor, $F$, from the category $\mathscr{A}$ to the category of sets.

In general, the functor $F$ will not be representable—even if $X_0$ is a manifold (see [14]). However, Kuranishi has essentially proved the following (see [7], [8] and [3]):

THEOREM 1.1 (KURANISHI). *If $X_0$ is a compact complex manifold, $F$ the deformation functor of $X_0$, then there is a pair $(A, \xi)$ with $A \in \mathscr{A}$, $\xi \in F(A)$ such that $h_A(B) \to F(B), f \mapsto F(f)(\xi)$ is*

(i) *surjective for all $B \in \mathscr{A}$,*

(ii) *bijective for $B = \boldsymbol{C}[t]/(t^2)$,*

*where $h_A(B) = \mathrm{Hom}_{\mathscr{A}}(A, B)$.*

*Notation.* We will always use $D$ to denote $\boldsymbol{C}[t]/(t^2)$ or the corresponding germ of analytic space.

For $A \in \mathscr{A}$, $h_A(D)$ is naturally isomorphic to the Zariski tangent space of the analytic germ corresponding to $A$. If $X_0$ is a complex manifold, $F(D)$ may be identified with $H^1(X_0, \Theta_0)$ (see [4] and [14]). $h_A(D) \to F(D)$ is, with these identifications, the Kodaira-Spencer map. Thus, for arbitrary $X_0$, $h_A(D) \to F(D)$ plays the role of the Kodaira-Spencer map. $F(D)$ has, in fact, a vector space structure (see [10]), as does $h_A(D)$; the map is a vector space homomorphism.

DEFINITION 1.2. A covariant functor, $F$, from $\mathscr{A}$ to (Sets) is called *quasi-representable* if there is a pair $(A, \xi)$, as in Theorem 1.1, such that (i), (ii) hold. The pair $(A, \xi)$ is said to *quasi-represent* $F$. If only (i) is satisfied, $(A, \xi)$ is called *complete*.

*Notation.* If $A \in \mathscr{A}$, let $A^{(n)} = A/\mathfrak{M}_A^{n+1}$. If $f\colon A \to B$ is a morphism in $\mathscr{A}$, let $f^{(n)}\colon A^{(n)} \to B^{(n)}$ denote the induced morphism. If $\xi \in F(A)$, $F$ a covariant functor, let $\xi^{(n)}$ denote the image of $\xi$ in $F(A^{(n)})$.

We will now collect some results about morphisms in $\mathscr{A}$.

PROPOSITION 1.3. *If $f\colon A \to B$ is a morphism in $\mathscr{A}$ such that $f^{(1)}$ is surjective, then $f$ is surjective.*

**Proof.** Take a basis for $\mathfrak{M}_A/\mathfrak{M}_A^2$ and pick representatives in $\mathfrak{M}_A$. We obtain an epimorphism $C\{x\} \to A$ which is an isomorphism mod $\mathfrak{M}_A^2$. Thus, for $A \in \mathscr{A}$, we may find a regular $\tilde{A} \in \mathscr{A}$ and an epimorphism $\alpha: \tilde{A} \to A$ such that $\alpha^{(1)}$ is an isomorphism.

In the situation above, the morphism $f$ may be extended to a morphism $\tilde{f}: \tilde{A} \to \tilde{B}$. Since $f^{(1)}$ is surjective, so is $\tilde{f}^{(1)}$. It follows from the Implicit Function Theorem that $\tilde{f}$ is surjective ($\tilde{A}$, $\tilde{B}$ are convergent power series rings). Hence $f$ is surjective.

PROPOSITION 1.4. *If $A, B \in \mathscr{A}$, $f: A \to B$, $g: B \to A$ morphisms such that $f^{(1)}$, $g^{(1)}$ are surjective, then $f$, $g$ are isomorphisms.*

**Proof.** Proposition 1.3 shows that $f$ and $g$ are surjective. Thus $g \circ f: A \to A$ is surjective. It must therefore be an isomorphism (if $K = \ker(g \circ f)$, $K \subset (g \circ f)^{-1}(K) \subset (g \circ f)^{-2}(K) \subset \cdots$ would be a nonstationary chain unless $K = 0$). Thus $f$ is an isomorphism. Similarly $g$ is an isomorphism.

