SYMMETRIC MASSEY PRODUCTS AND A HIRSCH FORMULA IN HOMOLOGY(1),(2)

BY
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Abstract. A Hirsch formula is proved for the singular chains of a second loop space and is applied to show that the symmetric Massey product \langle x \rangle^p is defined for \( x \) an odd dimensional mod \( p \) homology class of a second loop space with \( p \) an odd prime. \langle x \rangle^p is then interpreted in terms of the Dyer-Lashof and Browder operations.

1. Introduction. When using the Eilenberg-Moore spectral sequence for odd primes \( p \), it is often desirable to interpret the Massey product with a symmetric defining system \( \langle x \rangle^p, \deg x \text{ odd} \), in terms of operations on \( x \). We will show that under suitable technical hypotheses, there are operations \( \beta Q \) and \( \lambda \) defined on the homology of the cobar construction on a differential graded Hopf algebra and

\[
\langle x \rangle^p = -\beta Q(x) + \text{ad}^p_{\beta Q}(-x)\beta x.
\]

We will apply this result to the homology of second and higher loop spaces where we will identify \( \beta Q \) with the first nontrivial Dyer-Lashof operation and \( \lambda \) with the first Browder operation. Then we will apply this result to the cohomology of any topological space where we will identify \( \beta Q \) with the last nontrivial Steenrod operation to obtain a new proof of a theorem of D. Kraines. In our study of the former situation we will show that the singular chains of the Moore loops of a topological monoid satisfy the Hirsch formula

\[
(ab) \cup_1 c = (-1)^{\deg b a(b \cup_1 c)} + (-1)^{\deg c \deg b}(a \cup_1 c)b,
\]

which G. Hirsch [8] proved for \( a, b \) and \( c \) singular cochains on a topological space.

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2. A Hirsch formula in homology. Before we turn our attention to the Hirsch formula, we recall a few definitions.

If \((Y, \ast)\) is a based topological space then we define the space of loops on \( Y \) as in J. Moore [17, p. 18-03] with the convention that if \((r, f) \in \Omega Y\) then \( f \) has domain

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$R$ with $f(t)=0$ if $t \leq 0$ or $t \geq r$. If $Y$ is a topological monoid then define the Pontryagin product on the singular chains of $\Omega Y$ to be the composite

$$C_k(\Omega Y) \otimes C_h(\Omega Y) \xrightarrow{\eta} C_{k+h}(\Omega Y \times \Omega Y) \xrightarrow{P} C_{k+h}(\Omega Y)$$

where $\eta$ is the shuffle map and $P((r_1,f_1),(r_2,f_2))=(r_1+r_2,f)$ with $f(t)=f_1(t)f_2(t)$. $C_*(\Omega Y)$ thus becomes a differential Hopf algebra.

**DEFINITION 1.** Let $A$ be a homotopy commutative DGA-algebra. Any homotopy on $A$ which shows that the product is homotopy commutative is called a cup-one ($\cup_1$) product. In symbols, for all $a, b \in A$

$$\partial(a \cup_1 b) + \partial(a) \cup_1 b + (-1)^{\deg a} a \cup_1 \partial(b) = [a, b].$$

Let $\pi = \{e, T\}$ be the cyclic group of order two. The join of $\pi$ with itself is

$$J^{2\pi} = \{(tx, (1-t)y) \mid t \in [0, 1] \text{ and } x, y \in \pi\}/\sim$$

where $\sim$ is the equivalence relation generated by identifying $0e$ with $0T$ (see J. Milnor [16, p. 430]). $J^{2\pi}$ has the $\pi$-action

$$z \cdot \{tx, (1-t)y\} = \{ttx, (1-t)ty\} \quad \text{for } x, y, z \in \pi \text{ and } t \in [0, 1].$$

There are singular 1-simplexes $\tau_1$ and $\tau_2$ on $J^{2\pi}$ defined by $\tau_1(t, 1-t) = [(1-t)e, te]$ and $\tau_2(t, 1-t) = [(1-t)T, te]$ for $t \in [0, 1]$. Let $\tau = \tau_2 - \tau_1 \in C_*(J^{2\pi})$.

**DEFINITION 2 (E. Dyer and R. Lashof [5, p. 37]).** If $Y$ is a topological monoid and $X=\Omega Y$ then the following composite defines a cup-one product on the singular chains of $X$:

$$C_*(X) \otimes C_*(X) \xrightarrow{\iota} C_*(J^{2\pi}) \otimes_\pi C_*(X) \otimes C_*(X) \xrightarrow{1 \otimes \eta} C_*(J^{2\pi}) \otimes_\pi C_*(X \times X)$$

$$\xrightarrow{\eta} C_*(J^{2\pi} \times_\pi X \times X) \xrightarrow{\theta} C_*(X)$$

where $\iota$ is the inclusion map of degree one defined by $\iota(a \otimes b) = \tau \otimes a \otimes b$, $\eta$ is the shuffle map and $\theta: J^{2\pi} \times X \times X \to X$ is the $\pi$-equivariant map defined by $\theta(((1-t)x, te), (r_0, f_0), (r_1, f_1))(s) = f_0^t(s)f_1^{t+1}(s)$.

Note that $X$ is given the trivial $\pi$-action while $X \times X$ and $C_*(X) \otimes C_*(X)$ have a $\pi$-action by permuting their factors.

The object of this section is to prove the following theorem:

**THEOREM 3.** If $a, b$ and $c$ are singular chains on the Moore loops of a topological monoid and $\cup_1$ denotes the cup-one product of E. Dyer and R. Lashof [5] then

$$(ab) \cup_1 c = (-1)^{\deg a(b \cup_1 c)} + (-1)^{\deg b \cdot \deg c(a \cup_1 c)b}. $$
The following definition and theorem from S. Eilenberg and S. Mac Lane [7] will be used in the proof of Theorem 3 to manipulate the shuffle product.

**Definition 4.** (a) Let \( \Sigma_n \) be the symmetric group on \( n \) letters which acts on the set \( \{0, \ldots, n-1\} \). Then

\[
\text{sh} (n_1, \ldots, n_t) = \left\{ \sigma \in \Sigma_n \mid \sigma(u) < \sigma(v) \text{ if } \sum_{i=1}^{j-1} n_i \leq u < \sum_{i=1}^{j} n_i \text{ for some } 1 \leq j \leq t \right\}
\]

where \( \sum_{i=1}^{n_i} n_i = n \) and \( n_i \geq 0, 1 \leq i \leq t \).

