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ENTROPY-EXPANSIVE MAPS

BY
RUFUS BOWEN

Abstract. Let $f: X \to X$ be a uniformly continuous map of a metric space. f is called h-expansive if there is an $\varepsilon > 0$ so that the set $\Phi_\varepsilon(x) = \{y : d(f^n(x), f^n(y)) \leqq \varepsilon$ for all $n \geqq 0\}$ has zero topological entropy for each $x \in X$. For X compact, the topological entropy of such an f is equal to its estimate using ε: $h(f) = h(f, \varepsilon)$. If X is compact finite dimensional and μ an invariant Borel measure, then $h_\mu(f) = h_\mu(f, A)$ for any finite measurable partition A of X into sets of diameter at most ε. A number of examples are given. No diffeomorphism of a compact manifold is known to be *not* h-expansive.

Let $f: X \to X$ be a homeomorphism of a metric space. For $\varepsilon > 0$ and $x \in X$ define

$$\Gamma_\varepsilon(x) = \{y \in X : d(f^n(y), f^n(x)) \leqq \varepsilon \text{ for all } n \in Z\}.$$

f is called *expansive* if for some ε these sets are as small as possible, i.e. if $\Gamma_\varepsilon(x) = x$ for all x. We are concerned with entropy and shall call f *h-expansive* provided that for some $\varepsilon > 0$ the $\Gamma_\varepsilon(x)$ are negligible in terms of entropy, i.e. if the topological entropy $h(f, \Gamma_\varepsilon(x)) = 0$ for all x.

We have two main results for h-expansive maps with X compact. First, the topological entropy satisfies $h(f) = h(f, \varepsilon)$. Second, assuming X is finite dimensional, $h_\mu(f) = h_\mu(f, A)$ when μ is an f-invariant normalized Borel measure on X and A is a finite measurable partition of X into sets of diameter at most ε. Both these results are well known in case f is expansive (see [11] and [14] respectively). Arov [2] noted that the second statement was true for f an endomorphism of a torus and μ Haar measure when he calculated $h_\mu(f)$ for this case (see Example 1.2).

1. **Definitions and examples.** We now review the definition of topological entropy given in [4]. For X compact this definition was given independently by Dinaburg [7]; is related to the ε-entropy of Kolmogorov [12]. Topological entropy was defined first in [1].

Let $f: X \to X$ be uniformly continuous on the metric space X. For $E, F \subset X$ we say that E *(n, δ)-spans* F (with respect to f), if for each $y \in F$ there is an $x \in E$ so that $d(f^k(x), f^k(y)) \leqq \delta$ for all $0 \leqq k < n$. We let $r_n(F, \delta) = r_n(F, \delta, f)$ denote the minimum cardinality of a set which (n, δ)-spans F. If K is compact, then the continuity of f guarantees $r_n(K, \delta) < \infty$. For compact K we define

$$\bar{r}_f(K, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \delta)$$

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and

$$h(f, K) = \lim_{\delta \to 0} \bar{r}_f(K, \delta)$$

(notice that $\bar{r}_f(K, \delta)$ increases as δ decreases). Finally let $h(f) = \sup_K h(f, K)$ where K varies over all compact subsets of X. If X is compact, then $h(f) = h(f, X)$ and we write $h(f, \delta) = \bar{r}_f(X, \delta)$.

Let $\Phi_\varepsilon(x) = \bigcap_{n \geqq 0} f^{-n} B_\varepsilon(f^n(x)) = \{y : d(f^n(x), f^n(y)) \leqq \varepsilon \text{ for } n \geqq 0\}$ and $h_f^*(\varepsilon) = \sup_{x \in X} h(f, \Phi_\varepsilon(x))$. f is called *h-expansive* if $h_f^*(\varepsilon) = 0$ for some $\varepsilon > 0$. In case f is a homeomorphism we set

$$\Gamma_\varepsilon(x) = \bigcap_{n \in Z} f^{-n} B_\varepsilon(f^n(x))$$

and

$$h^*_{f,\text{homeo}}(\varepsilon) = \sup_{x \in X} h(f, \Gamma_\varepsilon(x)).$$

REMARK. For f a homeomorphism, $\Gamma_\varepsilon(x) \subset \Phi_\varepsilon(x)$ and so $h^*_{f,\text{homeo}}(\varepsilon) \leqq h_f^*(\varepsilon)$. The definition of h-expansiveness for homeomorphisms mentioned in the introduction, namely $h^*_{f,\text{homeo}}(\varepsilon) = 0$, is actually equivalent to the above one in case X is compact. For in 2.3 we prove $h_f^*(\varepsilon) = h^*_{f,\text{homeo}}(\varepsilon)$ when X compact.