PROPOSITION 1.5. *If $f: A \to B$ is a morphism in $\mathscr{A}$, $B$ regular, $f^{(1)}$ injective, then $f$ has a left inverse and, moreover, $A$ is regular.*

**Proof.** As in the proof of Proposition 1.3, we have $\alpha: \tilde{A} \to A$. $f^{(1)} \circ \alpha^{(1)}$ is injective, so, by the Implicit Function Theorem, $f \circ \alpha$ has a left inverse $g: B \to \tilde{A}$. Since $\alpha$ is surjective, $\alpha \circ g$ is a left inverse for $f$. Moreover, $f \circ \alpha$ is injective, so $\alpha$ is injective and hence an isomorphism.

DEFINITION 1.6. If $F$ is a covariant functor from $\mathscr{A}$ to (Sets), a *pair* $(A, \xi)$ consists of an $A \in \mathscr{A}$ and $\xi \in F(A)$. A *morphism* of pairs $(A, \xi) \to (B, \eta)$ is a morphism $f: A \to B$ such that $\eta = F(f)(\xi)$.

THEOREM 1.7. *If $F$ is quasi-representable by a pair $(A, \xi)$, the pair is unique up to (noncanonical) isomorphism.*

**Proof.** Let $(A, \xi)$, $(B, \eta)$ quasi-represent $F$. By condition (i) there are morphisms $f: (A, \xi) \to (B, \eta)$, $g: (B, \eta) \to (A, \xi)$. Condition (ii) implies that $f^{(1)}$ and $g^{(1)}$ are bijective. By Proposition 1.4, $f$ and $g$ are isomorphisms.

We will now show how the Theorem of Completeness follows from Kuranishi's Theorem.

THEOREM 1.8. *If $F$ is quasi-representable, $(B, \eta)$ a pair with $B$ regular such that $h_B(D) \to F(D)$ is surjective, then $(B, \eta)$ is complete.*

**Proof.** Let $(A, \xi)$ quasi-represent $F$. We have an $f: (A, \xi) \to (B, \eta)$, so the diagram

$$\begin{tikzcd} h_B(D) \arrow[rr] \arrow[dr] & & h_A(D) \arrow[dl] \\ & F(D) & \end{tikzcd}$$

is commutative. $h_B(D) \to F(D)$ is surjective while $h_A(D) \to F(D)$ is bijective. Thus, $h_B(D) \to h_A(D)$ is surjective or, what is the same, $f^{(1)}$ is injective. By Proposition 1.5, $f$ is left invertible. Let $g$ be a left inverse. We have $g\colon (B, \eta) \to (A, \xi)$ and hence a commutative diagram

$$\begin{tikzcd} h_A(C) \arrow[rr] \arrow[dr] & & h_B(C) \arrow[dl] \\ & F(C) & \end{tikzcd}$$

for any $C \in \mathscr{A}$. Since $h_A(C) \to F(C)$ is surjective, so is $h_B(C) \to F(C)$.

2. **Solutions of analytic equations.** In Artin's paper [1, Theorem 1.5a], it has been shown that if

$$\begin{tikzcd} A & & B \\ & C \arrow[ul, "v"] \arrow[ur, "w"'] & \end{tikzcd}$$

is a diagram in $\mathscr{A}$, and if there is a morphism $\bar{u}\colon \hat{B} \to \hat{A}$ with $\bar{u}\hat{w} = \hat{v}$, then there is also a $u\colon B \to A$ making the diagram commute (here $\hat{}$ denotes $\mathfrak{M}$-adic completion). Moreover, for any $c$, $u$ can be chosen such that $\hat{u} \equiv \bar{u} \bmod \mathfrak{M}^c$. The hypothesis means that we are given a compatible sequence of maps $u_n\colon B^{(n)} \to A^{(n)}$ such that $u_n w^{(n)} = v^{(n)}$. The purpose of this section is to prove Theorem 2.4 which removes from us the obligation of finding a *compatible* sequence of maps: it is enough that there exists a $u_n$ for each $n$.