(b) Let \( X \) be a topological space. For \( 1 \leq j \leq t \) define the natural degeneracy operators \( \rho_j : \text{sh} (n_1, \ldots, n_t) \to \text{Hom} (C_n(X), C_n(X)) \) as follows: Let

\[
\sigma \in \text{sh} (n_1, \ldots, n_t)
\]

and let

\[
\{m_1, \ldots, m_{n-n_j}\} = \{0, \ldots, n-1\} - \left\{ \sigma \left( h + \sum_{i=1}^{j-1} n_i \right) \mid 0 \leq h < n_j \right\}
\]

with \( m_k < m_{k+1} \) for \( 1 \leq k < n_n-j \). Then

\[
\rho_j(\sigma) = S_{m_{n-j}} \cdots S_{n_j}.
\]

**Theorem 5.** Let \( X_1, \ldots, X_t \) be topological spaces. Define the shuffle map

\[
\eta_t : [C_\ast (X_1) \otimes \cdots \otimes C_\ast (X_t)]_n \to C_\ast (X_1 \times \cdots \times X_t)
\]

by

\[
\eta_t(a_1 \otimes \cdots \otimes a_t) = \sum_{\sigma \in \text{sh} (n_1, \ldots, n_t)} (\text{sgn} \ \sigma) \left[ \rho_1(\sigma)(a_1) \right] \times \cdots \times \left[ \rho_t(\sigma)(a_t) \right]
\]

where \( a_i \in C_n(X_i) \) for \( 1 \leq i \leq t \) and \( n = \sum_{i=1}^{t} n_i \). Then \( \eta_t \) is a natural chain equivalence. Furthermore,

\[
\eta_t(a_1 \otimes \cdots \otimes a_t) = \sum_{i=2}^{t} \sum_{\sigma \in \text{sh} (n_1, \ldots, n_i)} \eta_2(\eta_2(\eta_2(\eta_2(\cdots \eta_2(\eta_2(a_1 \otimes a_2) \otimes a_3) \cdots) \otimes a_i).
\]

That is, the following diagram commutes:

\[
\begin{array}{ccc}
C_\ast (X_1) \otimes \cdots \otimes C_\ast (X_t) & \xrightarrow{\eta_t} & C_\ast (X_1 \times \cdots \times X_t) \\
\downarrow \eta_2 \otimes 1 & & \uparrow \eta_2 \\
C_\ast (X_1 \times X_2) \otimes C_\ast (X_3) \otimes \cdots \otimes C_\ast (X_t) & & C_\ast (X_1 \times \cdots \times X_{t-1}) \otimes C_\ast (X_t)
\end{array}
\]

We now apply all of the preceding definitions and Theorem 5 to evaluate \( a(b \cup_1 c), (a \cup_1 c)b \) and \( (ab) \cup_1 c \).
Lemma 6. Let $Y$ be a topological monoid with $X = \Omega Y$ and $a: \Delta^\alpha \to X$, $b: \Delta^\beta \to X$ and $c: \Delta^\gamma \to X$ singular simplexes. Then

\[
(a \cup_1 b)(b \cup_1 c) = \sum_{\sigma \in sh(\alpha, \beta, \gamma, 1)} (-1)^{\alpha + \gamma} (\text{sgn } \sigma) P_* \left[ [\rho_1(\sigma)(a)] \times [\rho_1(\sigma)(a) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)] \right],
\]

(i)

\[
(a \cup_1 b) b = \sum_{\sigma \in sh(\alpha, \beta, \gamma, 1)} (-1)^{\alpha + \beta + \gamma} (\text{sgn } \sigma)
\]

\[
\cdot P_* \left[ \theta_* \left[ [\rho_4(\sigma)(\tau) \times \rho_1(\sigma)(a) \times \rho_3(\sigma)(c)] \times \rho_2(\sigma)(b) \right] \right],
\]

(ii)

\[
(ab) \cup_1 c = \sum_{\sigma \in sh(\alpha, \beta, \gamma, 1)} (-1)^{\alpha + \beta + \gamma} (\text{sgn } \sigma)
\]

\[
\cdot \theta_* \left[ [\rho_4(\sigma)(\tau) \times \rho_3(\sigma)(c)] \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c) \right].
\]

(iii)

Proof. (i)

\[
a(b \cup_1 c) = (P_\ast \circ \eta) \circ (1 \otimes \theta_\ast) \circ (1 \otimes \eta) \circ (1 \otimes 1 \otimes \eta) \circ (1 \otimes \nu) [a \otimes (b \otimes c)]
\]

by Definition 2. Hence

\[
a(b \cup_1 c) = \sum_{\mu \in sh(\alpha, \beta + \gamma + 1)} \sum_{\nu \in sh(1, \beta + \gamma)} \sum_{\xi \in sh(\eta, \mu)} (\text{sgn } \mu)(\text{sgn } \nu)(\text{sgn } \xi)
\]

\[
\cdot P_* \left[ [\rho_1(\mu)(a)] \times \theta_* \left[ [\rho_2(\mu) \circ \rho_1(\nu)(\tau) \times \rho_2(\nu) \circ \rho_1(\xi)(b) \times \rho_2(\mu) \circ \rho_2(\nu) \circ \rho_2(\xi)(c)] \right] \right],
\]

by Definition 4. Hence

\[
a(b \cup_1 c) = \sum_{\mu \in sh(\beta + \gamma + 1, \alpha)} \sum_{\nu \in sh(\beta, \gamma, 1)} \sum_{\xi \in sh(\beta, \nu)} (-1)^{\alpha + \beta + \gamma + \tau} (\text{sgn } \mu)(\text{sgn } \nu)(\text{sgn } \xi)
\]

\[
\cdot P_* \left[ [\rho_2(\mu)(a)] \times \theta_* \left[ [\rho_1(\mu) \circ \rho_2(\nu)(\tau) \times \rho_1(\nu) \circ \rho_1(\xi)(b) \times \rho_1(\mu) \circ \rho_1(\nu) \circ \rho_2(\xi)(c)] \right] \right],
\]

since there is a one-to-one correspondence $F: sh(m, n_1, \ldots, n_t) \to sh(n_1, \ldots, n_t, m)$ with $\text{sgn } F(\sigma) = (-1)^{m(n_1 + \cdots + n_t)} \text{sgn } \sigma$ defined by

\[
F(\sigma)(k) = \sigma(k + m) \quad \text{if } 0 \leq k < n_1 + \cdots + n_t,
\]

\[
= \sigma(k - n_1 - \cdots - n_t) \quad \text{if } n_1 + \cdots + n_t \leq k < n_1 + \cdots + n_t + m.
\]

Hence

\[
a(b \cup_1 c) = \sum_{\sigma \in sh(\beta, \gamma, 1, \alpha)} (-1)^{\beta + \gamma + \tau} (\text{sgn } \sigma)
\]

\[
\cdot P_* \left[ [\rho_4(\sigma)(a)] \times \theta_* \left[ [\rho_4(\sigma)(\tau) \times \rho_3(\sigma)(b) \times \rho_3(\sigma)(c)] \right] \right],
\]

by Theorem 5. Thus,

\[
a(b \cup_1 c) = \sum_{\sigma \in sh(\alpha, \beta, \gamma, 1)} (-1)^{\beta + \gamma} (\text{sgn } \sigma)
\]

\[
\cdot P_* \left[ [\rho_4(\sigma)(a)] \times \theta_* \left[ [\rho_4(\sigma)(\tau) \times \rho_3(\sigma)(b) \times \rho_3(\sigma)(c)] \right] \right],
\]

by using the above one-to-one correspondence $F$ again.