EXAMPLE 1.0. Expansive maps.

EXAMPLE 1.1. If $f: R^n \to R^n$ is linear and d comes from a norm, then $h_f^*(\varepsilon) = 0$ for every ε.

Proof. f decomposes into a direct sum of linear maps $f = f_1 \oplus f_2 : E_1 \oplus E_2 \to E_1 \oplus E_2$ where f_1's eigenvalues have norm at most 1 and f_2's have norm greater than 1. If $u \in E_2$, $u \neq 0$, then $d(f_2^n(u), 0) \to \infty$ as $n \to \infty$. It follows that $\Phi_\varepsilon(0) \subset E_1$. But $h(f \mid E_1) = h(f_1) = 0$ by Theorem 15 of [4]. So $h(f, \Phi_\varepsilon(0)) = 0$. But $\Phi_\varepsilon(x) = \Phi_\varepsilon(0) + x$ and $h(f, K + x) = h(f, K)$ for any compact set K.

EXAMPLE 1.2. An endomorphism f of a Lie group G is h-expansive.

Proof. Here we use a right invariant metric d. Then one checks $\Phi_\varepsilon(x) = \Phi_\varepsilon(e)x$ and $h(f, Kx) = h(f, K)$ for compact K. So it is enough to see $h(f, \Phi_\varepsilon(e)) = 0$ for some ε. Now

$$\begin{tikzcd} T_eG \arrow[r, "df"] \arrow[d, "\exp"] & T_eG \arrow[d, "\exp"] \\ G \arrow[r, "f"] & G \end{tikzcd}$$

commutes and exp is a homeomorphism of a small neighborhood $B_\alpha(0) \subset T_eG$ onto a neighborhood of some $B_\varepsilon(e)$. Then $\Phi_\varepsilon(e, f) \subset \exp \Phi_\alpha(0, df)$ and since $f \mid \exp \Phi_\alpha(0, df)$ is a quotient of $df \mid \Phi_\alpha(0, df)$ one has

$$h(f, \Phi_\varepsilon(e, f)) \leqq h(df, \Phi_\alpha(0, df)) = 0.$$

EXAMPLE 1.3. Suppose f is h-expansive and T a uniformly continuous map so that $(T \cdot f)^n = T_n \cdot f^n$ for $n \geqq 0$ where the T_n are isometries. Then $T \cdot f$ is h-expansive.

Proof. One checks easily that $\Phi_\varepsilon(x, T\cdot f)=\Phi_\varepsilon(x,f)$ and that a set which (n, δ)-spans some $F\subset X$ with respect to f also (n, δ)-spans F with respect to $T\cdot f$. It follows that

$$h(T\cdot f, \Phi_\varepsilon(x, T\cdot f)) \leqq h(f, \Phi_\varepsilon(x,f)) = 0.$$

EXAMPLE 1.3*. Let G be a Lie group and for $g, u \in G$ define $L_g(u)=gu$ and $R_g(u)=ug$. If f is an endomorphism of G and $g\in G$, then the affine maps $R_g\cdot f$, $f\cdot R_g$, $L_g\cdot f$ and $f\cdot L_g$ are all h-expansive.

Proof. If we set $g_1=g$ and $g_{n+1}=f(g_n)g$, one sees that $(R_g\cdot f)^n=R_{g_n}\cdot f^n$. As we use a right invariant metric, R_{g_n} is an isometry and 1.3 applies. Now $(L_g\cdot f)(u)$ $=gf(u)=(gf(u)g^{-1})g=(R_g\cdot f^*)(u)$ where $f^*(u)=gf(u)g^{-1}$ is an endomorphism. We leave $f\cdot R_g$ and $f\cdot L_g$ to the reader.

EXAMPLE 1.4. Suppose H is a uniformly discrete subgroup of the Lie group G, i.e. G/H is compact and $\pi: G\to G/H$ given by $\pi(x)=xH$ is a covering. For f an endomorphism of G with $f(H)\subset H$ and $g\in G$ define f^* on G/H by $f^*(uH)=gf(u)H$. Then f^* is h-expansive.

Proof. For δ small enough π maps $B_\delta(x)$ isometrically onto $B_\delta(xH)$ for every x and $\pi\Phi_\delta(x, L_g\cdot f)=\Phi_\delta(xH,f^*)$. Then

$$h(f^*, \Phi_\delta(xH,f^*)) \leqq h(L_g\cdot f, \Phi_\delta(x, L_g\cdot f)) = 0.$$

So f^* is h-expansive (see [4] for some more details).