The proof of this and related theorems depends on the following lemma:

LEMMA 2.1. *Let $K$ be an algebraically closed, uncountable field,*

$$(*) \qquad \cdots \to V_n \xrightarrow{\phi_n} V_{n-1} \to \cdots \to V_0$$

*a chain of affine varieties over $K$ and regular maps. If $V_n \neq \varnothing$ for all $n$, then*

$$\operatorname{proj\,lim} V_n \neq \varnothing.$$

**Proof.** Adjoin to the prime field the coefficients of the defining equations for the $V_n$ and maps $\phi_n$. We obtain a field $k \subset K$ over which $K$ has infinite transcendence degree ($K$ is a universal domain). $k$ is the ground field for specializations, etc. Consideration of the irreducible components of the $V_n$ shows that we may reduce to the case in which the $V_n$ are irreducible (pigeon-hole principle). Let $g_n$ be a generic point for $V_n$ and $P_n$ its image in $V_0$. $P_{n+1}$ is a specialization of $P_n$. Thus there is an $N$ so that the $P_n$ with $n \geqq N$ are generic specializations of each other. Since $P_n$ comes from $V_n$, so does $P_N$ if $n \geqq N$. Thus $P_N$ is in the image of all the maps $V_n \to V_0$. Replace (*) by

$$(**) \qquad \cdots \to V'_n \to \cdots \to V'_1$$

where $V_n'$ is the subvariety of $V_n$ consisting of points whose image in $V_0$ is $P_N$. The previous argument produces a $v_1 \in V_1$ which is in the image of all the maps $V_n \to V_1$ and whose image in $V_0$ is $v_0 = P_N$. It should now be clear how to obtain, by induction, a point in proj lim $V_n$.

THEOREM 2.2. *Let $f_i(x, y) \in \boldsymbol{C}\{x, y\}$, $i = 1, \ldots, m$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_N)$. Suppose that for each $k$ we have $y_\nu^{(k)}(x) \in \boldsymbol{C}\{x\}$ with no constant term such that*

$$(1)_k \qquad f(x, y^{(k)}(x)) \equiv 0 \mod \mathfrak{M}^{k+1}.$$

*Then there are $y_\nu(x) \in \boldsymbol{C}\{x\}$ with no constant term such that $f(x, y(x)) = 0$. ($\mathfrak{M}$ denotes the maximal ideal of $\boldsymbol{C}\{x\}$.)*

**Proof.** Observe that in the expansion of $y_\nu^{(k)}$, only the coefficients of $x^I$ with $|I| \leqq k$ (multi-index notation!) make any contribution to the problem. Thus, we may write

$$(2) \qquad y_\nu^{(k)} = \sum_{|I| \leqq k} y_{\nu I}^{(k)} x^I.$$

The $y_{\nu I}^{(k)}$ may be taken as coordinates for a point in an affine space $\boldsymbol{C}^{N_k}$. $(1)_k$ imposes polynomial relations on the coordinates of this point. Hence there is an affine variety $V_k$ in $\boldsymbol{C}^{N_k}$ whose points correspond, via (2), to solutions of $(1)_k$. The projection map $\boldsymbol{C}^{N_{k+1}} \to \boldsymbol{C}^{N_k}$ takes $V_k$ to $V_{k-1}$. We obtain a sequence of affine varieties over $\boldsymbol{C}$ as in Lemma 2.1. By the Lemma, there is a point in proj lim $V_k$ which, clearly, corresponds to a formal solution of $f(x, y) = 0$. Now apply Artin's Theorem [1, Theorem 1.2].

We will need a more general result:

THEOREM 2.3. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_m, \mathfrak{B}_1, \ldots, \mathfrak{B}_N$ be proper ideals in $\boldsymbol{C}\{x\}$. Let*

$$f_i(x, y) \in \boldsymbol{C}\{x, y\}, \qquad i = 1, \ldots, m.$$

*Let $u_1, \ldots, u_N \in \boldsymbol{C}\{x\}$. Suppose that for each $k$ we have $y_\nu^{(k)} \in \boldsymbol{C}\{x\}$ with no constant term such that*

$$(1)_k \qquad f_i(x, y^{(k)}(x)) \in \mathfrak{A}_i + \mathfrak{M}^{k+1}, \qquad y_\nu^{(k)} \equiv u_\nu \mod (\mathfrak{B}_\nu + \mathfrak{M}^{k+1}).$$

*Then there are $y_\nu \in \boldsymbol{C}\{x\}$ with no constant term such that*

$$(1)_\infty \qquad f_i(x, y(x)) \in \mathfrak{A}_i, \qquad y_\nu \equiv u_\nu \mod \mathfrak{B}_\nu.$$