(ii) and (iii) are proved in the same way as (i).
Proof of Theorem 3. By the previous lemma, it suffices to show that if
\[ \sigma \in \text{sh} \ (\alpha, \beta, \gamma, 1) \]
then
\[ \theta_*(p_4(\sigma)(\tau) \times P_*[(p_1(\sigma)(a) \times (p_2(\sigma)(b)) \times p_3(\sigma)(c))] \]
\[ = P_*[(p_1(\sigma)(a) \times \theta_*(p_4(\sigma)(\tau) \times p_2(\sigma)(b) \times p_3(\sigma)(c))] + P_*[\theta_*(p_4(\sigma)(\tau) \times p_1(\sigma)(a) \times p_3(\sigma)(c)) \times [p_2(\sigma)(b)]. \]

Since \( \tau = \tau_2 - \tau_1 \) this follows from the following three identities:

1. \[ \theta_*[(p_1(\sigma)(a) \times (p_2(\sigma)(b)) \times p_3(\sigma)(c)) \times [p_2(\sigma)(b)] \]
\[ = P_*[(p_1(\sigma)(a) \times \theta_*(p_4(\sigma)(\tau_1) \times p_2(\sigma)(b) \times p_3(\sigma)(c)] \]
\[ = P_*[\theta_*(p_4(\sigma)(\tau_2) \times p_1(\sigma)(a) \times p_3(\sigma)(c)) \times [p_2(\sigma)(b)], \]

2. \[ P_*[(p_1(\sigma)(a) \times \theta_*(p_4(\sigma)(\tau_2) \times p_2(\sigma)(b) \times p_3(\sigma)(c))] \]
\[ = P_*[\theta_*(p_4(\sigma)(\tau_1) \times p_1(\sigma)(a) \times p_3(\sigma)(c)) \times [p_2(\sigma)(b)]. \]

We will only prove (1) since the proofs of (2) and (3) are similar to the proof of (1). We will show that the two singular \((\alpha + \beta + \gamma + 1)\)-simplexes in (1) are the same by evaluating them on a point \((t_0, \ldots, t_{\alpha + \beta + \gamma + 1})\) of \(\Delta^{\alpha + \beta + \gamma + 1}\) and then by evaluating the resulting two points of \(\Omega Y\) on \(s \in [0, \infty). \)

\[ \theta_*(p_4(\sigma)(\tau_1) \times P_*[(p_1(\sigma)(a) \times (p_2(\sigma)(b)) \times p_3(\sigma)(c))(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}))(s) \]
\[ = \theta_*(\tau_1(t_1, 1 - t_1), a(t'_0, \ldots, t'_\alpha) \cdot b(t'_0, \ldots, t'_\beta), c(t'_0, \ldots, t'_\gamma))(s) \]
\[ = [a(t'_0, \ldots, t'_2)(s)]b(t'_0, \ldots, t'_2)(s)][c(t'_0, \ldots, t'_2)(s)] \]
\[ \text{by Definitions 2 and 4 where} \]
\[ p_1(\sigma)(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}) = (t'_0, \ldots, t'_2), \]
\[ p_2(\sigma)(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}) = (t'_0, \ldots, t'_\beta), \]
\[ p_3(\sigma)(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}) = (t'_0, \ldots, t'_\gamma) \]
\[ \text{and} \]
\[ p_4(\sigma)(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}) = (t, 1 - t). \]

\[ P_*[(p_1(\sigma)(a) \times \theta_*(p_4(\sigma)(\tau_1) \times p_2(\sigma)(b) \times p_3(\sigma)(c))(t_0, \ldots, t_{\alpha + \beta + \gamma + 1}))(s) \]
\[ = [a(t'_0, \ldots, t'_2) \cdot \theta_*(\tau_1(t_1, 1 - t_1), b(t'_0, \ldots, t'_\beta), c(t'_0, \ldots, t'_\gamma))(s)](s) \]
\[ = [a(t'_0, \ldots, t'_2)(s)]b(t'_0, \ldots, t'_2)(s)][c(t'_0, \ldots, t'_2)(s)]. \]

These two calculations prove (1).

As in G. Hirsch [8], we have the following corollary to the Hirsch formula.
Corollary 7. Assume that $Y$ is a topological monoid and let $x, y \in H_*(\Omega Y; \mathbb{Z}_2)$.
If $xy = 0$ then $\langle x, y, x \rangle$ is defined and
\[
\langle x, y, x \rangle = yQ^{1 + \deg x}(x) \quad \text{modulo } xH_*(\Omega Y; \mathbb{Z}_2).
\]


Definition 8. Let $A$ be a GDA-algebra over $\mathbb{Z}_p$ with $x \in H_{2n-1}(A)$. The symmetric Massey product $\langle x \rangle^k$ is said to be defined if there exist elements $a_1, \ldots, a_{k-1} \in A$, called a defining system of $\langle x \rangle^k$, such that \{a_1\} = x and $d(a_t) = \sum_{i-t} a_{a_t-1}$ if $2 \leq t \leq k-1$. If $\langle x \rangle^k$ is defined then it equals the set of all homology classes \{\sum_{i-t} a_{a_t} - i \} where $a_1, \ldots, a_{k-1}$ varies over all defining systems of $\langle x \rangle^k$. If $\langle x \rangle^k$ is defined and equal to a single homology class then $\langle x \rangle^k$ is said to be defined with zero indeterminacy.

Lemma 9. Let $A$ be a DGA-algebra over $\mathbb{Z}_p$ which has a cup-one product that satisfies the Hirsch formula. If $x$ is an odd dimensional homology class of $A$ then $\langle x \rangle^p$ is defined with zero indeterminacy.

The proof of Lemma 9 is a direct generalization of the proof of Theorems 15 and 17 in D. Kraines [10] where Lemma 9 is proved for the special case when $x$ is an odd dimensional cohomology class.

Definition 10. Let $A$ be a DGA-coalgebra over a commutative ring $R$. Define the cobar construction $FA$ on $A$ as the tensor algebra $T(sIA)$ on the augmentation ideal $sIA$ where the $s$ indicates that the degree of each element is decreased by one. Elements of $T(sIA)$ will be denoted $[a_1| \cdots |a_k]$ for $a_1, \ldots, a_k \in sIA$, and $[ ]$ will denote the identity element of $T(sIA)$. $FA$ has three differentials $d, d_1$ and $d_2$ where
\[
d = d_1 + d_2,
\]
\[
d_1[a_1| \cdots |a_k] = \sum_{i=1}^{k} [\bar{a}_1| \cdots |\bar{a}_{i-1}|d(a_i)|\bar{a}_{i+1}| \cdots |a_k],
\]
\[
d_2[a_1| \cdots |a_k] = \sum_{i=1}^{k} \sum [\bar{a}_1| \cdots |\bar{a}_{i-1}|\bar{a}_i|a_i|\bar{a}_{i+1}| \cdots |a_k],
\]
a_1, \ldots, a_k \in sIA, \bar{a}_i = (-1)^{1 + \deg a_i}a_i and \bar{a}_i = \sum a_i \otimes a_i. FA is a differential algebra under $d$. By J. F. Adams [2, p. 36] and J. P. May [14], if $A$ is a differential Hopf algebra then $FA$ has a cup-one product which satisfies the Hirsch formula.

Definition 11. Let $A$ be a connected DGA-Hopf algebra over $\mathbb{Z}_p$. Define the primitive elements of $A$ by $PA = \{a \in A \mid \psi(a) = a \otimes 1 + 1 \otimes a\}$. For $a \in A$, let $\tilde{\psi}(a) = \psi(a) - a \otimes 1 - 1 \otimes a$. There is a natural inclusion of complexes $[PA] \rightarrow FA$. When this map induces an isomorphism in homology we will write $H_*([PA]) = H_*(FA)$.