EXAMPLE 1.5. For the case of X compact define the nonwandering set

$$\Omega(f) = \left\{x\in X : \text{for every neighborhood } U \text{ of } x,\ U\cap \bigcup_{n>0} f^n(U) \neq \varnothing\right\}.$$

Then $f\Omega(f)\subset\Omega(f)$. If $f|\Omega(f)$ is h-expansive, then so is f. An example of this is one of Smale's Axiom A diffeomorphisms [15], where $f|\Omega(f)$ is expansive.

Proof. Splice together the proof of Theorem 2.4 in [5] and that of 2.2 below for $f|\Omega$, $a=0$ and x staying in a neighborhood of Ω up to time n.

EXAMPLE 1.6. Suppose $\Phi=\{\varphi_t: X\to X\}_{t\in R}$ is a continuous flow on a compact metric space X. Suppose also that there are $\varepsilon>0$ and $s>0$ so that

$$\begin{aligned}\Gamma_\varepsilon(x, \Phi) &= \{y\in X : d(\varphi_t(y), \varphi_t(x)) \leqq \varepsilon \text{ for all } t\in R\}\\ &\subset \varphi_{[-s,s]}(x) = \{\varphi_r(x) : |r| \leqq s\}.\end{aligned}$$

Then each φ_t is h-expansive.

Proof. For any $t\in R$ there is a δ so that $d(x, y)\leqq\delta$ implies $d(\varphi_r(x), \varphi_r(y))\leqq\varepsilon$ for all $|r|\leqq|t|$. Then

$$\Gamma_\delta(x, \varphi_t) \subset \Gamma_\varepsilon(x, \Phi) \subset \varphi_{[-s,s]}(x).$$

For $\beta>0$ choose $\alpha>0$ such that for all $x\in X$ and all $|r|\leqq\alpha$ we have $d(x, \varphi_r(x))\leqq\delta$ (here we use X compact). Let K be a set of numbers so that every point in $[-s, s]$ is within α of one of them. Then $\{\varphi_u(x) : u \in K\}$ (n, δ)-spans $\varphi_{[-s,s]}(x)$ with respect to φ_t. Hence

$$r_n(\varphi_{[-s,s]}(x), \delta, \varphi_t) \leqq \text{card } K$$

and

$$h(\varphi_t, \Gamma_\delta(x, \varphi_t)) \leqq h(\varphi_t, \varphi_{[-s,s]}(x)) = 0.$$

EXAMPLE 1.6*. Let $\Phi=\{\varphi_t\}$ be one of Smale's Axiom A flows [15]. Then $\Phi|\Omega(\Phi)$ satisfies the condition of 1.6 [9]. By 1.5 and 1.6, each φ_t is h-expansive.

Problem. Find some differentiable maps which are *not* h-expansive.

2. **Calculating topological entropy.**

Assumption. For the remainder of the paper X is compact.

LEMMA 2.1. *Suppose* $0=t_0<t_1<\cdots<t_{r-1}<t_r=n$ *and* $E_i(t_{i+1}-t_i, \alpha)$*-spans* $f^{t_i}(F)$ *for* $0\leqq i<r$. *Then*

$$r_n(F, 2\alpha) \leqq \prod_{0\leqq i<r} \text{card } E_i.$$

Proof. For $x_i \in E_i$ write

$$V(x_0,\ldots,x_{r-1}) = \{x\in F : d(f^{t+t_i}(x), f^t(x_i)) \leqq \alpha \text{ for } 0 \leqq t < t_{i+1}-t_i, 0 \leqq i < r\}.$$

If $x, y \in V(x_0,\ldots,x_{r-1})$, then by the triangle inequality $d(f^s(x),f^s(y))\leqq 2\alpha$ for $0\leqq s<n$. Since $F=\bigcup V(x_0,\ldots,x_{r-1})$ we get an $(n, 2\alpha)$-spanning set for F by taking one element from each nonempty $V(x_0,\ldots,x_{r-1})$.

PROPOSITION 2.2. *Let* $a=h_f^*(\varepsilon)$ *or* $h^*_{f,\text{homeo}}(\varepsilon)$ (*in case f is a homeomorphism*). *Then for every* $\delta>0$ *and* $\beta>0$ *there is a c such that*

$$r_n\left(\bigcap_{k=0}^{n-1} f^{-k}B_\varepsilon(f^k(x)), \delta\right) \leqq ce^{(a+\beta)n}$$

for all $x\in X$.