**Proof.** Let $a_{i\mu} \in \boldsymbol{C}\{x\}$ generate $\mathfrak{A}_i$, $b_{\nu\tau} \in \boldsymbol{C}\{x\}$ generate $\mathfrak{B}_\nu$. As before, we write $y_\nu^{(k)} = \sum_{|I| \leqq k} y_{\nu I}^{(k)} x^I$. The conditions $(1)_k$ are equivalent to the existence of $c_{i\mu}^{(k)}$, $d_{\nu\tau}^{(k)} \in \boldsymbol{C}$, $z_{i\mu}^{(k)}$, $w_{\nu\tau}^{(k)} \in \boldsymbol{C}\{x\}$ with no constant term, such that

$$(2)_k \qquad \begin{aligned} &f_i(x, y^{(k)}(x)) - \sum (c_{i\mu}^{(k)} + z_{i\mu}^{(k)}(x)) a_{i\mu} \equiv 0 \quad (\mathfrak{M}^{k+1}), \\ &y_\nu^{(k)}(x) - u_\nu(x) - \sum (d_{\nu\tau}^{(k)} + w_{\nu\tau}^{(k)}(x)) b_{\nu\tau} \equiv 0 \quad (\mathfrak{M}^{k+1}). \end{aligned}$$

We take $c_{i\mu}^{(k)}$, $d_{\nu\tau}^{(k)}$, $y_{\nu}^{(k)}{}_{I}$, $z_{i\mu}^{(k)}{}_{I}w_{\nu\tau}^{(k)}{}_{I}$ as coordinates of a point in an affine space and proceed exactly as in the proof of Theorem 2.2. The result is a formal solution of the equations $(2)_\infty$ and hence, by Artin's Theorem, there is a convergent solution.

THEOREM 2.4. *Let $A, B, C \in \mathscr{A}$, $v: C \to A$, $w: C \to B$ morphisms. Suppose, for each $k$, a morphism $u_k: B^{(k)} \to A^{(k)}$ such that $u_k w^{(k)} = v^{(k)}$. Then we can find a morphism $u: B \to A$ such that $uw = v$. Moreover, if the $u_k$, $k \geqq c$, coincide* mod $\mathfrak{M}^{c+1}$, *then $u$ can be chosen to coincide with $u_c$* mod $\mathfrak{M}^{c+1}$.

**Proof.** Let $A = \boldsymbol{C}\{x\}/\mathfrak{A}$, $B = \boldsymbol{C}\{y\}/\mathfrak{B}$, $C = \boldsymbol{C}\{z\}/\mathfrak{C}$ with $\mathfrak{A} = (f_1, \ldots, f_a)$, $\mathfrak{B} = (g_1, \ldots, g_b)$, $\mathfrak{C} = (h_1, \ldots, h_c)$. The map $w$ is described by $z_\nu \in \boldsymbol{C}\{y\}$ without constant term such that $h_k(z(y)) \in \mathfrak{B}$ for all $k$. Similarly, $v$ is described by $'z_\nu \in \boldsymbol{C}\{x\}$ such that $h_k('z(x)) \in \mathfrak{A}$. A map $u$ is described by $y_\mu \in \boldsymbol{C}\{x\}$ such that $g_i(y(x)) \in \mathfrak{A}$ and the condition $uw = v$ is exactly $z_\nu(y(x)) \equiv 'z_\nu(x)$ mod $\mathfrak{A}$. Thus, we must show the existence of a convergent solution of

$$g_i(y(x)) \in \mathfrak{A}, \qquad z_\nu(y(x)) - 'z_\nu(x) \in \mathfrak{A}.$$

We are given solutions mod $\mathfrak{M}^{k+1}$ for each $k$. So, by Theorem 2.3, there is a convergent solution. The second assertion is proved by applying Theorem 2.3 with $\mathfrak{B}_\nu = \mathfrak{M}^{c+1}$.

REMARK. Generalizations of the other Artin Theorems can be proved similarly.

3. **A theorem of completeness.** In §1 we showed how to obtain a completeness theorem for a quasi-representable functor $F$. No special assumptions were made about the nature of $F$. If $F$ is not known to be quasi-representable, it is more difficult to prove a completeness theorem. In this section we will prove the completeness theorem when $F$ is the deformation functor of a compact analytic space $X_0$. This proof uses (1) a sort of formal quasi-representability for $F$ which follows from the work of Schlessinger [10]; (2) the relative form of Douady's Theorem as proved in [9] and [12]; and (3) the generalization of Artin's Theorems which we proved in §2. If the full strength of Schlessinger's results are used, it is possible to avoid the use of the results of §2 by showing directly that maps at each level can be chosen compatibly. There seemed to be no point in doing this, however.

We first prove a generalization of Proposition 1.5.