There are two important examples of connected DGA-Hopf algebras $A$ over $\mathbb{Z}_p$ with $H_*([PA]) = H_*(FA)$. They are $A = C_*(\Omega S^2 X)$ for $X$ a connected topological space and $A = C_*(K(\mathbb{Z}_p, n))$, $n \geq 2$. In fact, by D. Kraines [11] these two examples have the stronger property that every decomposable $d_2$-cycle of $FA$ is a $d_2$-boundary.
Definition 12. Let $A$ be a connected DGA-Hopf algebra over $\mathbb{Z}_p$. Define two operations $Q: H_{2n-1}(PA) \to H_{2n-1}(FA)$ for $n \geq 1$ and $\lambda: H_r([PA] \otimes [PA]) \to H_{r+s+1}(FA)$ for $r \geq 0$ and $s \geq 0$ by $Q(a) = [a^p]$, $\lambda(1 \otimes x) = \lambda(x \otimes 1) = 0$ and $\lambda([b] \otimes [c]) = \{bc - (1 - 1)^{r+s+1}cb\}$ where $a \in PA_{2n}$, $x \in H_*(FA)$, $b \in PA_{r+1}$ with $r > 0$, $c \in PA_{s+1}$ with $s > 0$, $d(a) = 0$, $d(b) = 0$ and $d(c) = 0$.

We will show in Lemma 15 that $\lambda$ is well defined and if $A = A' \otimes \mathbb{Z}_p$ with $A'$ a $\mathbb{Z}$-free connected DGA-Hopf algebra then $\beta Q$ is well defined. Thereafter, we will prove the following theorem.

Theorem 13. Let $A'$ be a $\mathbb{Z}$-free connected DGA-Hopf algebra with $A = A' \otimes \mathbb{Z}_p$ and assume that every decomposable $d_2$-cycle of $FA$ is a $d_2$-boundary. If $x \in H_{2n-1}(FA)$, $n \geq 1$, then $\langle x \rangle^p$ is defined with zero indeterminacy and

$$\langle x \rangle^p = -\beta Q(x) + ad\lambda^{-1}(x)(\beta x).$$

Recall the use of the notation $ad\lambda$:

\[ad\lambda^k(x)(y) = \lambda(x \otimes y) \quad \text{and} \quad ad\lambda^k(x)(y) = \lambda(x \otimes ad\lambda^{k-1}(x)(y)) \quad \text{if } k \geq 2.\]

Similarly in any algebra over $\mathbb{Z}_p$ we may define $ad\lambda^k(x)(y)$ by substituting the commutator operation for $\lambda$. Thus,

$$ad\lambda^{-1}(x)(y) = [x, [x, \ldots, [x, y]\ldots] \quad (x \text{ taken } p-1 \text{ times})$$

if deg $x$ or deg $y$ is even by N. Jacobson [9, p. 186]. Observe that if $x = \{a\}$, $y = \{b\}$, $a, b \in PA$, $d(a) = 0$ and $d(b) = 0$ then $ad\lambda^{-1}(x)(y) = \{ad\lambda^{-1}(a)(b)\}$.

Lemma 14. Let $A$ be a connected DGA-Hopf algebra over $\mathbb{Z}_p$. If $a \in PA_{2n}$ with $d(a) = 0$ then $\langle \{a\} \rangle^p \in H_{2n-2}(FA)$ is defined with zero indeterminacy and

$$\langle \{a\} \rangle^p = \left\{ \sum_{i=1}^{p-1} \frac{(i, p-i)}{p} [a^i a^{p-i}] \right\}.$$ 

Proof. By Lemma 9, $\langle \{a\} \rangle^p$ is defined with zero indeterminacy. Let $a_i = (-1)^i(1/i!) [a^i]$ if $1 \leq i \leq p-1$. Then $a_1, \ldots, a_{p-1}$ is a defining system for $\langle \{a\} \rangle^p$, and

$$\sum_{i=1}^{p-1} a_i a_{p-i} = \sum_{i=1}^{p-1} \frac{(i, p-i)}{p} [a^i a^{p-i}].$$

Lemma 15. Let $A'$ be a $\mathbb{Z}$-free connected DGA-algebra with $A = A' \otimes \mathbb{Z}_p$. Then $\beta Q$ and $\lambda$ are well defined.

Proof. To prove that $\beta Q$ is well defined, we must show that if $b \in PA_{2n+1}$, $a \in PA_{2n}$ and $d(a) = 0$ then $\beta([a^p]) = \beta([a + d(b)]^p)$. By N. Jacobson [9, p. 187], $(a + d(b))^p - a^p - d(b)^p$ is in the sub-Lie algebra $S$ of $PA$ generated by $a$ and $d(b)$. Clearly, every element of $S$ except for $\mathbb{Z}_p$-multiples of $a$ is a boundary. Hence
\[(a + d(b))^p \] is homologous to \([ap] + [dp]^p\]. Therefore \(\beta\{(a + d(b))^p\} = \beta\{[ap]\}\) since 

\[
\beta\{[dp]^p\} = -\left(\sum_{i=1}^{p-1} \frac{(i, p - i)}{p} \left[dp\right] \left[dp\right]^{p - i}\right)
\]

\[
= -\langle\{[dp]\}\rangle^p \quad \text{by Lemma 14}
\]

\[
= -\langle\{dp\}\rangle^p = 0.
\]

\(\lambda\) is well defined since if \(a \in PA_r, b \in PA_s, c \in PA_{r+1}, e \in PA_{s+1}\) with \(r, s \geq 2, d(a) = 0\) and \(d(b) = 0\) then 

\[
[(a + d(c))(b + d(e)) - (-1)^{rs}(b + d(e))(a + d(c))]
\]

\[
= [ab - (-1)^{rs}ba] + d([cb - (-1)^{(r+1)s}bc] + (-1)^s[ae - (-1)^{(r+1)d}ae] + [cd(e) - (-1)^{(r+1)d}d(e)c]).
\]

**Proof of Theorem 13.** We will compute \(\beta Q(x)\) by using the hypothesis that every decomposable \(d_2\)-cycle is a \(d_2\)-boundary and then relate the result to \(\langle x \rangle^p\) by applying Lemma 14. Since \(H_*(PA) = H_*(FA)\), write \(x = [a]\) with \(a \in PA_{2n}\) and \(d(a) = 0\). Choose any \(a' \in A_{2n}\) which reduces to \(a \mod p\). Then \(d(a') = pb'\) and \(\tilde{\psi}(a') = p \sum_{i=1}^e a'_i \otimes b'_i\) for some \(b', a_i' \in A\). Let \(b', a_i' \in A\) respectively. Then \(\beta([a]) = \{(b) + \sum_{i=1}^e [a_i, b_i]\}\). \(\sum_{i=1}^e [a_i b_i]\) is a decomposable \(d_2\)-cycle which by hypothesis must bound. Hence there exists \(c \in A_{2n}\) with \(\tilde{\psi}(c) = \sum_{i=1}^e a_i \otimes b_i\). Thus, \(\beta([a]) = \{(b - d(c))\}\). Choose \(c' \in A_{2n}\) which reduces to \(c \mod p\). Let \(a'' = a' - pc'\) and \(b'' = b' - d(c')\). Then \(a''\) and \(b''\) reduce mod \(p\) to \(a\) and \(b\) respectively, \(d(a'') = pb''\) and \(\tilde{\psi}(a'') = p e'\) for some \(e' \in (A' \otimes A')_{2n}\).