Proof. We do the case where f is a homeomorphism and $a=h^*_{f,\text{homeo}}(\varepsilon)$. The case where $a=h_f^*(\varepsilon)$ is slightly simpler and we leave the necessary modifications to the reader.

For each $y\in X$ pick $m(y)$ so that $a+\beta\geqq(1/m(y))\log\text{card } E(y)$ where $E(y)$ is a set which $(m(y), \frac{1}{4}\delta)$-spans $\Gamma_\varepsilon(y)$. Then $U(y)=\{w\in X : \exists\, z\in E(y)$ such that $d(f^k(w),f^k(z))<\frac{1}{2}\delta$ for all $0\leqq k<m(y)\}$ is an open neighborhood of the compact set $\Gamma_\varepsilon(y)$. Let $S_M=\bigcap_{|j|\leqq M} f^{-j}B_\varepsilon(f^j(y))$. Then $S_0\supset S_1\supset\cdots$ is a decreasing chain of compact sets with intersection $\Gamma_\varepsilon(y)$; hence there is an integer $N(y)$ so that $S_{N(y)}\subset U(y)$. Consider the compact sets $W_\gamma=\bigcap_{|j|\leqq N(y)} f^{-j}B_\gamma(f^j(y))$. Then $\bigcap_{\gamma>\varepsilon} W_\gamma = W_\varepsilon = S_{N(y)}\subset U(y)$; hence, $W_\gamma\subset U(y)$ for some $\gamma>\varepsilon$. Let $V(y)$ be a neighborhood of y such that $d(f^j(u),f^j(y))<\gamma-\varepsilon$ for $|j|\leqq N(y)$ when $u\in V(y)$. Then $B_\varepsilon(f^j(u)) \subset B_\gamma(f^j(y))$ and

$$\bigcap_{|j|\leqq N(y)} f^{-j}B_\varepsilon(f^j(u)) \subset U(y).$$

Let $V(y_1),\dots, V(y_s)$ cover the compact space X and

$$N = \max\{N(y_1),\dots, N(y_s), m(y_1),\dots, m(y_s)\} + 1.$$

Consider now any $x \in X$ and $F_n = \bigcap_{j=0}^{n-1} f^{-j}B_\varepsilon(f^j(x))$. For any $t \in [N, n-N]$, $f^t(x)$ is in some $V(y_i)$ and

$$f^t(F_n) = \bigcap_{k=-t}^{n-t-1} f^{-k}B_\varepsilon(f^k(f^t(x))) \subset \bigcap_{|k| \leqq N(y_1)} f^{-k}B_\varepsilon(f^k(f^t(x))) \subset U(y_i).$$

Now $E(y_i)$ $(m(y_i), \frac{1}{2}\delta)$-spans $U(y_i)$, so it does $f^t(F_n)$ also.

We shall define integers $0 = t_0 < t_1 < \cdots < t_r = n$. If $n \leqq N$, let $r=1$ and $t_1 = n$. If $n > N$, take $t_1 = N$ and pick $V(y_{i_1})$ containing $f^{t_1}(x)$. Suppose we have chosen $t_1,\dots, t_k$ and $y_{i_1},\dots, y_{i_k}$ (with $t_k < n$). If $t_k > n-N$, then set $r = k+1$ and $t_r = N$. If $t_k \leqq n-N$, then set $t_{k+1} = t_k + m(y_{i_k}) < n$ and choose $V(y_{i_{k+1}})$ containing $f^{t_{k+1}}(x)$. Eventually this process stops.

Let K be a set which $(N, \frac{1}{2}\delta)$-spans X. Then K $(t_1 - t_0, \frac{1}{2}\delta)$-spans F_n and also $(t_r - t_{r-1}, \frac{1}{2}\delta)$-spans $f^{t_{r-1}}(F_n)$. From the way the t_k's and y_{i_k}'s were chosen we see that, for $0 < k < r-1$, $E(y_{i_k})$ $(t_{k+1} - t_k, \frac{1}{2}\delta)$-spans $f^{t_k}(F_n)$. Lemma 2.1 applies to give

$$\begin{aligned} r_n(F_n, \delta) &\leqq (\text{card } K)^2 \prod_{0<k<r-1} \text{card } E(y_{i_k}) \\ &\leqq (\text{card } K)^2 \prod_{0<k<r-1} \exp((a+\beta)(n(y_{i_k}))) \leqq (\text{card } K)^2 e^{(a+\beta)n}. \end{aligned}$$