PROPOSITION 3.1. *Let $A, B \in \mathscr{A}$, $B$ regular. If $f_n: A^{(n)} \to B^{(n)}$ is a morphism with $f_n^{(1)}$ injective, then $f_n$ has a left inverse.*

**Proof.** Let $\alpha: \tilde{A} \to A$, $\beta: \tilde{B} \to B$ be as in the Proof of Proposition 1.3. We may assume $\tilde{B} = B$, $\beta = \mathrm{id}$. The map $f_n$ extends to $\tilde{f}_n: \tilde{A} \to B$ with $\tilde{f}_n^{(1)}$ injective. Thus $\tilde{f}_n$ has a left inverse $\tilde{g}_n$. The composition $g_n': B \to \tilde{A} \to A^{(n)}$ sends $\mathfrak{M}_B^{n+1}$ to 0. Thus $g_n'$ defines $g_n: B^{(n)} \to A^{(n)}$. It is easy to see that $g_n$ is a left inverse for $f_n$.

Let $\mathscr{A}_0$ denote the subcategory of $\mathscr{A}$ consisting of Artinian local $\boldsymbol{C}$-algebras. Let $R$ be a complete local $\boldsymbol{C}$-algebra such that $R^{(n)} \in \mathscr{A}_0$ for all $n$. Let

$$\xi = \{\xi_n\} \in \operatorname{proj\,lim} F(R^{(n)}).$$

For each $B \in \mathscr{A}_0$, $\xi$ defines a map $h_R(B) \to F(B)$.

DEFINITION 3.2 [10]. If $F$, $G$ are covariant functors from $\mathscr{A}_0$ to (Sets), a functorial morphism $F \to G$ is called smooth if for every epimorphism $B \to C$ in $\mathscr{A}_0$, the map $F(B) \to F(C) \times_{G(C)} G(B)$ is surjective.

DEFINITION 3.3 [10]. If $F$ is a covariant functor $\mathscr{A}_0$ to (Sets), a pair $(R, \xi)$ as above is called a *hull* for $F$ if

(i) $h_R \to F$ is smooth,

(ii) $h_R(D) \to F(D)$ is bijective.

If $F(C)$ is a point (which is the case for a deformation functor), it follows that

(iii) $h_R(B) \to F(B)$ is surjective for all $B \in \mathscr{A}_0$.

Thus, for $F$ a deformation functor, $(R, \xi)$ represents a formal family which is formally complete.

In what follows, $F$ is a covariant functor from $\mathscr{A}$ to (Sets) such that $F(C)$ is a point. We will say $F$ has a hull if its restriction to $\mathscr{A}_0$ has a hull.

DEFINITION 3.4. A pair $(A, \xi)$ is called *formally complete* (for $F$) if $h_A(B) \to F(B)$ is surjective for all $B \in \mathscr{A}_0$.

THEOREM 3.5. *If $F$ has a hull, $(A, \xi)$ a pair such that $A$ is regular and $h_A(D) \to F(D)$ is surjective, then $(A, \xi)$ is formally complete.*

REMARK. It should be observed that only properties (ii) and (iii) of a hull are used. We do not use the full strength of the hypothesis.

**Proof of Theorem 3.5.** Let $(R, \eta)$, $\eta = \{\eta_n\}$, be a hull for $F$. For each $n$ we obtain $f_n\colon R^{(n)} \to A^{(n)}$ such that $F(f_n)(\eta_n) = \xi^{(n)}$. As in the proof of Theorem 1.8 we deduce that $f_n^{(1)}$ is injective. By Proposition 3.1, $f_n$ has a left inverse $g_n$. $\eta_n = F(g_n)(\xi^{(n)})$ so that

$$\begin{tikzcd} h_{R^{(n)}}(B) \arrow[rr] \arrow[dr] & & h_{A^{(n)}}(B) \arrow[dl] \\ & F(B) & \end{tikzcd}$$

commutes. If $B \in \mathscr{A}_0$ such that $\mathfrak{M}_B^{n+1} = 0$, then $h_A(B) \to F(B)$ is surjective since $h_A(B) = h_{A^{(n)}}(B)$ for such $B$. For any $B \in \mathscr{A}_0$ there is an $n$ such that $\mathfrak{M}_B^{n+1} = 0$. Hence $(A, \xi)$ is formally complete.