\[
d_1[a'^p] = p [ad^{p-1}(a'')(b'')]
\]

since 

\[
d(a'^p) = \sum_{i=1}^p a'^{i-1} d(a'') a'^{-i} = p \sum_{i=1}^p a'^{i-1} b'' a'^{-i} = p \ ad^{p-1}(a'')(b'')
\]

by N. Jacobson [9, p. 186].

\[
d_2[a'^p] = -p \sum_{i=1}^{p-1} (i, p - i) \left[ad^{p-1}\right](a'' a'^{-i}) \quad \text{modulo } p^2 A' \otimes A'
\]

since 

\[
\psi(a'^p) = (a' \otimes 1 + 1 \otimes a'' + p^2 e')^p
\]

\[
= (a^p \otimes 1 + 1 \otimes a'^{p-1}) \quad \text{modulo } p^2 A' \otimes A'
\]

\[
= \sum_{i=1}^{p-1} (i, p - i) a'^i \otimes a'^{p-1} \quad \text{modulo } p^2 A' \otimes A'.
\]

Hence 

\[
\beta Q(x) = \beta([a'^p]) = \left[\left\{ad^{p-1}(a')(b')\right\} - \sum_{i=1}^{p-1} (i, p - i) \left[ad^{p-1}\right][a'' a^{-i}]\right] \quad \text{modulo } p^2 A' \otimes A'.
\]

\[
= ad^{p-1}(x) (\beta x) - \langle x \rangle^p.
\]

by Lemma 14 and Definition 12.

When we apply Theorem 13 to the homology of a second loop space and to the cohomology of any topological space, the major task will be to identify the operations $\beta Q$ and $\lambda$ with the Dyer-Lashof operation $\beta Q^n$ and the first Browder operation $\lambda_1$ in homology and with the Steenrod operation $\beta \mathcal{P}^n$ and 0 in cohomology. We will first show in Theorem 17 that $\beta Q$ and $\lambda$ have all the properties that one expects. Then under suitable technical hypotheses we will prove a uniqueness theorem for $\beta Q$.

**Definition 16.** Let $A$ be a connected DGA-Hopf algebra over $\mathbb{Z}_p$. Define the suspension map $\sigma : IH_n([PA]) \to H_{n+1}(A)$, $n \geq 0$ by $\sigma([a]) = \{a\}$ if $a \in PA$ with $d(a) = 0$. Note that $\sigma$ is well defined, the image of $\sigma$ is a submodule of $PH^{n+1}(A)$ and if $H_*(PA) = H_*(FA)$ then $IH_*(FA)^2$ is contained in the kernel of $\sigma$.

**Theorem 17.** Let $\mathcal{C}$ be the category of $\mathbb{Z}$-free connected DGA-Hopf algebras $A'$ such that every decomposable $d_2$-cycle of $FA$ is a $d_2$-boundary where $A = A' \otimes \mathbb{Z}_p$. Then $\beta Q$ and $\lambda$ satisfy the following properties on the objects of $\mathcal{C}$:

(a) $\lambda$ is natural.
(b) $\beta Q$ is natural.
(c) $\lambda$ is linear.
(d) $\beta Q(x+y) = \beta Q(x) + \beta Q(y) + \sum_{i=1}^{n-1} \beta d^i(x \otimes y)$ and $\beta Q(kx) = k\beta Q(x)$ if $x, y \in H_{2n-1}(FA)$ and $k \in \mathbb{Z}_p$ where $d^i(x \otimes y)$ is defined by

$$i d^i(x \otimes y) = \sum \text{ad}_1^i(x) \text{ad}_n^{k_1}(y) \cdots \text{ad}_t^i(x) \text{ad}_n^{k_t}(y)(x),$$

the sum taken over all $(j_1, k_1, \ldots, j_t, k_t)$ such that $k_t \geq 1, j_1 \geq 0, j_t \geq 1$ if $t > 1$, $\sum j_t = n - 1$ and $\sum k_t = p - i$. The important points to note are:

1. $\beta Q(x+y) - \beta Q(x) - \beta Q(y)$ is in the image under the Bockstein of the sub-$\lambda$-algebra of $H_*(FA)$ generated by $x$ and $y$.
2. By (f), $\beta Q$ is additive if $A$ has a cup-one product such that $a, b \in PA$ implies $a \cup b \in PA$.
3. $\lambda$ suspends to the commutator operation on $H_*(A)$.
4. The relation $Q$ suspends to the $p$th power operation on $H_*(A)$.
5. (External Cartan Formula) Let $A', B', C' \in \mathcal{C}$ and assume that $f' : A' \otimes B' \to C'$ is a map of DGA-Hopf algebras with $f_* : H_*(A \otimes B) \to H_*(C)$ an isomorphism. Let $\eta : FA \otimes FB \to F(A \otimes B)$ be the canonical map. Then $\eta_*^{-1} \circ F(f)_*^{-1}$ defines $\lambda$ on $H_*(FA) \otimes H_*(FB)$ and

$$\lambda((x \otimes y) \otimes (u \otimes v)) = (-1)^{\deg y(1 + \deg u)} \lambda(x \otimes u) \otimes yv + (-1)^{\deg u(1 + \deg y)} xu \otimes \lambda(y \otimes v)$$

where $x, u \in H_*(FA)$ and $y, v \in H_*(FB)$. 

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(d)' (Internal Cartan Formula). If \( A \in \mathcal{C} \) and \( x, y, u, v \in H_*(FA) \) all have positive degree then

\[
\lambda(xy \otimes uv) = (-1)^{\deg x(1 + \deg u)} \lambda(x \otimes u) y v + (-1)^{\deg y(1 + \deg v)} x u \lambda(y \otimes v)
\]

and

\[
\lambda(y \otimes uv) = \lambda(y \otimes u) v + (-1)^{\deg u \deg v} \lambda(y \otimes v) u.
\]

(d) (Cartan Formula). Let \( A', B', C' \in \mathcal{C} \) and assume that \( f': A' \otimes B' \to C' \) [or \( g': C' \to A' \otimes B' \)] is a map of DGA-Hopf algebras with \( f_*: H_*(A \otimes B) \to H_*(C) \) [or \( g_*: H_*(C) \to H_*(A \otimes B) \)] an isomorphism. Then \( \eta^{-1} \circ F(f)_* \circ \eta^{-1} \circ F(g)_* \) defines \( \beta Q \) on \( H_*(FA) \otimes H_*(FB) \) and [if \( \lambda = 0 \) on \( H_*(FA), H_*(FB) \) and \( H_*(FC) \) then]

\[
\beta Q(x \otimes y) = x^p \otimes \beta Q(y) + \beta Q(x) \otimes y^p
\]

where \( x \in H_*(FA), y \in H_*(FB) \) and \( \deg x \otimes y \) is odd.