Corollary 2.3. *If f is a homeomorphism, then $h^*_{f,\text{homeo}}(\varepsilon) = h^*_f(\varepsilon)$.*

Proof. Let $a = h^*_{f,\text{homeo}}(\varepsilon)$. Fixing β, δ the proposition gives us $r_n(\Phi_\varepsilon(x), \delta) \leqq ce^{(a+\beta)n}$. Hence $\bar{r}_f(\Phi_\varepsilon(x), \delta) \leqq a + \beta$ and $h(f, \Phi\varepsilon(x)) \leqq a+\beta$. As $\beta > 0$ was arbitrary, $h(f, \Phi_\varepsilon(x)) \leqq a$ and $h^*_f(\varepsilon) \leqq a = h^*_{f,\text{homeo}}(\varepsilon)$. The reverse inequality we noted before.

Theorem 2.4. *$h(f) \leqq h(f, \varepsilon) + h^*_f(\varepsilon)$. In particular, $h(f) = h(f, \varepsilon)$ if ε is an h-expansive constant for f.*

Proof. Let $\delta > 0$ and $\beta > 0$. Let $E_n(n, \varepsilon)$-span X, i.e.

$$X = \bigcup_{x \in E_n} \bigcap_{k=0}^{n-1} f^{-k}B_\varepsilon(f^k(x)).$$

By Proposition 2.2 there is a constant c so that each of the sets in the above union can be (n, δ)-spanned by using at most $ce^{(a+\beta)n}$ elements (where $a = h^*_f(\varepsilon)$). Hence $r_n(X, \delta) \leqq \text{card } E_n ce^{(a+\beta)n} \leqq r_n(X, \varepsilon)ce^{(a+\beta)n}$. It follows that $h(f, \delta) \leqq h(f, \varepsilon) + a + \beta$. Letting $\beta \to 0$, $h(f, \delta) \leqq h(f, \varepsilon) + a$. Now letting $\delta \to 0$ we get our result.

If $h^*_f(\varepsilon) = 0$, then $h(f) \leqq h(f, \varepsilon)$. But $h(f) \geqq h(f, \varepsilon)$ from the definition of $h(f)$; hence $h(f) = h(f, \varepsilon)$.

Corollary 2.5. *If $h(f) = h(f, \varepsilon) + h^*_f(\varepsilon)$, then $(1/n) \log r_n(X, \varepsilon) \to h(f, \varepsilon)$. In particular, if $h^*_f(\varepsilon) = 0$, then $(1/n) \log r_n(f, \varepsilon) \to h(f)$.*

Proof. Otherwise there is an increasing sequence of integers $\{n_k\}$ so that $(1/n_k) \log r_{n_k}(X, \varepsilon) \to b < h(f, \varepsilon)$. Let $a = h^*_f(\varepsilon)$. Then $h(f) > a + b$ and, for $\gamma > 0$

small enough, $h(f, \gamma) > a+b$. Choose $\beta > 0$ so that $h(f, \gamma) > a+b+\beta$. For some c, as in the proof of the theorem, we have $r_{n_k}(X, \frac{1}{2}\gamma) \leqq r_{n_k}(X, \varepsilon)c \exp((a+\beta)n_k)$. So

$$\limsup_{k\to\infty} \frac{1}{n_k} \log r_{n_k}(X, \tfrac{1}{2}\gamma) \leqq b + a + \beta < h(f, \gamma).$$

Choose R so that $(1/R)\log r_R(X, \frac{1}{2}\gamma) = \alpha < h(f, \gamma)$. This means there is an $(R, \frac{1}{2}\gamma)$-spanning set for X with $e^{R\alpha}$ elements. By Lemma 2.1 (using $t_k = kR$) one gets $r_{Rp}(X, \gamma) \leqq (e^{R\alpha})^p$. For $0 \leqq q \leqq R$,

$$r_{Rp+q}(X, \gamma) \leqq r_{R(p+1)}(X, \gamma) \leqq e^{R\alpha(p+1)}.$$

Hence

$$h(f, \gamma) = \limsup_{n=Rp+q\to\infty} \frac{1}{n} \log r_n(X, \gamma) \leqq \limsup_{p\to\infty} \frac{(p+1)R\alpha}{Rp} = \alpha.$$

But we chose $\alpha < h(f, \gamma)$, a contradiction.

If $h_f^*(\varepsilon) = 0$, then $h(f) = h(f, \varepsilon)$ by the theorem, and so the first statement applies.