THEOREM 3.6. *If $F$ is the deformation functor for a compact analytic space $X_0$, then $F$ has a hull.*

**Proof.** Schlessinger provides necessary and sufficient conditions for the existence of a hull [10, Theorem 2.11]. To show that a deformation functor satisfies these conditions, one follows the same plan as in Schlessinger's paper for the algebraic deformation functor. The only task is to show that the sheaves of rings define there place on $|X_0|$ the structure of analytic space. This task is technical but not difficult.

THEOREM 3.7. *If $F$ is the deformation functor for a compact analytic space $X_0$, $(A, \xi)$ a pair which is formally complete, then $(A, \xi)$ is complete.*

**Proof.** It is convenient here to switch to the dual formulation. Thus $(A, \xi)$ corresponds to a family $Y/T$ of deformations of $X_0$. Let $X/S$ be another family of deformations. We have, for each $n$, a morphism $f_n: S^{(n)} \to T^{(n)}$ so that the family $Y_{f_n}$ induced by $f_n$ is isomorphic to $X^{(n)}$ ($X^{(n)}$ is the restriction of $X$ to $S^{(n)}$, $T^{(n)}$ the space corresponding to $A^{(n)}$, etc.).

Consider the functor

$$\mathbf{Isom}_{T\times S}(X\times_S(T\times S),\ Y\times_T(T\times S))$$

(see [12]). The maps $X\times_S(T\times S)\to T\times S$, $Y\times_T(T\times S)\to T\times S$ are flat and proper. Thus, by the relative Douady Theorem (see [2], [9] and [12]), this functor is representable by an analytic space $\mathscr{S}$ over $T\times S$. The maps $f_n: S^{(n)}\to T$, $i_n: S^{(n)}\to S$ (inclusion), provide a morphism $S^{(n)}\to T\times S$. The isomorphism $X^{(n)}\cong Y_{f_n}$ provides a morphism $\tilde{f}_n: S^{(n)}\to\mathscr{S}$ over $T\times S$. In particular, the composition $S^{(n)}\to\mathscr{S}\to T\times S\to S$ is the inclusion. Moreover, on $S^{(0)}$ all the $\tilde{f}_n$ coincide with the morphism $S^{(0)}\to\mathscr{S}$ corresponding to the isomorphism given as part of the structure of the families. By Theorem 2.4 we obtain $\tilde{f}: S\to\mathscr{S}$ such that the composition $S\to\mathscr{S}\to T\times S\to S$ is the identity. Let $f$ denote the composition $S\to\mathscr{S}\to T\times S\to T$. Then $\tilde{f}$ corresponds to an isomorphism of the families $X$ and $Y_f$.

Combining the previous results we obtain

THEOREM OF COMPLETENESS 3.8. *Let $F$ be the deformation functor of a compact analytic space $X_0$. Let $(A, \xi)$ be a pair such that $A$ is regular and $h_A(D)\to F(D)$ is surjective, then $(A, \xi)$ is complete.*

## 4. Applications.

1. *Rigidity theorem for analytic spaces* (*see* [11]).

THEOREM 4.1. *If $F$ is the deformation functor for a compact analytic space $X_0$ and $F(D)$ is a point, then $F(B)$ is a point for all $B\in\mathscr{A}$.*

**Proof.** Let $A=\mathbf{C}$, $\xi$ the unique point of $F(A)$. The conditions of Theorem 3.8 are satisfied by $(A, \xi)$. But $h_{\mathbf{C}}(B)$ is a point for any $B\in\mathscr{A}$.

2. *Hypersurfaces in projective space.* Consider the family $\mathscr{V}_{n,k}$ of hypersurfaces of degree $k$ in $\mathbf{P}^n$ (see [5, §14]). This family is parametrized by $\mathbf{P}^N$ with $N=\binom{n+k}{k}-1$.

THEOREM 4.2. *If $X_0$ is a hypersurface of degree $k$ in $\mathbf{P}^n$, $\mathscr{V}_{n,k}$ is a complete family of deformations of $X_0$ for $n\geqq 4$ and for $n=3$, $k\leqq 3$.*

**Proof.** If $X/D$ is a family of deformations of $X_0$, $\mathscr{L}_0$ the very ample invertible sheaf defined by the imbedding of $X_0$ in $\mathbf{P}^n$, the obstruction to extending $\mathscr{L}_0$ to $X$ lies in $H^2(X_0, \mathcal{O}_{X_0})$. This group vanishes under the hypotheses of this theorem. If the extension exists it is very ample by [4, Theorem 2.1, 15-06]. Thus $X/D$ is induced by the family $\mathscr{V}_{n,k}$.

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