(e) \( \lambda(x \otimes y) = (-1)^{\deg x \deg y + \deg x + \deg y} \lambda(y \otimes x) \) if \( x, y \in H_*(FA) \).

(f) If \( A \) has a cup-one product such that \( a, b \in PA \) implies \( a \cup_1 b \in PA \) then \( \lambda = 0 \) on \( H_*(FA) \).

(g) \[ (-1)^{1 + \deg z} \lambda(x \otimes (y \otimes z)) + (-1)^{1 + \deg y(1 + \deg x)} \lambda(y \otimes (z \otimes x)) + (-1)^{(1 + \deg o(1 + \deg y)} \lambda(z \otimes (x \otimes y)) = 0 \] if \( x, y, z \in H_*(FA) \).

Proof. (a), (a)', (b), (c), (c)', (e) and (g) are clear from Definitions 12 and 16. (b)' Write \( x = \{a\} \) and \( y = \{b\} \) with \( a, b \in PA_{2n}, d(a) = 0 \) and \( d(b) = 0 \). Then

\[
\beta Q(x + y) = \beta([a + b]^p) = \beta([a^p]) + \beta([b^p]) + \sum_{i=1}^{p-1} \beta(s_i(a \otimes b))
\]

by N. Jacobson [9, p. 187]. \( s_i(a \otimes b) \) is defined in the same way as \( d^i(a \otimes b) \) by replacing \( \lambda \) with the commutator operation. Hence

\[
\beta Q(x + y) = \beta Q(x) + \beta Q(y) + \sum_{i=1}^{p-1} \beta d^i(x \otimes y).
\]

\[
\beta Q(kx) = \beta([k^p a^p]) = k \beta([a^p]) = k \beta Q(x).
\]

(d) Observe that \( F(f)_* \) is an isomorphism by the naturality of the Eilenberg-Moore spectral sequence.

Case 1. \( x, y, u \) and \( v \) all have positive degree. Let \( x = \{a\}, y = \{b\}, u = \{c\} \) and \( v = \{e\} \) with \( a \in PA_a, b \in PB_b, c \in PA_c, e \in PB_e, d(a) = 0, d(b) = 0, d(c) = 0 \) and \( d(e) = 0 \). Then \( F(f) \circ \eta([a] \otimes [b]) = [f(a \otimes 1)] f(1 \otimes b) \) and \( F(f) \circ \eta([c] \otimes [e]) = [f(c \otimes 1)] f(1 \otimes e) \). Since \([f(a \otimes 1)] f(1 \otimes b)\) and \([f(c \otimes 1)] f(1 \otimes e)\) are decomposable \( d_2 \)-cycles, there are \( g, h \in C \) so that \([f(a \otimes 1)] f(1 \otimes b)\) is homologous to \([d(g)]\) and \([f(c \otimes 1)] f(1 \otimes e)\) is homologous to \([d(h)]\). Moreover, \( d(g) \)
and \( d(h) \) are primitive, \( \psi(g) = (-1)^q f(a \otimes 1) \otimes f(1 \otimes b) \) and \( \psi(h) = (-1)^r f(c \otimes 1) \otimes f(1 \otimes e) \). Hence

\[
\lambda(x \otimes y) \otimes (u \otimes v) = \eta^{-1}_* \circ F(f)_*^{-1} [(d(g) d(h) - (-1)^{\alpha + \beta + 1} \epsilon + 1) d(h) d(g)]
\]

\[
= (-1)^{\psi + 1}[(ac - (-1)^{\alpha + \beta} a \otimes b) [b] \otimes \eta_\psi^{-1} \circ F(f)_*^{-1} [d([gd(h) - (-1)^{\alpha + \beta + 1} \epsilon + 1} d(g) - [g] h
\]

\[
+ (-1)^{\alpha + \beta + 1} \epsilon + 1) h] (y + 1) d(g) \}
\]

\[
= (-1)^{\psi + 1} \lambda(x \otimes u) \otimes y v + (-1)^{\psi + 1} x u \otimes \lambda(y \otimes v).
\]

**Case 2.** \( x = 1 \) and \( y, u, v \) have positive degree. With the notation of Case 1,

\[
\lambda((1 \otimes y) \otimes (u \otimes v)) = \eta_*^{-1} \circ F(f)_*^{-1} [(f(1 \otimes b) d(h) - (-1)^{\psi + 1} \epsilon + 1) d(h) f(1 \otimes b)]
\]

\[
= (-1)^{\psi + 1} (\psi + 1) [c] \otimes \eta_*^{-1} \circ F(f)_*^{-1} [d([(-1)^{\psi + 1} \epsilon + 1) f(1 \otimes b)]
\]

\[
= (-1)^{\psi + 1} u \otimes \lambda(y \otimes v).
\]

**Case 3.** \( x = 1, \) \( u = 1 \) and \( y, u \) have positive degree. With the notation of Case 1,

\[
\lambda((1 \otimes y) \otimes (u \otimes 1)) = \eta_*^{-1} \circ F(f)_*^{-1} [(f(1 \otimes b) f(c \otimes 1) - (-1)^{\psi + 1} \epsilon + 1) f(c \otimes 1)]
\]

\[
= \eta_*^{-1} \circ F(f)_*^{-1} [(-1)^{\psi + 1} f(c \otimes b) - (-1)^{\psi + 1} f(c \otimes b)] = 0.
\]

(d)* To prove the first assertion apply Case 1 of the proof of (d) with \( A, B \) and \( C \) all equal to \( A \) and \( F(f) \) replaced by the multiplication \( \mu \) of \( F\lambda \). The computation which concludes the argument must be appropriately altered to take place in \( H_*(FA) \) rather than in \( H_*(FA) \otimes H_*(FA) \) since \( \mu_* \) is not an isomorphism. We now prove the second assertion with the notation of Case 1 of the proof of (d).

\[
\lambda(y \otimes u v) = \{(bd(h) - (-1)^{\psi + 1} \epsilon + 1) d(h) b\} - \{(d(-1)^{\psi + 1} d(h) b)\}
\]

\[
= \{bc - (-1)^{\psi + 1} c e\} + (1)^{\psi + 1} \psi + 1 d(c \otimes b) - (1)^{\psi + 1} d(c \otimes b)]
\]

\[
= \lambda(y \otimes u) v + (1)^{\psi + 1} \psi + 1 \lambda(y \otimes u) v u.
\]

\[
(d)' \text{ We will show that if deg} \ x \text{ is even and deg} \ y \text{ is odd then} \ \beta Q(x \otimes y) = x^p \otimes \beta Q(y). \ \xi: FA \otimes FB \rightarrow FC \text{ or} F(f) : FA \otimes FB \rightarrow FC \text{ [or} F(g) : FC \rightarrow (FA \otimes FB)] \text{ are maps of DGA-algebras which induce isomorphisms in homology. Hence by J. P. May [13, p. 340] any one of} FA \otimes FB, FA \otimes FB \text{ or} FC \text{ can be used to calculate Massey products in} H_*(FA \otimes FB). \text{ Let} x = \{[a]\} \text{ and} y = \{[b]\} \text{ with} a \in FA, \ b \in PB, \ d(a) = 0, \text{ and} d(b) = 0. \text{ Then} (-1)^{i + 1}(1/i)! [a \cdots [a] \otimes [b]] \text{ for}
$1 \leq i \leq p-1$ is a defining system in $FA \otimes FB$ for $\langle x \otimes y \rangle^p$. Hence $\langle x \otimes y \rangle^p = x^p \otimes \langle y \rangle^p$.