REMARKS. For expansive homeomorphisms the second part of 2.4 was proved in [11] and the second part of 2.5 in [6]. In the original definition of topological entropy using open covers [1] certain limits existed whose analogues might not exist when one uses spanning sets. 2.5 is a technical result giving us conditions which insure that these limits exist. It has an application in counting periodic orbits of the Axiom A diffeomorphisms and flows of Smale (see [6]).

3. **Measures.** We continue to assume $f: X \to X$ is continuous and X a compact metric space. μ denotes a Borel measure on X with $\mu(X) = 1$ which is f-invariant, i.e. $\mu(f^{-1}(E)) = \mu(E)$ for Borel sets E.

We call $\boldsymbol{A} = \{A_1, \ldots, A_r\}$ a (finite) *Borel partition* provided the A_i are pairwise disjoint Borel sets whose union is X. (Note that any finite μ-measurable partition is μ-equivalent to a Borel partition.) We write

$$H_\mu(\boldsymbol{A}) = \sum_{i=1}^{r} -\mu(A_i) \log \mu(A_i).$$

If $\boldsymbol{A}, \boldsymbol{B}$ are two Borel partitions, so is $\boldsymbol{A} \vee \boldsymbol{B} = \{A \cap B : A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$. Setting $\boldsymbol{A}^n = \boldsymbol{A}_f^n = \boldsymbol{A} \vee f^{-1}\boldsymbol{A} \vee \cdots \vee f^{-(n-1)}\boldsymbol{A}$, one defines the entropies (of Kolmogorov and Sinai, see [3])

$$h_\mu(f, \boldsymbol{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\boldsymbol{A}^n) \quad \text{and} \quad h_\mu(f) = \sup_{\boldsymbol{A}} h(f, \boldsymbol{A}).$$

An important device for calculating $h_\mu(f)$ in some examples is Goodwyn's theorem [8]: $h_\mu(f) \leqq h(f)$. We shall use his ideas to prove a stronger statement for the case of *X finite dimensional*: $h_\mu(f) \leqq h_\mu(f, \boldsymbol{A}) + h_f^*(\varepsilon)$ where $\boldsymbol{A}$ is a finite Borel partition with diam $\boldsymbol{A} = \max\{\text{diam } A : A \in \boldsymbol{A}\} \leqq \varepsilon$. This reduces to Goodwyn's theorem when we take $\boldsymbol{A} = \{X\}$ and $\varepsilon = \text{diam } X$. If ε is an h-expansive constant for f, it gives $h_\mu(f) = h_\mu(f, \boldsymbol{A})$.

LEMMA 3.1. *If* $a_1, \dots, a_n \geqq 0$ *and* $s = \sum_{i=1}^{n} a_i \leqq 1$, *then*

$$\sum -\mu(a_i) \log \mu(a_i) \leqq s(\log n - \log s).$$

Proof. This is a well-known case of Jensen's inequality [13, pp. 11–12].

LEMMA 3.2. *Let* $A_1, A_2, \dots$ *be finite Borel partitions of* X *with* diam $A_m \to 0$. *Then* $h_\mu(f, A_m) \to h_\mu(f)$.

Proof. This is a slight variation of a well-known result of Rohlin. Looking at 6.3, 8.6 and 9.5 of [16], one sees that our lemma is implied by the following statement:

Given a Borel partition $\beta = \{B_1, \dots, B_n\}$ and $\varepsilon > 0$, then for large m we can find a partition $\alpha = \{C_1, \dots, C_n\}$ coarser than A_m (i.e. each C_i is the union of members of A_m) so that $\mu(B_i \triangle C_i) < \varepsilon$ for $1 \leqq i \leqq n$.

We now prove this statement. Since μ is a Borel measure, one can choose compact sets $K_i \subset B_i$ with $\mu(B_i \backslash K_i) < \varepsilon/n$. Choose $\delta > 0$ so that $d(K_i, K_j) > \delta$ for $i \neq j$ and suppose diam $A_m < \delta$. Form $\alpha = \{C_1, \dots, C_n\}$ coarser than A_m by putting $A \in A$ into

(a) C_i if $A \cap K_i \neq \varnothing$ or

(b) C_k if $A \cap K_i = \varnothing$ for all i.

This makes sense, for if $x \in A \cap K_i$ and $y \in A \cap K_j$, then

$$d(K_i, K_j) \leqq d(x, y) \leqq \operatorname{diam} A < \delta$$

and so $i = j$.

Clearly $C_i \supset K_i$. Hence $\mu(B_i \backslash C_i) \leqq \mu(B_i \backslash K_i) < \varepsilon/n$. Since $C_i \backslash B_i \subset \bigcup_{j \neq i} (B_j \backslash C_j)$, $\mu(C_i \backslash B_i) < (n-1)\varepsilon/n$. Thus $\mu(B_i \triangle C_i) < \varepsilon$.

REMARK. We used 3.2 in [4] but stated there (in the introduction) instead a stronger form—which we cannot prove.

Suppose now that $\mathfrak{B}$ is any finite cover of X. For $E \subset X$ let

$$F(E, \mathfrak{B}) = \{B \in \mathfrak{B} : B \cap E \neq \varnothing\}.$$

We give a very slight modification of Proposition 2 of [8].

LEMMA 3.3. *Let* $\mathfrak{B}$ *be a finite cover of* X *by closed sets such that each point* $x \in X$ *lies in at most* m *elements of* $\mathfrak{B}$. *There is a* $\delta > 0$ *so that* card $F(E, \mathfrak{B}^n) \leqq r_n(\delta, E)m^n$ *for all* $E \subset X$, $n \geqq 0$.

Proof. For each $x \in X$ choose a neighborhood U_x intersecting at most m elements of $\mathfrak{B}$. Let $U_{x_1}, \dots, U_{x_r}$ cover X and $\delta > 0$ be a Lebesgue number for this open cover. For each n let K_n be a set which (n, δ)-spans E and has $r_n(E, \delta)$ elements. For each $\beta \in F(E, \mathfrak{B}_f^n)$ pick $p(\beta) \in E \cap \beta$ and $q(\beta) \in K_n$ so that $d(f^t(q(\beta)), f^t(p(\beta))) \leqq \delta$ for $0 \leqq t \leqq n$. If $\beta = \bigcap_{t=0}^{n-1} B_{i_t}$, $B_{i_t} \in \mathfrak{B}$, then $f^t p(\beta) \in B_\delta(f^t q(\beta)) \cap B_{i_t} \neq \varnothing$. Since $B_\delta(f^t q(\beta))$ lies inside some U_{x_j}, for a given $q(\beta)$ there are at most m possibilities for B_{i_t}. It follows that, for $z \in K_n$, card $q^{-1}(z) \leqq m^n$. Hence card $F(E, \mathfrak{B}^n) \leqq (\text{card } K_n)m^n$.

DEFINITION. For A, B two Borel partitions let

$$b(\boldsymbol{A}, \boldsymbol{B}) = \max_{A \in \boldsymbol{A}} \operatorname{card} F(A, \boldsymbol{B}).$$

LEMMA 3.4. $$h_\mu(f, \boldsymbol{A} \vee \boldsymbol{B}) \leqq h_\mu(f, \boldsymbol{A}) + \liminf_{n\to\infty} \frac{1}{n} \log b(\boldsymbol{A}^n, \boldsymbol{B}^n).$$

Proof. Since $(\boldsymbol{A} \vee \boldsymbol{B})^n = \boldsymbol{A}^n \vee \boldsymbol{B}^n$,

$$H_\mu((\boldsymbol{A} \vee \boldsymbol{B})^n) = \sum_{\alpha \in \boldsymbol{A}^n} \sum_{\beta \in F(\alpha, \boldsymbol{B}^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta).$$

By Lemma 3.1

$$\sum_{\beta \in F(\alpha, \boldsymbol{B}^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta) \leqq \mu(\alpha)(\log b(\alpha, \boldsymbol{B}^n) - \log \mu(a))$$

and so

$$H_\mu((\boldsymbol{A} \vee \boldsymbol{B})^n) \leqq \log b(\boldsymbol{A}^n, \boldsymbol{B}^n) + H_\mu(\boldsymbol{A}^n).$$

Divide by n and let $n \to \infty$.

THEOREM 3.5. *Assume X is finite dimensional. Let A be a Borel partition of X with* diam $\boldsymbol{A} \leqq \varepsilon$. *Then $h_\mu(f) \leqq h_\mu(f, \boldsymbol{A}) + h_f^*(\varepsilon)$ for any normalized f-invariant Borel measure μ. If ε is an h-expansive constant for f, then $h_\mu(f) = h_\mu(f, \boldsymbol{A})$.*

Proof. Say dim $X = m-1$. Then for each $\gamma > 0$ we can find a finite closed cover $\mathfrak{B} = \mathfrak{B}(\gamma)$ with diameter $< \gamma$ and no point of X in more than m elements of $\mathfrak{B}$ (see [10]). Let M be a fixed positive integer.