$\beta Q(x \otimes y) = -\langle x \otimes y \rangle^p + \text{ad}^{-1}_p (x \otimes y)(\beta x \otimes y + x \otimes \beta y)$ by Theorem 13

$= -x^p \otimes \langle y \rangle^p + \text{ad}^{-1}_p (x \otimes y)(x \otimes \beta y)$ by (d)

since $y^2 = 0$ and $A(y \otimes y) = 0$ [\(A = 0\)]. Since $\lambda(y \otimes y \beta(y)) = y \lambda(y \otimes \beta(y))$ by (d)$^*$ and $y^2 = 0$,

$\text{ad}^{-1}_p (x \otimes y)(x \otimes \beta y)$

$= x^p \otimes \text{ad}^{-1}_p (y)(\beta y) + \sum_{i=1}^{p-1} x^{p-i} \lambda(x \otimes x^{p-i}) \otimes y \text{ad}^{-2}_p (y)(\beta y)$

$= x^p \otimes \text{ad}^{-1}_p (y)(\beta y) + \sum_{i=1}^{p-1} (p-i)x^{p-2} \lambda(x \otimes x) \otimes y \text{ad}^{-2}_p (y)(\beta y)$ by (d)$^*$

$= x^p \otimes \text{ad}^{-1}_p (y)(\beta y)$

since $\sum_{i=1}^{p-1} p-i = \frac{1}{2}p(p-1)$ which is divisible by $p$. Thus,

$\beta Q(x \otimes y) = -\langle x \otimes y \rangle^p + \text{ad}^{-1}_p (y)(\beta y)) = x^p \otimes \beta Q(y)$.

(f) Let $x = \{a\}, y = \{b\} \in H_\bullet(FA)$ with $a, b \in PA$, $d(a) = 0$ and $d(b) = 0$. Then $a \cup_1 b \in PA$ and $d(a \cup_1 b) = ab - (-1)^{\text{deg}a \text{deg}b} ba$ by Definition 1. Hence

$\lambda(x \otimes y) = \{[ab - (-1)^{\text{deg}a \text{deg}b} ba]\} = \{[d(a \cup_1 b)]\} = \{d[a \cup_1 b]\} = 0$.

A second loop space $B = \Omega^2 C$ has two homology operations defined on $H_\bullet(B; \mathbb{Z}_p)$: a Dyer-Lashof operation $Q^\bullet: H_{2n-1}(B) \rightarrow H_{2n-1}(B)$, $n \geq 1$, and a Browder operation $\lambda^\bullet: H_j(B) \otimes H_k(B) \rightarrow H_{j+k+1}(B)$, $j \geq 0$, $k \geq 0$. W. Browder [3] and J. P. May [15, §6] have shown that these two operations satisfy the analogues of properties (a)–(g) and (a)$^*$–(d)$^*$ of Theorem 17.

The following theorem shows that $\beta Q$ is uniquely determined by several of its properties.

**Theorem 18.** Let $A', B' \in \mathcal{C}$ and assume that $H_\bullet(A) = \mathbb{E}\{L^-\} \otimes P\{L^+\}$ as coalgebras where $L^-$ is a set of odd degree elements and $L^+$ is a set of even degree elements. Let $\Delta: A \rightarrow B$ be a map of DGA-Hopf algebras and $f: A \otimes A \rightarrow B$ [or $g: B \rightarrow A \otimes A$] a map of DGA-Hopf algebras with $f_*: H_\bullet(A \otimes A) \rightarrow H_\bullet(B)$ [or $g_*: H_\bullet(B) \rightarrow H_\bullet(A \otimes A)$] an isomorphism [and $\lambda = 0$ on $H_\bullet(A)$ and on $H_\bullet(B)$]. Assume that $R: H_{2n-1}(FX) \rightarrow H_{2n-1}(FX)$ for all $n \geq 1$ is defined for $X = A$ and $X = B$ such that

1. $R \circ F(\Delta)_* = F(\Delta)_* \circ R$.
2. $R(x + y) = R(x) + R(y) + \sum_{i=1}^{p-1} d^i(x \otimes y)$ and $R(kx) = kR(x)$ if $k \in \mathbb{Z}_p$, $x, y \in H_{2n-1}(FX)$ for $X = A$ and $X = B$.

3. $R$ suspends to the $p$th power operation for $X = A$ and $X = B$.
4. $R$ satisfies the Cartan formula on $H_\bullet(FA) \otimes H_\bullet(FA)$.

Then $\beta Q = \beta R$ on $H_\bullet(FA)$. 

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Proof. Assume that \( \beta Q \neq \beta R \) on \( H_*(FA) \) and let \( x \in H_{2n-1}(FA) \) be an element of smallest degree for which \( \beta Q(x) \neq \beta R(x) \). By the naturality of the Eilenberg-Moore spectral sequence \( F(f)_* : H_*(FA) \otimes H_*(FA) \to H_*(FB) \) [or \( F(g)_* : H_*(FB) \to H_*(FA) \otimes H_*(FA) \)] is an isomorphism, and hence we can define a coproduct on \( H_*(FA) \) by \( \psi = F(f)_*^{-1} \circ F(\Delta)_* \). If \( \psi(x) = \sum x' \otimes x'' \) then \( \psi \circ \beta Q(x) = \sum \beta Q(x') \otimes x'' + \sum x' \otimes \beta Q(x') \) and \( \psi \circ \beta R(x) = \sum \beta R(x') \otimes x'' + \sum x' \otimes \beta R(x') \) by Theorem 17(d)' and hypothesis (4) of this theorem. Hence \( \beta Q(x) - \beta R(x) \) is primitive. Therefore it is indecomposable since \( \text{deg} (\beta Q(x) - \beta R(x)) = 2np - 2 \), and the only decomposable primitive elements of \( H_*(FA) \) are \( p \)-th powers. Let \( Q(x) \) denote a fixed choice of the relation \( Q \) on \( x \). Then \( Q(x) - R(x) \) is in the kernel of the suspension map \( \sigma \). Furthermore, \( Q(x) - R(x) \) is indecomposable because its image under \( \beta \) is indecomposable. By J. P. May [12], \( H_*(FA) = P(sL^-) \otimes P(tL^+) \) as algebras. Hence \( tL^+ \) is a \( Z_p \)-basis of \( (\text{Kernel } \sigma) \cap QH_*(FA) \), all of whose elements lie in degrees congruent to \( p - 2 \) mod \( p \). This is a contradiction since \( \text{deg} (Q(x) - R(x)) = 2np - 1 \neq p - 2 \) mod \( p \). Hence \( \beta Q = \beta R \) on \( H_*(FA) \).