Let $\boldsymbol{B} = \{B_1^*, \ldots, B_r^*\}$ be a Borel partition of X where $B_i^* \subset B_i$ and $\mathfrak{B} = \{B_1, \ldots, B_r\}$. We consider f^M with respect to the partition $\boldsymbol{B} \vee \boldsymbol{A}_f^M$. If $\alpha \in (\boldsymbol{A}_f^M)_{f^M}^n$ and $x \in \alpha$, then $\alpha \subset \bigcap_{s=0}^{nM-1} f^{-s} B_\varepsilon(f^s(x))$. Let $\delta > 0$ be as in Lemma 3.3 and $\beta > 0$ arbitrary. By 2.2 we have

$$r_n(\alpha, \delta, f^M) \leqq r_{Mn}(\alpha, \delta, f) \leqq c e^{(a+\beta)nM}$$

where $a = h_f^*(\varepsilon)$. Using Lemma 3.3 we get (the first inequality is obvious)

$$\operatorname{card} F(\alpha, \boldsymbol{B}_{f^M}^n) \leqq \operatorname{card} F(\alpha, \mathfrak{B}_{f^M}^n) \leqq c e^{(\alpha+\beta)nM} m^n$$

Applying Lemma 3.4,

$$h(f^M, \boldsymbol{B} \vee \boldsymbol{A}_f^M) \leqq h_\mu(f^M, \boldsymbol{A}_f^M) + M(a+\beta) + \log m.$$

Letting $\gamma \to 0$, diam $\boldsymbol{B} \vee \boldsymbol{A}_f^M \leqq$ diam $\boldsymbol{B} \leqq$ diam $\mathfrak{B}(\gamma) \to 0$ and so by Lemma 3.2

$$h_\mu(f^M) \leqq h_\mu(f^M, \boldsymbol{A}_f^M) + M(a+\beta) + \log m.$$

Now $h_\mu(f^M) = M h_\mu(f)$ and $h_\mu(f^M, \boldsymbol{A}_f^M) = M h_\mu(f, \boldsymbol{A})$. So

$$h_\mu(f) \leqq h_\mu(f, \boldsymbol{A}) + a + \beta + \frac{1}{M} \log m.$$

Letting $\beta \to 0$ and then $M \to \infty$, we get our result.

References

1. R. Adler, A. Konheim and M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319. MR **30** #5291.

2. D. Arov, *Calculation of entropy for a class of group endomorphisms*, Zap. Meh.-Mat. Fak. Har'kov. Gos. Univ. i Har'kov. Mat. Obšč. (4) **30** (1964), 48–69. (Russian) MR **35** #4368.

3. P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965. MR **33** #254.

4. R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1970), 401–414.

5. ———, *Topological entropy and Axiom* A, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 23–41. MR **41** #7066.

6. ———, *Periodic points and measures for Axiom* A *diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397.

7. E. Dinaburg, *A correlation between topological entropy and metric entropy*, Dokl. Akad. Nauk SSSR **190** (1970), 19–22 = Soviet Math. Dokl. **11** (1969), 13–16. MR **41** #425.

8. L. Goodwyn, *Topological entropy bounds measure theoretic entropy*, Proc. Amer. Math. Soc. **23** (1969), 679–688. MR **40** #299.

9. C. Pugh and M. Shub, *The Ω-stability theorem for flows*, Invent. Math. **11** (1970), 150–158.

10. W. Hurewicz and H. Wallman, *Dimension theory*, Chap. 5, Princeton Univ. Press, Princeton, N. J., 1941. MR **3**, 312.

11. H. Keynes and J. Robertson, *Generators for topological entropy and expansiveness*, Math. Systems Theory **3** (1969), 51–59. MR **40** #300.

12. A. Kolmogorov and V. Tihomirov, *ε-entropy and ε-capacity of sets in function spaces*, Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3–86; English transl., Amer. Math. Soc. Transl. (2) **17** (1961), 277–364. MR **22** #2890; MR **23** #A2031.

13. W. Parry, *Entropy and generators in ergodic theory*, Benjamin, New York, 1969. MR **41** #7071.

14. Ja. G. Sinaĭ, *On the concept of entropy for a dynamic system*, Dokl. Akad. Nauk SSSR **124** (1959), 768–771. (Russian) MR **21** #2036a.

15. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37** #3598.

16. V. Rohlin, *Lectures on the entropy theory of transformations with invariant measure*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 3–56 = Russian Math. Surveys **22** (1967), no. 5, 1–52. MR **36** #349.

Department of Mathematics, University of California, Berkeley, California 94720