Theorem 19. Let \( x \in H_{2n-1}(X), X = \Omega^2Z, X \text{ connected and } n \geq 1 \). Then \( \langle x \rangle^p \) is defined with zero indeterminacy and
\[
\langle x \rangle^p = -\beta Q^n(x) + \text{ad}^p_{\beta^n-1}(x)(\beta x).
\]

Proof. By Theorem 3 and Lemma 9, \( \langle x \rangle^p \) is defined with zero indeterminacy. The following diagram commutes and the vertical map is a morphism of second loop spaces:

\[
\begin{array}{ccc}
X & \longrightarrow & \Omega^2S^2X \\
\downarrow & & \downarrow \\
1_X & \longleftarrow & X
\end{array}
\]

Hence it suffices to evaluate \( \langle x \rangle^p \) on \( H_*(X) \subset H_*(\Omega^2S^2X) \). By J. F. Adams [1], there is a map \( \phi : FC_*(\OmegaS^2X) \to C_*(\Omega^2S^2X) \) of differential algebras which induces an isomorphism in homology. By D. Kraines [11], the hypotheses of Theorem 13 are satisfied. Hence \( \langle x \rangle^p = -\beta Q(x) + \text{ad}^p_{\beta^n-1}(x)(\beta x) \). It remains to show that \( \beta Q \) and \( \lambda \) correspond to \( \beta Q^n \) and \( \lambda_1 \) under the isomorphism \( \phi_*. \) As a first step, we will prove that the Adams map \( \phi_* \) commutes with the suspension map and the external product. That is, we will show that Figures 1 and 2 commute. Figure 1 clearly commutes. Note that
\[
\eta : C_*(\Omega^2S^X) \otimes C_*(\Omega^2Y) \to C_*(\Omega(S^2X \times S^2Y))
\]
is a map of DGA-Hopf algebras and hence induces an algebra homomorphism \( F(\eta) \). Figure 3 shows that Figure 2 commutes on elements of \( H_*(FC_*(\Omega^2S^2X)) \otimes 1 \).

Similarly Figure 2 commutes on elements of \( 1 \otimes H_*(FC_*(\Omega^2S^2Y)) \), and hence Figure 2 commutes on all elements of \( H_*(FC_*(\Omega^2S^2X)) \otimes H_*(FC_*(\Omega^2S^2Y)) \) since
all the maps in Figure 2 are algebra isomorphisms. J. P. May [unpublished] has proved that the first Browder operation $\lambda_1$ is the unique operation on the category of second loop spaces which satisfies the analogues of the properties (a)-(g) of Theorem 17. Hence $\phi_* \circ \lambda \circ (\phi_*^{-1} \otimes \phi_*^{-1})$ must equal $\lambda_1$. Recall that $H_*(\Omega S^2 X) = T(H_*(S X)) = P(L^+) \otimes E(L^-)$ where $L = L^+ \cup L^-$ is the free Lie algebra generated by $H_*(S X)$. Hence by Theorem 18, $\beta Q$ equals $\phi_*^{-1} \circ \beta Q^* \circ \phi_*$ on $H_*(\Omega S^2 X)$.

**Corollary 20.** Let $x$ be a $(2n-1)$-dimensional homology class of a connected third loop space, $n \geq 1$. Then $\langle x \rangle^p$ is defined with zero indeterminacy and $\langle x \rangle^p = -\beta Q^*(x)$.

**Proof.** This result follows from Theorem 19 and the fact that the first Browder operation is zero on the homology of a third loop space.

Observe that $\text{ad}^p \circ \text{id}^{-1} \circ (x)(\beta x)$ may be nonzero. For example, let $M$ be the Moore space with $H_{2n-1}(M; \mathbb{Z}_p) = \mathbb{Z}_p x$, $H_{2n-2}(M; \mathbb{Z}_p) = \mathbb{Z}_p \beta x$ and $\tilde{H}_k(M; \mathbb{Z}_p) = 0$ otherwise, $k \geq 1$. Then $\text{ad}^p \circ \text{id}^{-1} \circ (x)(\beta x) \neq 0$ in $H_*(\Omega^2 S^2 M; \mathbb{Z}_p)$ since its suspension in $H_*(\Omega S^2 M; \mathbb{Z}_p)$ is $\text{ad}^p \circ \text{id}^{-1} \circ (x)(\beta x) \neq 0$ in $H_*(\Omega S^2 M; \mathbb{Z}_p)$ which is nonzero because $H_*(\Omega S^2 M; \mathbb{Z}_p)$ contains the free Lie algebra generated by $H_*(SM; \mathbb{Z}_p)$.

We can also use the machinery of this section to prove the following theorem of D. Kraines [10]:

**Theorem 21.** Let $x$ be a $(2k+1)$-dimensional cohomology class of any topological space, $n \geq 0$. Then $\langle x \rangle^p$ is defined with zero indeterminacy and $\langle x \rangle^p = -\beta \mathfrak{P}^n(x)$.

**Proof.** $\langle x \rangle^p$ is defined with zero indeterminacy by Lemma 9. By the naturality of $\langle \rangle^p$ and $\beta \mathfrak{P}^n$, it suffices to show that $\langle i \rangle^p = -\beta \mathfrak{P}^n(i)$ where $i$ is the fundamental class of $K(\mathbb{Z}_p, 2n+1)$. Assume first that $n \geq 1$. By S. Eilenberg and S. Mac Lane [6, p. 94] there is a map $\gamma: C^*(K(\mathbb{Z}_p, 2n+1)) \to FC^*(K(\mathbb{Z}_p, 2n))$ of differential algebras which induces an isomorphism in homology. By D. Kraines [11] the hypotheses of Theorem 13 are satisfied. Hence $\langle i \rangle^p = -\beta Q^*(i) + \text{ad}^p \circ \text{id}^{-1} \circ (i)(\beta i)$. We will now show that $\lambda = 0$. Let $Y$ be a topological space with $u_i \in H^n(Y)$ represented by $\hat{u}_i: Y \to K(\mathbb{Z}_p, n)$ for $i = 1, 2$. In $H^*(K(\mathbb{Z}_p, n_1) \times K(\mathbb{Z}_p, n_2))$,

$\lambda((\iota_{n_1} \otimes 1) \otimes (1 \otimes \iota_{n_2})) = 0$

by Theorem 17(d). Hence $\lambda(u_1 \otimes u_2) = (\hat{u}_1 \times \hat{u}_2)^* \circ \lambda((\iota_{n_1} \otimes 1) \otimes (1 \otimes \iota_{n_2})) = 0$. Thus, $\langle i \rangle^p = -\beta Q^*(i)$. It remains to show that $\beta Q$ corresponds to $\beta \mathfrak{P}^n$ under $\gamma_*$. Reasoning as in the proof of Theorem 19, one sees that $\gamma_*$ commutes with the suspension map and the external product. It is well known that $H^*(K(\mathbb{Z}_p, 2n))$ is an exterior Hopf algebra on odd degree elements tensored with a polynomial Hopf algebra on even degree elements. Hence by Theorem 18, $\beta Q = \beta \mathfrak{P}^n(i)$. If $n = 0$ then in $H^*(K(\mathbb{Z}_p, 1) \times K(\mathbb{Z}_p, 2))$, $\langle \iota_1 \rangle^p \otimes \iota_2^p = \langle \iota_1 \otimes \iota_2 \rangle^p = -\beta \mathfrak{P}^1(\iota_1 \otimes \iota_2) = -\beta \iota_1 \otimes \iota_2^p$. Hence $\langle \iota_1 \rangle^p = -\beta \iota_1$. 

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